2-Diameter of de Bruijn Graphs

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This paper shows that in the undirected binary de Bruijn graph of dimension n, $UB(n)$, which has diameter n, there exist at least two internally vertex disjoint paths of length at most n between any two vertices. In other words, the 2-diameter of $UB(n)$ is equal to its diameter n.

1. INTRODUCTION AND NOTATION

Binary strings are used as vertex labels or components of vertex labels. We will abbreviate "the vertex with label x" as "vertex x."

The *binary directed de Bruijn graph* of dimension n, denoted $B(n)$, has 2^n vertices which are labeled with the binary strings of length n. There is an arc from any vertex $x_1x_2 \cdots x_n$ to the vertices $x_2x_3 \cdots x_n0$ and $x_2x_3 \cdots x_n1$. We say that the ith coordinate of x is x_i. Any x_i is equal to 0 or 1, and $\bar{x}_i = 1 - x_i$.

The undirected binary de Bruijn graph $UB(n)$ is obtained from $B(n)$ by deleting the orientation of the arcs and omitting multiple edges and loops. It is well known that $UB(n)$ is 2-connected and that its diameter (maximum of the distances between all pairs of vertices) is equal to n. Due to their bounded maximum degree equal to 4 and their low diameter, de Bruijn graphs have been proposed as a possible good interconnection network for a parallel architecture [1]. They have also been shown to solve a wide class of problems [2].

For a k-connected graph, Hsu [4] introduced the notion of the k-diameter as follows: Given any two vertices x and y, a k-container between x and y is a set of k vertex disjoint paths between x and y. The length of a k-container is the length of the longest of the k paths. The k-distance $d_k(x, y)$ between x and y is the minimum of the lengths of all the k-containers. Then, the k-diameter of a graph G is the maximum of $d_k(x, y)$ for all pairs of vertices x and y. General results on the k-diameters of k-regular k-connected graphs can be found in [4] or [5]. Results for some particular classes of graphs and many open problems can also be found in [4].

It was shown by Pradhan and Reddy [7] that the 2-diameter of $UB(n)$ is less than or equal to $2n$. The 2-diameter is also obviously at least n. The aim of this paper was to prove that it is exactly n. In other words, between any two vertices of $UB(n)$, there exist two internally vertex disjoint paths of length at most n. This gives a good measure of the fault tolerance of the graph [6].

Let us first introduce some notation and recall some properties of the de Bruijn digraph $B(n)$:

Given two vertices x and y, we will denote $\boldsymbol{P}[x, y]$ as a shortest path $\boldsymbol{P}$ from x to y. The length of this path, denoted by $|\boldsymbol{P}[x, y]|$, is the number of edges in the path and is also the distance $d(x, y)$ from x to y.

$\boldsymbol{P}[x, y]$ also represents the set of vertices of the path, including its extremities.

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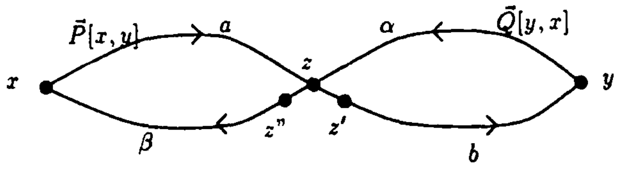

Fig. 1. The shortest paths $P[x, y]$ and $Q[y, x]$.

$P(x, y)$ will denote the set of vertices of the path excluding the extremities x and y.

$P(x, y]$ is the set of vertices including y, and excluding x (and similarly for $P[x, y)$).

Property 1.1. *Given any two vertices $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_n$ of $B(n)$ (the directed de Bruijn graph), there is a unique shortest path from x to y, and the distance $d(x, y)$ is equal to the smallest $i \le n - 1$ such that $x_{i+1} \cdots x_n = y_1y_2 \cdots y_{n-i}$ if it exists and to n otherwise.*

Property 1.2. *If C is a closed walk of length $l < n$ in the de Bruijn graph $B(n)$ and $z_1z_2 \cdots z_n$ is a vertex on C, then $z_i = z_{i+l}$ for all $1 \le i \le n - l$.*

2. PRELIMINARY RESULTS

Let x and y be two nonadjacent vertices of $B(n)$, and let us denote by $P = P[x, y]$ the shortest path from x to y in $B(n)$ with $|P| = p$, and by $Q = Q[y, x]$, the shortest path from y to x with $|Q| = q$.

In this section, if $P(x, y) \cap Q(y, x) \neq \emptyset$ and then if $z = z_1z_2 \cdots z_n$ is any vertex in the intersection, we will denote $|P[x, z]| = a$, $|P[z, y]| = b$, $|Q[y, z]| = \alpha$, and $|Q[z, x]| = \beta$ (see Fig. 1).

Let us first prove the following easy results.

