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Note

A sufficient condition for equality of arc-connectivity and minimum degree of a digraph ☆

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Abstract

Let D be a simple digraph with order p, strong arc-connectivity $\lambda(D)$ and minimum degree $\delta(D)$. It is shown here that in order to obtain the equality $\lambda(D)=\delta(D)$, it is sufficient that there are $\lfloor \frac{1}{2}p \rfloor$ pairs of vertices x_i, y_i in D such that $d_D(x_i)+d_D(y_i)\geqslant p$ $(i=1,2,\ldots,\lfloor \frac{1}{2}p \rfloor)$. This is a generalization of a theorem of Goldsmith and White (1978) for graphs.

Goldsmith and White [1] proved the following theorem.

Theorem 1. *Let G be a simple graph with order p, edge-connectivity $\lambda(G)$ and minimum degree $\delta(G)$. If there are $\lfloor \frac{1}{2}p \rfloor$ pairs of vertices x_i, y_i in G such that*

$$d_G(x_i)+d_G(y_i)\geqslant p, \quad (i=1,2,\ldots,\lfloor \tfrac{1}{2}p \rfloor),$$

then $\lambda(G)=\delta(G)$.

Let $D=(V,A)$ be a simple digraph with the vertex-set $V=V(D)$ and the arc-set $A=A(D)$. For $x\in V(D)$, the degree of vertex x, denoted by $d_D(x)$, is the minimum value of its out-degree and in-degree. The minimum degree of D, denoted by $\delta(D)$, is the minimum value of $d_D(x)$ for all $x\in V(D)$. In the present note we will generalize Theorem 1 to digraphs. Our proof would seem simple than that given by Goldsmith and White for the unidirected case.

Theorem 2. *Let D be a simple digraph with order p, strong arc-connectivity $\lambda(D)$ and minimum degree $\delta(D)$. If there are $\lfloor \frac{1}{2}p \rfloor$ pairs of vertices x_i, y_i of D such that*

$$d_D(x_i)+d_D(y_i)\geqslant p, \quad (i=1,2,\ldots,\lfloor \tfrac{1}{2}p \rfloor),$$

then $\lambda(D)=\delta(D)$.

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Proof. Suppose that the theorem is false; let D be a digraph satisfying the conditions of the theorem, but for which $\lambda(D)<\delta(D)$. (It is always true, of course, that $\lambda(D)\leqslant\delta(D)$.) Moreover, among all such digraphs of order p, let D be one with the maximum number of arcs.

We will denote the $\lfloor \frac{1}{2}p \rfloor$ pairs of vertices in D as described in the statement of the theorem by $\pi(D)$. If the two vertices x, y of D are paired in $\pi(D)$ we will write $\{x, y\}\in\pi(D)$. Without loss of generality, we may suppose that the unpaired vertex is one with minimum degree when p is odd.

Let X and Y be two disjoint nonempty subsets of $V(D)$. We denote by (X, Y) the set of the arcs in D joining X to Y and by $D[X]$ the subdigraph of D induced by X.

Since the strong arc-connectivity of D is $\lambda(D)$, there exists a bipartition $\{X, Y\}$ of $V(D)$ such that $|(X, Y)|=\lambda(D)$. Let $|X|=m$ and $|Y|=n$, and so $m+n=p$. By the maximality of $|A(D)|$, it is clear that $D[X]=K_m$, $D[Y]=K_n$ and $(y, x)\in A(D)$ for all $x\in X$ and $y\in Y$. The symbol K_p denotes the complete symmetric digraph on p vertices. Consequently,

$$d_D^-(x)=p-1, \quad d_D(x)=d_D^+(x), \qquad \text{for every } x\in X,$$

$$d_D^+(y)=p-1, \quad d_D(y)=d_D^-(x), \qquad \text{for every } y\in Y.$$

Note that if D is changed to its inverse digraph $\vec{D}$ obtained from D by reversing the directions on all arcs, then $\lambda(D)=\lambda(\vec{D})$. Without loss of generality, we may, therefore, suppose $m\leqslant n$, otherwise we may consider $\vec{D}$. It does not add to the difficulties of the statement below whether the unpaired vertex, if it exists, is in X or Y. We may, therefore, suppose that the unpaired vertex is in X when p is odd. Thus

$$\lambda(D)<\delta(D) = m-1. \tag{1}$$

Let $x, x'\in X$ and $\{x, x'\}\in\pi(D)$. Since $d_D(x)+d_D(x')\geqslant p=m+n$, we have

$$|(\{x, x'\}, Y)|=d_D^+(x)+d_D^+(x')-2(m-1)$$

$$\geqslant d_D(x)+d_D(x')-2(m-1)\geqslant n-m+2.$$

Assume that there are r pairs of vertices x, x' of X for which $\{x, x'\}\in\pi(D)$ and denotes by X' the set of these vertices in X. Then $|X'|=2r$ and $|(X', Y)|\geqslant r(n-m+2)$. Let

$$|(X', Y)|=r(n-m+2)+s. \tag{2}$$

Let $X_i=\{x\in X\setminus X'\colon d_D^+(x)=m-1+i\}$, $i=0, 1$, and let $X_2=X\setminus(X'\cup X_0\cup X_1)$. So

