A Very Short Proof of a Conjecture Concerning Set-to-Set Broadcasting*

Qiao Li, Zhongliang Zhang, and Junming Xu

Department of Mathematics, University of Science and Technology of China, Hefei, 230026, The People's Republic of China

We present a very short proof of a conjecture of Richards and Liestman concerning set-to-set broadcasting.

Richards and Liestman considered in [2] the following information dissemination problem that they called set-to-set broadcasting: Given two sets A and B, which may intersect, each member of A knows a unique message and is ignorant of the messages of the other members at the start of the process. A broadcasting from A to B is defined as a sequence of calls such that at the end of the sequence of calls every member in B learns every message from the set A. A call can be defined as a two-member subset of $A \cup B$, and during each call, the involved two members exchange all of the information they know at that moment.

Let $F(A,B)$ be the length of the shortest sequence of calls that completes broadcasting from A to B, assuming that all calls are possible. Richards and Liestman [2] proved that

$$F(A,B) = \begin{cases} |A| + |B| - |A \cap B| - 1, & \text{for } 0 \le |A \cap B| \le 2; \\ |A| + |B| - 3, & \text{for } |A \cap B| = 3. \end{cases}$$

$$F(A,B) \le |A| + |B| - 4, \qquad \text{for } |A \cap B| \ge 4. \quad (1)$$

They also conjectured that the equality holds in (1). In this note, we give a surprisingly simple proof of this conjecture. Our proof is based on the following facts:

(a) $F(A,B) = F(B,A)$. This is because the reversal of a valid calling scheme from A to B is a valid calling scheme from B to A.

(b) $F(A,A) = 2|A| - 4$ for $|A| \ge 4$. Since broadcasting from A to A is known as gossiping among members of A, the result is well known (e.g., see [1]).

(c) $F(A,B \cup \{x\}) \ge F(A,B) + 1$ for $x \notin A \cup B$.

Proof. Let S_x be a sequence of $F(A,B \cup \{x\})$ calls that completes broadcasting from A to $B \cup \{x\}$. Let $\{x,x_1\}, \{x,x_2\}, \ldots, \{x,x_m\} (m \ge 1)$ be the subsequence of S_x consisting of all calls containing x. We construct a sequence S of $F(A,B \cup \{x\}) - 1$ calls from S_x by deleting $\{x,x_1\}$ and replacing $\{x,x_i\}$ with $\{x_1,x_i\}$ for $i = 2, \ldots, m$. It is easy to verify that S completes broadcasting from A to B; then, we have $F(A,B \cup \{x\}) - 1 \ge F(A,B)$.

(c′) $F(A,B \cup C) \ge F(A,B) + |C|$ for $C \cap (A \cup B) = \varnothing$.

Proof of the conjecture. For $|A \cap B| \ge 4$, we have

$$F(A,B) = F(A,(A \cap B) \cup (B \setminus (A \cap B)))$$

$$\ge F(A,A \cap B) + |B \setminus (A \cap B)| \qquad \text{by } (c'),$$

*This work was partially supported by a grant of the Ministry of Education of China.

CCC 0028-3045/93/040449-02

$$F(A,A \cap B) = F(A \cap B,A)$$
$$= F(A \cap B,(A \cap B) \cup (A \setminus (A \cap B)))$$
$$\geq F(A \cap B,A \cap B) + |A \setminus (A \cap B)| \quad \text{by } (a) \text{ and } (c'),$$

$$F(A \cap B,A \cap B) \geq 2|A \cap B| - 4 \qquad \text{by } (b).$$

Then,

$$F(A,B) \geq F(A,A \cap B) + |B \setminus (A \cap B)|$$
$$\geq F(A \cap B,A \cap B) + |A \setminus (A \cap B)| + |B \setminus (A \cap B)|$$
$$\geq 2|A \cap B| - 4 + |A \setminus (A \cap B)| + |B \setminus (A \cap B)|$$
$$= |A| + |B| - 4. \tag{2}$$

The equality $F(A,B) = |A| + |B| - 4$ follows from (1) and (2).

REFERENCES

[1] S. M. Hedetniemi, S. T. Hedetmiemi, and A. L. Liestman, A survey of gossiping and broadcasting in communication networks. *Networks* **18** (1988) 319–349.

[2] D. Richards and A. L. Liestman, Generalizations of broadcasting and gossiping. *Networks* **18** (1988) 125–138.