Lemma 2.1. *For any two vertices x and y of the de Bruijn graph $B(n)$, if the shortest path from x to y intersects the shortest path from y to x in a vertex other than x and y, then, necessarily, the sum of the lengths of the two paths is strictly more than n.*

Proof. Let us consider a vertex z of the intersection such that its outneighbors z' and z'', respectively, on P and Q are distinct (see Fig. 1). With the notation introduced above, since z can be reached in a steps from x and its distance to y is b, then it can be written as

$$z = x_{a+1} \cdots x_n y_{n-b-a+1} \cdots y_{n-b} \tag{1}$$

and $z' = x_{a+2} \cdots x_n y_{n-b-a+1} \cdots y_{n-b}y_{n-b+1}$. But, also, z can be reached in α steps from y and its distance to x is β; thus, it can be written as

$$z = y_{\alpha+1} \cdots y_n x_{n-\beta-\alpha+1} \cdots x_{n-\beta} \tag{2}$$

and $z'' = y_{\alpha+2} \cdots y_n x_{n-\beta-\alpha+1} \cdots x_{n-\beta}x_{n-\beta+1}$.

We will show that if $|P| + |Q| = p + q \le n$, then $x_{n-\beta+1} = y_{n-b+1}$ and, thus, $z' = z''$, which leads to a contradiction.

Indeed, if $p + q \le n$, z is on a closed walk of length $a + \beta < p + q \le n$ and, thus, using Property 1.2 with the expression (2) of z, we have $y_{n-b+1} = y_{n-b+1-(a+\beta)} = y_{n-(p+q)+\alpha+1}$ and, thus, y_{n-b+1} is the $(n - (p + q) + 1)$th coordinate of z.

But, also, z is on a closed walk of length $b + \alpha < p + q \le n$ and, thus, using Property 1.2 again but with expression (1) of z, we have $x_{n-\beta+1} = x_{n-\beta+1-(b+\alpha)} = x_{n-(p+q)+a+1}$ and, therefore, $x_{n-\beta+1}$ is the $(n - (p + q) + 1)$th coordinate of z. ■

Proposition 2.2. *For any two vertices x and y of the de Bruijn graph $B(n)$, the union of the shortest path from x to y and the shortest path from y to x consists of at most three circuits.*

Proof. If we had at least four circuits, then, because we are considering shortest paths, there would be a vertex t in the intersection of the paths such that $P[x, t]$ and $Q[t, x]$ intersect in a vertex other than x and t and also $P[t, y]$ and $Q[y, t]$ intersect in a vertex other than t and y. Thus, by Lemma 2.1, we would have $|P[x, y]| + |Q[y, x]| = (|P[x, t]| + |Q[t, x]|) + (|P[t, y]| + |Q(y, t)|) > 2n$, a contradiction with the fact that a shortest path between any two vertices is of length at most n. ■

Let us now prove a lemma that will be repeatedly used in the proof of the main theorem and is illustrated partially in Figure 2.

Lemma 2.3. *With the notation given above, for any two vertices x and y of $B(n)$ and any vertex $z = z_1 \cdots z_n$ in $P(x, y) \cap Q(y, x)$, we have the following:*

If $b \le \beta < a$, then, if $u = \bar{z}_b \cdots \bar{z}_b z_{b+1} \cdots z_n$, and if $R_1 = R_1[u, y]$ is the shortest path from u to y and $R_2 = R_2[u, x]$ is the shortest path from u to x, then $|R_1| \le b$ and $|R_2| \le \beta$ and, moreover,

(*i*) $R_1 \cap P[x, y) = \emptyset$.

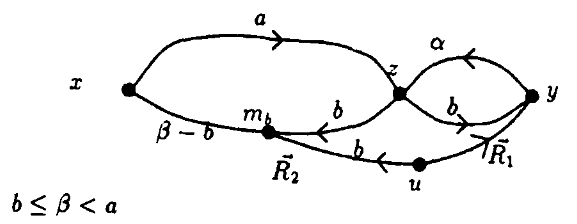

Fig. 2. The paths R_1 and R_2 of Lemma 2.3 when $|R_2| = \beta$.

(*ii*) $R_2 \cap P[z, y] = \emptyset$.
(*ii*) $R_2 \cap Q[y, z] = \emptyset$.
(*iv*) a. *If* $|R_2| < \beta$, *then* $|R_2| < b$ *and* $R_2 \cap P(x, z] = \emptyset$.
b. *If* $|R_2| = \beta$ *and* m_b *is the vertex at distance* b *from* z *on* $Q[z, x]$, *then* $R_2 \cap Q[z, x] = Q[m_b, x]$ *and* $R_2[u, m_b) \cap P(x, z] = \emptyset$.

Proof. Since $z = z_1 \cdots z_b z_{b+1} \cdots z_n$, $u = \bar{z}_b \cdots \bar{z}_b z_{b+1} \cdots z_n$, and $d(z, y) = b$, then, necessarily, $|R_1| = d(u, y) \le b$. Similarly, $d(z, x) = \beta$, and, by hypothesis, $\beta \ge b$, thus $|R_2| = d(u, x) \le \beta$.

(**i**) $R_1 \cap P[x, y) = \emptyset$.

Assume that there exists a vertex t in $R_1 \cap P[x, y)$, then, necessarily, t is on $P(z, y)$; otherwise, $P[x, y]$ would not be the shortest path from x to y. Let i be the distance from z to t on P and j be the distance from u to t, then, necessarily, from the definition of u and because $|R_1| \le b$, we have $j \le i < b$. Also, $|P[z, t]| = i$ implies that $t = z_{i+1} \cdots z_b z_{b+1} \cdots z_n \cdots$ and, thus, the $(b - i)th$ *coordinate of* t is z_b, and $|R_1[u, t]| = j$ implies that

$$t = \underbrace{\bar{z}_b \cdots}_{b-j} \bar{z}_b z_{b+1} \cdots z_n \cdots,$$

and since $j \le i$, the $(b - i)th$ coordinate of t is $\bar{z}_b$, which is a contradiction.

