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Local Strongly Arc-Connectivity in Regular Bipartite Digraphs*

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We show that for any vertex x of a d-regular bipartite digraph there are a vertex y, in the other class of the bipartition, and d (x, y)-paths and d (y, x)-paths such that all $2d$ of them are pairwise arc-disjoint. This result generalizes a theorem of Hamidoune and Las Vergnas for graphs. © 1993 Academic Press, Inc.

1. INTRODUCTION

Hamidoune and Las Vergnas [1] proved that for any vertex x of a d-regular bipartite graph G there is a vertex y of G, in the other class of the bipartition, joined to x by d pariwise edge-disjoint paths. In the present note we generalize this to digraphs by using the same techniques used in [1].

Let $D=(V, A)$ be a digraph (possibly with multiple arcs and loops) with the vertex-set $V=V(D)$ and the arc-set $A=A(D)$. A digraph D is said to be d-regular if both the outdegree $d_D^+(x)$ and the indegree $D_D^-(x)$ are equal to d for any $x \in V(D)$. Our result can be stated as follows:

THEOREM. *Let D be a d-regular bipartite digraph. For all vertex x of D there are a vertex y of D, in the other class of the bipartition, and d (x, y)-paths $P_1, P_2, \ldots, P_d$ and d (y, x)-paths $Q_1, Q_2, \ldots, Q_d$ such that all $2d$ of them are pairwise arc-disjoint.*

2. PROOF OF THEOREM

For two disjoint nonempty and proper subsets X and Y of $V(D)$ we denote by $A_D(X, Y)$ the set of the arcs of D joining X to Y and by $D[X]$ the subdigraph of D induced by X. Let $\bar{X}=V\backslash X$.

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LEMMA 1 (Kotzig [2], cf. Problem 6.47 in [3]). *Let x and y be two distinct vertices of a digraph D and the local strongly arc-connectivity be k between x and y. If the outdegree of each vertex of D is equal to its indegree, then there are k (x, y)-paths and k (y, x)-paths which are mutually arc-disjoint.*

LEMMA 2. *Let D be a digraph and X be a nonempty and proper subset of $V(D)$. If the outdegree of each vertex of D is equal to its indegree, then $|A_D(X, \bar{X})| = |A_D(\bar{X}, X)|$.*

Proof. In fact, since

$$\sum_{x \in X} d^+_{D[X]}(x) = \sum_{x \in X} d^-_{D[X]}(x),$$

and $d^+_D(x) = d^-_D(x)$ for each $x \in V(D)$ we have immediately that

$$|A_D(X, \bar{X})| = \sum_{x \in X} d^+_D(x) - \sum_{x \in X} d^-_{D[X]}(x)$$

$$= \sum_{x \in X} d^-_D(x) - \sum_{x \in X} d^-_{D[X]}(x) = |A_D(\bar{X}, X)|.$$

Proof of Theorem. The proof is by induction on $|V(D)|$. For $|V(D)| = 2$ the theorem is obvious (note that a bipartite digraph has no loops).

Let $V(D) = S \cup T$ be the bipartition of $V(D)$ and, without loss of generality, $x \in S$. If $|A_D(X, \bar{X})| \geqslant d$ for every proper subset X of $V(D)$ containing x, then the theorem follows immediately from Menger's theorem and Lemma 1. Next we suppose that there is a proper subset X of $V(D)$ such that $x \in X$ and $|A_D(X, \bar{X})| < d$. We can furthermore suppose that X is chosen in such a way that $|A_D(X, \bar{X})|$ takes the minimum possible value. Then $X \cap S \neq \emptyset$ and $X \cap T \neq \emptyset$ by our choice of X. Let

$$A_D(X, T \backslash X) = \{a_1, a_2, ..., a_s\}, \qquad A_D(X, S \backslash X) = \{a_{s+1}, a_{s+2}, ..., a_{s+t}\},$$

$$A_D(S \backslash X, X) = \{b_1, b_2, ..., b_{s'}\}, \qquad A_D(T \backslash X, X) = \{b_{s'+1}, b_{s'+2}, ..., b_{s'+t'}\}.$$

Then, by Lemma 2,

$$s + t = |A_D(X, \bar{X})| = |A_D(\bar{X}, X)| = s' + t'.$$

Considering that $X = (X \cap S) \cup (X \cap T)$ is the bipartition of the bipartite digraph $D[X]$ and counting the number of the arcs in $D[X]$ with initial vertices in $X \cap S$ and terminal vertices in $X \cap Y$, we have

$$d\,|X \cap S| - s = d\,|X \cap T| - s'.$$

Noting that $s+t=s'+t'<d$, we obtain, from the above,

$$s=s', \qquad t=t'.$$

Let D' be the digraph obtained from $D[X]$ by adding the arcs a_i' $(i=1, 2, ..., s+t)$ joining the initial vertex of a_i $(i=1, 2, ..., s+t)$ so the terminal vertex of b_i $(i=1, 2, ..., s'+t')$. Clearly D' is again a d-regular bipartite digraph and $|V(D')|<|V(D)|$. Hence by the induction hypothesis there are $y \in X \cap T$ and $d(x, y)$-paths $P_1', P_2', ..., P_d'$ and d (y, x)-paths $Q_1', Q_2', ..., Q_d'$ such that all $2d$ of them are pairwise arc-disjoint.

Let D'' be the digraph obtained from D by deleting all arcs in $D[X]$ and then identifying with x all vertices in X. Clearly the outdegree of each vertex of D'' is equal to its indegree. By the minimality of $|A_D(X, \bar{X})|=s+t$ and Lemma 2, the digraph D'' is strongly $(s+t)$-arc-connected. Hence by Lemma 1, for any $z \in \bar{X}$ there are $(s+t)$ (x, z)-paths and $(s'+t')$ (z, x)-paths which are mutually arc-disjoint in D''. Let $z \in \bar{X}$ and $X_1, X_2, ..., X_{s+t}$ and $Y_1, Y_2, ..., Y_{s'+t'}$ be these (x, z)-paths and, respectively, (z, x)-paths labelled such that $a_i \in A(X_i)$ $(i=1, 2, ..., s+t)$ and $b_i \in A(Y_i)$ $(i=1, 2, ..., s'+t')$. Then, for $j=1, 2, ..., d$, let P_j be the (x, y)-path of D obtained from P_j' by replacing each occurrence of an arc a_i', $i=1, 2, ..., s$, in P_j by $X_i \cap Y_i$. Similarly, for $j=1, 2, ..., d$, let Q_j be the (y, x)-path of D obtained from Q_j' by replacing each occurrence of an arc a_{s+i}, $i=1, 2, ..., t$ in Q_j by $X_{s+i} \cup Y_{s+i}$. Clearly $P_1, P_2, ..., P_d, Q_1, Q_2, ..., Q_d$ have the required properties which proves the theorem.

References

1. Y. O. Hamidoune and M. Las Vergnas, Local edge-connectivity in regular bipartite graphs, *J. Combin. Theory Ser. B* **44** (1988), 370–371.
2. A. Kotzig, Beitrag zur Theorie der endlichen gerichteten Graphen, *Wiss. Z. Martin-Luther-Univ. Hall-Wittenberg, Math-Natur. Reihe* **10** (1961–62), 118–125.
3. L. Lovász, "Combinatorial Problems and Exercises," North-Holland, Budapest, 1979.