$$|(X_1, Y)|=|X_1|, \tag{3}$$

and $|(X_2, Y)|\geqslant 2|X_2|$. Let

$$|(X_2, Y)|=|X_2|+t, \tag{4}$$

then

$$t \geqslant |X_2|. \tag{5}$$

It follows from (2)–(4) that

$$\begin{aligned}|X_0| &= |X| - |X'| - |X_2| - |X_1| \\ &= m - 2r - |(X_2, Y)| + t + |(X_1, Y)| \\ &= m - |(X', Y)| + r(n-m) + s - |(X_2, Y)| + t - |(X_1, Y)| \\ &= m - |(X', Y) \cup (X_2, Y) \cup (X_1, Y)| + r(n-m) + s + t \\ &= m - |(X, Y)| + r(n-m) + s + t,\end{aligned}$$

i.e.,

$$|X_0| = m - \lambda(D) + r(n-m) + s + t. \tag{6}$$

Let $Y' = \{y \in Y$: there is $y' \in Y$ such that $\{y, y'\} \in \pi(D)\}$ and let $Y_i = \{y \in Y$: there is $x \in X_i$ such that $\{x, y\} \in \pi(D)\}$, $i = 0, 1, 2$. Then

$$|Y_2| = |X_2|, \quad |Y_1| = |X_1|, \quad |Y_0| = |X_0| - q, \tag{7}$$

where

$$q = \begin{cases} 0 & \text{for } p \text{ even}, \\ 1 & \text{for } p \text{ odd}. \end{cases}$$

And so

$$\begin{aligned}|Y'| &= n - |Y_0| - |Y_1| - |Y_2| \\ &= n - (|X_0| - q) - |X_1| - |X_2| \\ &\geqslant m - |X_0 \cup X_1 \cup X_2| + q \\ &= |X'| + q.\end{aligned}$$

Observing that both $|X'|$ and $|Y'|$ are even, we have

$$|Y'| \geqslant 2(r+q). \tag{8}$$

Let $y, y' \in Y'$ with $\{y, y'\} \in \pi(D)$. Since $d_D(y) + d_D(y') \geqslant p = m + n$, we have

$$\begin{aligned}|(X, \{y, y'\})| &= d_D^-(y) + d_D^-(y') - 2(n-1) \\ &\geqslant d_D(y) + d_D(y') - 2(n-1) \\ &\geqslant m - n + 2.\end{aligned}$$

Thus we have

$$|(X, Y')| \geqslant (r+q)(m-n+2). \tag{9}$$

On the one hand, let $Y_0 = \{y_1, y_2, \ldots, y_k\}$, $X_0 = \{x_1, x_2, \ldots, x_k\}$ and $\{x_i, y_i\} \in \pi(D)$,

$i=1,2,\ldots,k$. Since $d_D^+(x_i)+d_D^-(y_i)\geqslant m+n$ for every $i=1,2,\ldots,k$, we have

$$\begin{aligned}|(X,Y_0)|&=\sum_{i=1}^{k}(d_D^-(y_i)-d_{D[Y]}^-(y_i))\\&\geqslant\sum_{i=1}^{k}((m+n-d_D^+(x_i))-(n-1))\\&=\sum_{i=1}^{k}(m+n-(m-1)-n+1)\\&=2|Y_0|.\end{aligned}$$

On the other hand, from (7), (2) and (4), we have

$$\begin{aligned}|(X,Y_O)|&=|(X,Y_O)|+|Y_1|-|X_1|\\&\leqslant|(X,Y_O)|+|(X,Y_1)|-|(X_1,Y)|\\&\leqslant|(X',Y)|+|(X_2,Y)|-|(X,Y')|\\&=r(n-m+2)+s+|X_2|+t-|(X,Y')|.\end{aligned}$$

Thus,

$$|Y_0|\leqslant\tfrac{1}{2}(r(n-m+2)+s+|X_2|+t-|(X,Y')|). \tag{10}$$

It follows from (6), (7) and (10) that

$$\lambda(D)\geqslant m+\tfrac{1}{2}(r(n-m)+s+t-|X_2|+|(X,Y')|)-r-q.$$

Considering that $s\geqslant 0$ and (5), we have

$$\lambda(D)\geqslant m+\tfrac{1}{2}(r(n-m)+|(X,Y')|)-r-q. \tag{11}$$

We will consider two cases, depending on the parity of p.

Suppose first that p is even. In this case $q=0$. If $n=m$, then we have $|(X,Y')|\geqslant 2r$ by (9). It follows from (11) that $\lambda(D)\geqslant m$, which contradicts (1). If $n\geqslant m+2$, then it is obvious from (11) that $\lambda(D)\geqslant m+r+|(X,Y')|\geqslant m$, but again this contradicts (1).

Suppose then that p is odd. In this case, $q=1$ and $n\geqslant m+1$. If $n=m+1$, we have $|(X,Y')|\geqslant r+1$ by (9). It follows from (11) that

$$\lambda(D)\geqslant m+\tfrac{1}{2}(r+r+1)-r-1>m-1,$$

which contradicts (1). If $n\geqslant m+2$, we immediately from (11) have

$$\lambda(D)\geqslant m+\tfrac{1}{2}(2r)-r-1=m-1,$$

which again contradicts (1).

The proof of Theorem 2 is completed. □

Reference

[1] D.L. Goldsmith and A.T. White, On graphs with equal edge-connectivity and minimum degree, Discrete Math. 23 (1978) 31–36.