(**ii**) $R_2 \cap P[z, y] = \emptyset$.

Assume that there exists a vertex t' in $R_2 \cap P[z, y]$, then, necessarily, $t' \neq y$ and z; otherwise, $Q[y, x]$ would not be a shortest path. Let $|P[z, t']| = i < b$ and $|R_2[u, t']| = j$. Then, $|Q[z, x]| < |P[z, t']| + |R_2[t', x]|$ because $Q[z, x]$ is the shortest path from z to x. So, we have $\beta < i + (|R_2| - j)$, which implies that $i - j > 0$ since $|R_2| \le \beta$. Also, therefore, $j < i < b$. But using the same argument as in (i), this leads to a contradiction.

(**iii**) $R_2 \cap Q[y, z] = \emptyset$.

Otherwise, since R_2 is of length at most β, $Q[y, x]$ would not be the shortest path from y to x.

(**iv**) If $|R_2| < \beta$, then $|R_2| < b$. Indeed, if we had $|R_2| = r_2$ with $b \le r_2 < \beta$, then, from the definition of u, we would have a path from z to x of length $r_2 < \beta$, but this would contradict the shortness of $Q[z, x]$ which has length β.

Let us now first investigate $R_2 \cap Q[z, x]$ in case b. If $|R_2| = \beta$, and m_b is the vertex at distance b from z on $Q[z, x]$, then $m_b = z_{b+1} \cdots z_n \cdots$. Thus, there is also a path of length b from u to m_b; this path followed by $Q[m_b, x]$ is a path of length $b + (\beta - b)$ from u to x and thus is the path R_2 (because of Property 1.1). Obviously, $R_2[u, m_b)$ does not intersect $Q(m_b, x]$; otherwise, we would have a shorter path from u to x. Also, $R_2[u, m_b)$ does not intersect $Q[z, m_b)$; otherwise, there would be a vertex at a distance strictly less than b from both z and u, which is impossible by the definition of u.

Let us consider the set $\{t \in R_2$ such that $R_2[u, t] < b\}$. Then, if $|R_2| < \beta$, this set is the same as R_2 since we have seen that in that case $|R_2| < b$. Also, if $|R_2| = \beta$, then it follows from the above that this set is $R_2[u, m_b)$. So, to conclude the proof of (iv)a and b, it is sufficient to show that $\{t \in R_2$ such that $R_2[u, t] < b\} \cap P(x, z) = \emptyset$ (since we already know that z is not on R_2).

Assume that there exists a vertex t in that intersection with $|R_2[u, t]| = j < b$.

Let $|P[x, t]| = k$, then $k < a$. Note that the vertices x and t belong to a closed walk of length $s = k + r_2 - j$ (where $r_2 = |R_2|$).

Since $d(z, y) = b$, then $z = \cdots y_1 y_2 \cdots y_{n-b}$, and thus by definition, $u = \cdots y_1 y_2 \cdots y_{n-b}$. Since t can be reached from x on $P[x, y]$ in k steps and also $d(t, y) = p - k$, then

$$t = \underbrace{x_{k+1} \cdots x_{k+b-j}}_{b-j} x_{k+b-j+1} \cdots x_n \underbrace{y_{n-p+1} \cdots y_{n-(p-k)}}_{k}. \tag{3}$$

Also, since t can be reached from u on R_2 in j steps, $j < b$, and $d(t, x) = r_2 - j$, we can also write

$$t = \underbrace{\cdots\cdots\cdots\cdots}_{b-j} y_1 y_2 \cdots y_{n-b} \underbrace{x_{n-r_2+1} \cdots x_{n-(r_2-j)}}_{j}. \tag{4}$$

Then, if $k \ge j$ and if we denote $p' = k + b - j$, we have $x_{p'+1} \cdots x_n = y_1 \cdots y_{n-p'}$ and, thus, from Property 1.1, $d(x, y) \le p'$. But $p' = b + k - j < b + a = p$, which contradicts the fact that $P[x, y]$ is a shortest path from x to y of length p.

So, we can assume that $k < j$, so that $p' < b$. Also, by hypothesis of the lemma, $\beta < a$ and, thus, $s = k + r_2 - j < r_2 \le \beta < a$.

Therefore, we have $p' + s < b + a = p$. Let c be the biggest integer such that $p'' = p' + cs < p$. We have $p'' = p' + (c + 1)s - s \ge p - s > p - a$ and, thus, $p'' > b$.

Looking at the expression of t given in (3) and (4), since $j > k$ and $n - p'' < n - b$, we have $y_1 y_2 \cdots y_{n-p''} = x_{p'+1} \cdots x_{n-p''+p'}$.

Since t is on a closed walk of length s, then, if $t = t_1 t_2 \cdots t_n$, we have $t_i = t_{i+s}$ for all $1 \le i \le n - s$ (see Property 1.2). Thus, applying that c times in expression (3) of t we get that $x_{p'+1} \cdots x_{n-p''+p'} = x_{p'+1+cs} \cdots x_{n-p''+p'+cs}$

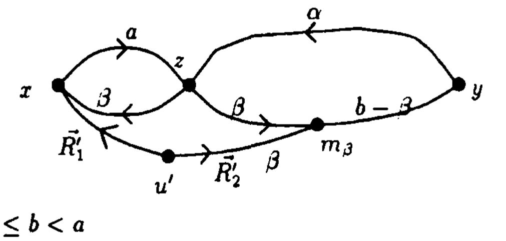

Fig. 3. The paths R'_1 and R'_2 of Lemma 2.4 when $|R'_2| = b$.

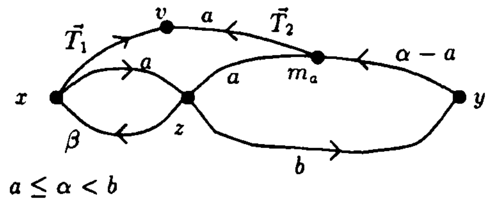

Fig. 4. The paths T_1 and T_2 of Lemma 2.5 when $|T_2| = \alpha$.

$= x_{p''+1} \cdots x_n$ and, thus, $y_1 y_2 \cdots y_{n-p''} = x_{p''+1} \cdots x_n$. From Property 1.1, this implies that $d(x, y) \le p''$. But since $p'' < p$, this leads to a contradiction. ∎

The following lemma is given without proof since it can clearly be deduced from the previous one by exchanging the roles of x and y (see Fig. 3).

Lemma 2.4. *If $\beta \le b < \alpha$, then if $u' = z_\beta \cdots z_\beta z_{\beta+1} \cdots z_n$, and if $R'_1 = R'_1[u', x]$ is the shortest path from u' to x, and $R'_2 = R'_2[u', y]$ is the shortest path from u' to y, then $|R'_1| \le \beta$ and $|R'_2| \le b$ and, moreover,*

(*i*) $R'_1 \cap Q[y, x) = \varnothing$.
(*ii*) $R'_2 \cap Q[z, x] = \varnothing$.
(*iii*) $R'_2 \cap P[x, z] = \varnothing$.
(*iv*) *a.* if $|R'_2| < b$, *then* $|R'_2| < \beta$ *and* $R'_2 \cap Q(y, z] = \varnothing$.
b. if $|R'_2| = b$ *and* m_β *is the vertex at distance* β *from* z *on* $P[z, y]$, *then* $R'_2 \cap P[z, y] = P[m_\beta, y]$ *and* $R'_2[u', m_\beta) \cap Q(y, z] = \varnothing$.

Now the following lemma can be deduced from Lemma 2.3 by symmetry and will also be given without proof (see Fig. 4).

Lemma 2.5. *If $a \le \alpha < b$, then if $v = z_1 \cdots z_{n-a} z_{n-a+1} \cdots z_{n-a+1}$, and if $T_1 = T_1[x, v]$ is the shortest path from x to v, and $T_2 = T_2[y, v]$ is the shortest path from y to v, then $|T_1| \le a$ and $|T_2| \le \alpha$ and, moreover,*

(*i*) $T_1 \cap P(x, y] = \varnothing$.
(*ii*) $T_2 \cap P[x, z] = \varnothing$.
(*iii*) $T_2 \cap Q[z, x] = \varnothing$.
(*iv*) *a. If* $|T_2| < \alpha$, *then* $|T_2| < a$ *and* $T_2 \cap P[z, y) = \varnothing$.
b. If $|T_2| = \alpha$ *and* m_a *is the vertex at distance* a *to* z *on* $Q[y, z]$, *then* $T_2 \cap Q[y, z] = Q[y, m_a]$ *and* $T_2(m_a, v] \cap P[z, y] = \varnothing$.

3. MAIN RESULT

In this section, if the union of $P[x, y]$ and $Q[y, x]$ consists of three circuits as shown in Figure 5, we will use the following notation: Let z^* (respectively, w) be the first (respectively, last) vertex that $P(x, y)$ has in common with $Q(y, x)$. Let w^* (respectively, z) be the first (respectively, last) vertex that $Q(y, x)$ has in common with $P(x, y)$.

Let $|P[x, z^*]| = a^*$, $|P[z^*, z]| = |Q[z^*, z]| = \epsilon$, $|P[z, w^*]| = l^*$, $|P[w^*, w]| = |Q[w^*, w]| = \omega$, $|P[w, y]| = b$, with $\alpha^* + \epsilon = a$, $l^* + \omega = l$, and $a + l + b = p$.

Let $|Q[y, w^*]| = \alpha^*$, $|Q[w, z^*]| = \lambda^*$, $|Q[z, x]| = \beta$, with $\alpha^* + \omega = \alpha$ and $\lambda^* + \epsilon = \lambda$, and $\alpha + \lambda + \beta = q$. All these integers are strictly positive except for ϵ and ω which may be equal to 0.

Before proceeding with the main result, we will first prove a lemma that will be useful in its proof.

Lemma 3.1. *If the union of $P[x, y]$ and $Q[y, x]$ consists of three circuits as shown in Figure 5, then the following inequalities are satisfied:*

(*i*) $\min\{a^*, b\} < \lambda^*$, $\min\{\alpha^*, \beta\} < l^*$.
(*ii*) $\alpha + \lambda > n - p + a$, $l + b > n - q + \beta$.

Proof of (i). Clearly, by symmetry, it is sufficient to prove the first inequality. Consider the sequence S of length $n + l + \epsilon$:

$$S = \overbrace{x_{a^*+1} x_{a^*+2} \cdots x_p}^{p - a^*} y_1 \cdots \overbrace{y_{n-p} y_{n-p+1} \cdots y_{n-b}}^{p - b}$$
$$= x_{a^*+1} x_{a^*+2} \cdots x_p x_{p+1} \cdots x_n y_{n-p+1} \cdots y_{n-b}.$$

The consecutive subsequences of length n of S represent the successive vertices of the shortest path from z^* to w along $P[x, y]$. The first n terms represent the vertex z^*, which is at distance a^* from x, and the last n terms represent the vertex w, whose distance to y is b.

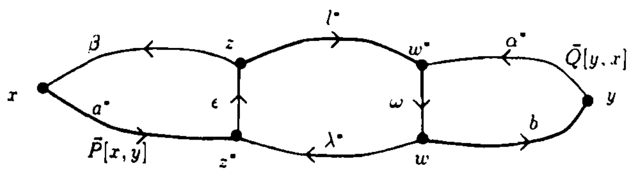

Fig. 5. $P[x, y] \cup Q[y, x]$ as a union of three circuits.

Since z^* and w are on the same closed walk of length $l + \lambda$, the sequence S has period $l + \lambda$, so if we shift the sequence to the left (or similarly to the right) by $l + \lambda$ positions, we must find equal subsequences.

First, if $a^* \leq b$, then assume by contradiction with (*i*) that $a^* \geq \lambda^*$.

If we shift S to the right by $l + \lambda$ positions in the second expression of S, since $l + \lambda \leq l + a = p - b \leq p - a^*$, we get that $x_{p+1-(l+\lambda)} \cdots x_n = y_1 \cdots y_{n-p+l+\lambda}$ and, thus, by Property 1.1, $d(x, y) \leq p - (l + \lambda)$. This contradicts the minimality of the path $P[x, y]$ of length p.

Now if $b < a^*$, then assume now again by contradiction with (*i*) that $b \geq \lambda^*$.

If we now shift S to the left by $l + \lambda$ positions in the first expression of S, since $l + \lambda \leq l + b + \epsilon = p - a^* < p - b$, then we get again that $x_{p+1-(l+\lambda)} \cdots x_n = y_1 \cdots y_{n-p+l+\lambda}$, which, as above, leads to a contradiction.

Proof of (ii) By Lemma 2.1, we have $|Q[y, z]| + |P[z, y]| > n$, which implies that $\alpha + \lambda + p - a > n$ or also that $q - \beta + l + b > n$. ■

Theorem 3.2. *The 2-diameter of the undirected de Bruijn graph $UB(n)$ is equal to its diameter n, which means that there exist two internally vertex disjoint paths of length at most n between any two vertices x and y.*

Proof. The result is trivially true for $n = 2$. It is also true for $n \geq 3$ if the vertices x and y are adjacent. Indeed, if $x = x_1x_2 \cdots x_n$ and $y = x_2 \cdots x_n\mu$, then let $y' = x_2 \cdots x_n\bar{\mu}$ *and* $x' = \bar{x}_1x_2 \cdots x_n$ (where y' may be x and, if not, x' may be y); then, the undirected paths $[x, y]$ and $[x, y', x', y]$ are two internally vertex disjoint paths of length at most 3. In other words, any two adjacent vertices belong to a cycle of length 3 or 4.

So, we can now assume that $n \geq 3$ and x and y are not adjacent. Let us denote by $P = P[x, y]$ the shortest path from x to y in $B(n)$ with $|P| = p$, and by $Q = Q[y, x]$, the shortest path from y to x with $|Q| = q$.

If P and Q are internally vertex disjoint, then the theorem is proved. If not, then, by Proposition 2.2, the union of P and Q consists of either two or three circuits. We will divide the proof by considering successively these two cases. In every subcase we will exhibit the two undirected paths P_1 and P_2 which give the solution. They will be described as the union of subdirected paths which have to be considered without their orientation. In all the figures which illustrate the different cases, P_1 is indicated with thick lines, and P_2, with dashed lines.

CASE 1. **$P \cup Q$ consists of two circuits.** Let z be the last common vertex of $P(x, y)$ and $Q(y, x)$ and z' and z'' its outneighbors on, respectively, P and Q. Let z^* be the first common vertex of $P(x, y)$ and $Q(y, x)$. Let $|P[x, z]| = a$, $|P[z, y]| = b$, $|Q[y, z]| = \alpha$, and $|Q[z, x]| = \beta$ (see Fig. 6).

1.1. If $a + \alpha \leq n$ and $b + \beta \leq n$, then, if $z^* \neq z$, we are obviously done by taking $P_1 = P[x, z^*] \cup Q[y, z^*]$ and $P_2 = Q[z, x] \cup P[z, y]$. Otherwise, let us consider the vertex $\hat{z} = \bar{z}_1z_2 \cdots z_n$ which has the same outneighbors as z; then, clearly, since every vertex of $B(n)$ has out-degree at most 2, $\hat{z}$ is not on P and not on Q. Thus, the undirected path $P_1 = P[x, z] \cup Q[y, z]$ of length $a + \alpha$ and the undirected path $P_2 = P[z'', y] \cup [z'', \hat{z}, z'] \cup Q[z', x]$ of length $b + \beta$ are vertex disjoint and of length at most n.

1.2. If $a + \alpha > n$ (a similar reasoning would occur if $b + \beta > n$), then $b + \beta = p + q - (a + \alpha) < p + q - n \leq p$.

If $b \leq \beta$, applying Lemma 2.3, we can take $P_1 = P$ and $P_2 = R_1 \cup R_2$, which are internally vertex disjoint and both of length $\leq n$ (see Fig. 2).

If $\beta < b$, then applying Lemma 2.4, we can take $P_1 = Q$ and $P_2 = R'_1 \cup R'_2$ (see Fig. 3).

CASE 2. **$P \cup Q$ consists of three circuits.**

If $a^* + \lambda^* + b \leq n$ and $\alpha^* + l^* + \beta \leq n$, then we are done. This is clear if $\epsilon > 0$ and $\omega > 0$. If $\epsilon = 0$, which means that $z = z^*$, then consider the vertex $\bar{z}_1z_2 \cdots z_n$ which has the same outneighbors as z. Also, if $\omega = 0$, which means that $w = w^*$, then consider the vertex $\bar{w}_1w_2 \cdots w_n$ which has same outneighbors as w. Because of the degrees of the vertices of $B(n)$, neither of these vertices is on $P[x, y]$ or $Q[y, x]$. Also, it is easy to see that we have two internally vertex disjoint undirected paths between x and y of length at most n (similar to the proof given in Case 1.1).

We now assume without loss of generality **that** $\alpha^* + l^* + \beta > n$ **and also that** $a^* \geq b$ (the case where $b \geq a^*$ can be treated similarly).

Then, since $(a + l^* + \omega + b) + (\alpha^* + \omega + \lambda + \beta) = p + q$, this implies that $a + \lambda + b + 2\omega < p + q - n$. We can now apply Lemma 2.3 to the paths between z and y. Indeed, the vertex w is on the intersection of $P(z, y)$ and $Q(y, z)$ of respective length $p - a$ and $q - \beta$. From the above, we have that

$$\lambda < l - 2\omega + q - n < l. \quad (5)$$

Also, by assumption $b \leq a^*$ and by Lemma 3.1, $\min\{a^*, b\} < \lambda^*$, so that $b \leq \lambda$.

Thus, the hypotheses of Lemma 2.3 are satisfied (with the corresponding parameters which do not necessarily

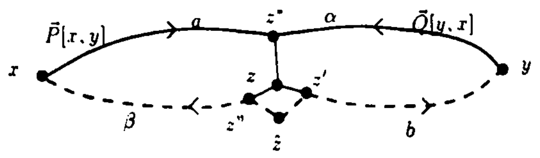

Fig. 6. Illustration of case 1: $P \cup Q$ consists of two circuits.

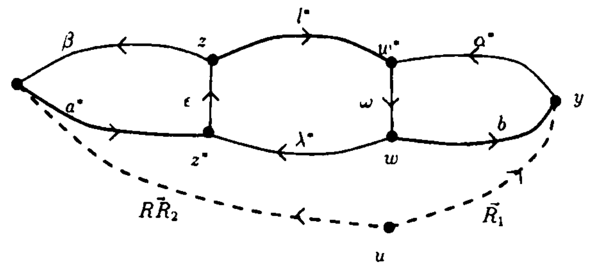

Fig. 7. Case 2 with $|R_2| < \lambda$.

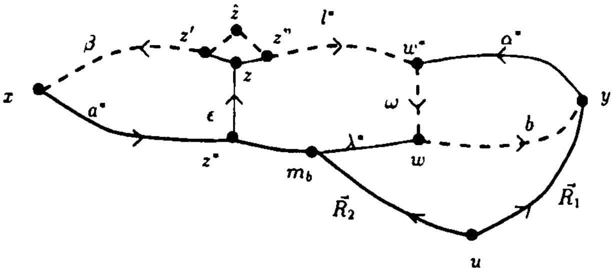

Fig. 8. Illustration of Case 2.1.1.

have the same label: Here, λ and l instead of, respectively, β and a in the lemma), so that if $u = \bar{w}_b \cdots \bar{w}_b w_{b+1} \cdots w_n$, $\boldsymbol{R}_1$ is the shortest path from u to y, and $\boldsymbol{R}_2$, the shortest path from u to z, then $|\boldsymbol{R}_1| \le b$, $|\boldsymbol{R}_2| \le \lambda$ and also $\boldsymbol{R}_1 \cap \boldsymbol{P}[z, y) = \varnothing$. But, clearly, $\boldsymbol{R}_1$ does not intersect $\boldsymbol{P}[x, z]$; otherwise, $\boldsymbol{P}[x, y]$ would not be a shortest path from x to y. So, $\boldsymbol{R}_1 \cap \boldsymbol{P}[x, y) = \varnothing$.

Assume that $|\boldsymbol{R}_2| < \lambda$; then, from Lemma 2.3, $|\boldsymbol{R}_2| < b$. Let us show that we also have $\beta + \lambda < a + l$. If not, using (5), we have $\beta \ge a + l - \lambda = p - \lambda - b > a \ge a^*$ and, thus, by hypothesis, $\beta > b$. So, we can consider the vertex x' at distance $b - |\boldsymbol{R}_2|$ from z on $\boldsymbol{Q}(z, x)$. Clearly, there is a path from u to x' of length b and, therefore, also a path from w to x' of length b. So, $d(w, x') \le b$. But, also, $|\boldsymbol{Q}[w, x']| > \lambda$. This contradicts the result of Lemma 3.1 which says that $b \le \lambda^*$.

So, we have $\beta + \lambda < a + l$. From Lemma 3.1, we also deduce that $b \le \beta + \lambda$. Therefore, we can now apply again Lemma 2.3 with w an intersection vertex of the paths $\boldsymbol{P}(x, y)$ and $\boldsymbol{Q}(y, x)$ (and the corresponding parameters, $\beta + \lambda$, $a + l$ instead of, respectively, β, a in the lemma). We have u and $\boldsymbol{R}_1$ as before, and if $\boldsymbol{RR}_2$ is the shortest path from u to x, then $|\boldsymbol{RR}_2| = d(u, x) \le d(u, z) + d(z, x) < b + \beta < \lambda + \beta$. Since we have a strict inequality, then by Lemma 2.3(iv), we have $\boldsymbol{RR}_2 \cap \boldsymbol{P}(x, y) = \varnothing$.

If we consider the undirected paths $P_1 = \boldsymbol{P}$ and $P_2 = \boldsymbol{R}_1 \cup \boldsymbol{RR}_2$, they are internally vertex disjoint of length at most n. Indeed, $|P_2| \le b + \lambda + \beta < b + a + l = p \le n$ (see Fig. 7).

We now assume that $|\boldsymbol{R}_2| = \lambda$.

Note that in this case we have $\boldsymbol{R}_2 \cap \boldsymbol{Q}[w, z] = \boldsymbol{Q}[m_b, z]$, where m_b is the vertex at distance b from w on $\boldsymbol{Q}[w, z]$. We have

(i) $m_b \in \boldsymbol{Q}(w, z^*)$, since $b < \lambda^*$.
(ii) $\boldsymbol{R}_2[u, m_b) \cap \boldsymbol{P}(x, y) = \varnothing$.

Indeed, if there was a vertex $t \in \boldsymbol{R}_2[u, m_b) \cap \boldsymbol{P}(x, y)$, then by Lemma 2.3, the vertex t would be on $\boldsymbol{P}(x, z^*)$. But the shortest path from u to z cannot go through a vertex t of $\boldsymbol{P}(x, z^*)$ and then through m_b since the shortest path from t to z follows the arcs of $\boldsymbol{P}$.

Now, let us assume that $\boldsymbol{R}_2[u, m_b) \cap \boldsymbol{Q}[z, x) \neq \varnothing$ and let x' be a vertex of the intersection. Then, $|\boldsymbol{R}_2[u, x']| = b' < b$, and, necessarily, $|\boldsymbol{Q}[x', x]| < b - b'$; otherwise, there would be a vertex x'' on $\boldsymbol{Q}(x', x]$ at distance $b - b'$ from x' and, thus, a path from u to x'' of length b, but then also a path from w to x'' of length b. Also, since $b < \lambda^*$, this would contradict the shortness of $\boldsymbol{Q}[y, x]$.

So, we can now assume that $\boldsymbol{R}_2[u, m_b) \cap \boldsymbol{Q}[z, x] = \varnothing$, and, hence,

(iii) $\boldsymbol{R}_2[u, m_b) \cap \boldsymbol{Q}[y, x] = \varnothing$.

SUBCASE 2.1. $\beta + l + b \le n$. First notice that $\boldsymbol{R}_1 \cap \boldsymbol{P}(x, z^*) = \varnothing$ by the shortness of $\boldsymbol{P}[x, y]$.

2.1.1. $\boldsymbol{R}_1 \cap \boldsymbol{Q}[z, x] = \varnothing$ (see Fig. 8).
Consider the undirected paths $P_1 = \boldsymbol{P}[x, z^*] \cup \boldsymbol{R}_2[u, z^*] \cup \boldsymbol{R}_1$ and $P_2 = \boldsymbol{Q}[z, x] \cup \boldsymbol{P}[z, y]$ if $z \neq z^*$ and $P_2 = \boldsymbol{Q}[z', x] \cup [\hat{z}, z'] \cup [\hat{z}, z''] \cup \boldsymbol{P}[z'', y]$ if $z = z^*$, where $\hat{z} = \bar{z}_1 z_2 \cdots z_n$. These paths are internally vertex disjoint and we have $|P_1| \le a^* + \lambda^* + b < p + q - n < n$ and $|P_2| = \beta + l + b \le n$ by hypothesis of this Subcase 2.1.

2.1.2. $\boldsymbol{R}_1 \cap \boldsymbol{Q}[z, x] \neq \varnothing$ (see Fig. 9).
If $x' \in \boldsymbol{R}_1 \cap \boldsymbol{Q}[z, x]$, then x' is different from z and from x by the shortness of $\boldsymbol{P}[x, y]$. Consider the undirected paths $P_1 = \boldsymbol{P}[x, y]$ and $P_2 = \boldsymbol{R}_1[x', y] \cup \boldsymbol{Q}[x',$

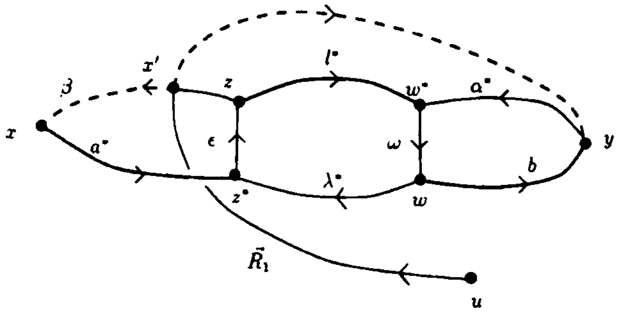

Fig. 9. Illustration of Case 2.1.2.

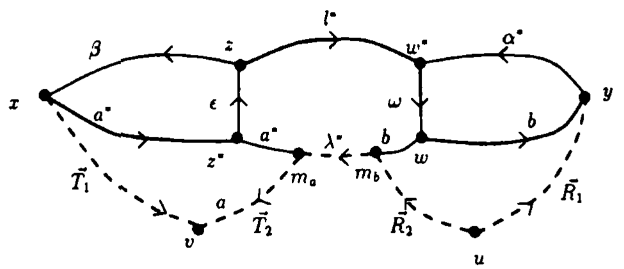

Fig. 10. Illustration of Case 2.2.1.

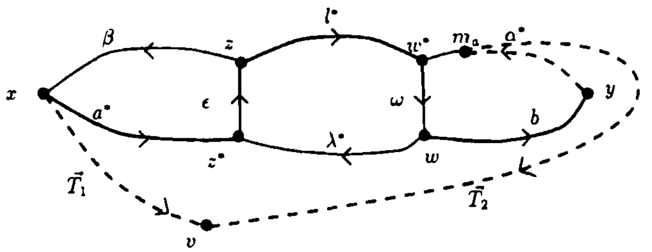

Fig. 11. Illustration of Case 2.2.2.

$x]$. They are internally vertex disjoint and $|P_2| < b + \beta < n$.

SUBCASE 2.2. $\beta + l + b > n \Leftrightarrow a + \lambda + \alpha < p + q - n \Leftrightarrow \alpha + \lambda < l + b + q - n$. By Lemma 3.1, we know that $a < \lambda + \alpha$. We will apply Lemma 2.5 to the paths $P(x, y)$ and $Q(y, x)$ which intersect in z (with the corresponding parameters; here, $\lambda + \alpha$, $l + b$ instead of, respectively, α, b in the lemma). If we consider the vertex $v = z_1 z_2 \cdots z_{n-a} \bar{z}_{n-a+1} \cdots \bar{z}_{n-a+1}$, $T_1 = T_1[x, v]$ the shortest path from x to v, and $T_2 = T_2[y, v]$ the shortest path from y to v, we have $|T_1| \le a$, $|T_2| \le \alpha + \lambda$ and $T_1 \cap P(x, y] = \emptyset$.

If $|T_2| < \alpha + \lambda$, then by Lemma 2.5, we also have $T_2 \cap P[x, y) = \emptyset$. So, we can take the two undirected paths $P_1 = P$ and $P_2 = T_1 \cup T_2$. They are internally vertex disjoint and $|P_2| < a + \alpha + \lambda$ and, thus, less than n by the hypothesis of this Subcase 2.2.

So, assume that $|T_2| = \alpha + \lambda$; then, by Lemma 2.5, if we call m_a the vertex of $Q(y, z)$ whose distance to z is equal to a, then $|T_2[m_a, v]| = a$. Also, m_a is on $Q[y, z^*)$ (indeed, we have $a > \epsilon$) and $T_2(m_a, v]$ has no vertex in common with P and Q.

We now distinguish several cases depending on the location of m_a on $Q[y, z^*)$.

2.2.1. $m_a \in Q[m_b, z^*)$ (see Fig. 10).

In that case, we can take for the two internally vertex disjoint undirected paths, $P_1 = P$ and $P_2 = T_1 \cup T_2[m_a, v] \cup R_2[u, m_a] \cup R_1[u, y]$. We have $|P_2| \le a + |Q[m_a, z]| + |Q[w, m_a]| + b = a + \lambda + b$, and thus from the assumption taken at the beginning, $|P_2| < p + q - n \le n$.

2.2.2. $m_a \in Q[y, w^*)$ (see Fig. 11).

In that case, we take $P_1 = P$ and $P_2 = T_1 \cup T_2$. We have $|P_2| \le a + \lambda + \alpha$, which is less than n by the hypothesis of Subcase 2.2.

2.2.3. $m_a \in Q[w^*, m_b)$ (see Fig. 12).

(a) There exists a vertex t in $T_2 \cap R_2$.

Then, $t \in T_2[m_a, v] \cap R_2[u, m_b]$. Consider the undirected paths $P_1 = P[x, y]$ and $P_2 = T_1 \cup T_2[t, v] \cup R_2[u, t] \cup R_1$.

We have $|T_2[m_a, t]| + |T_2[t, v]| = |T_2[m_a, v]| = |Q[m_a, z]| \le |T_2[m_a, t]| + |R_2[t, z]|$, and, thus, $|T_2[t, v]| + |R_2[u, t]| \le |R_2[t, z]| + |R_2[u, t]| = \lambda$. Therefore, we have $|P_2| \le a + \lambda + b$ and, thus, $|P_2|$ is of length less than n by (5).

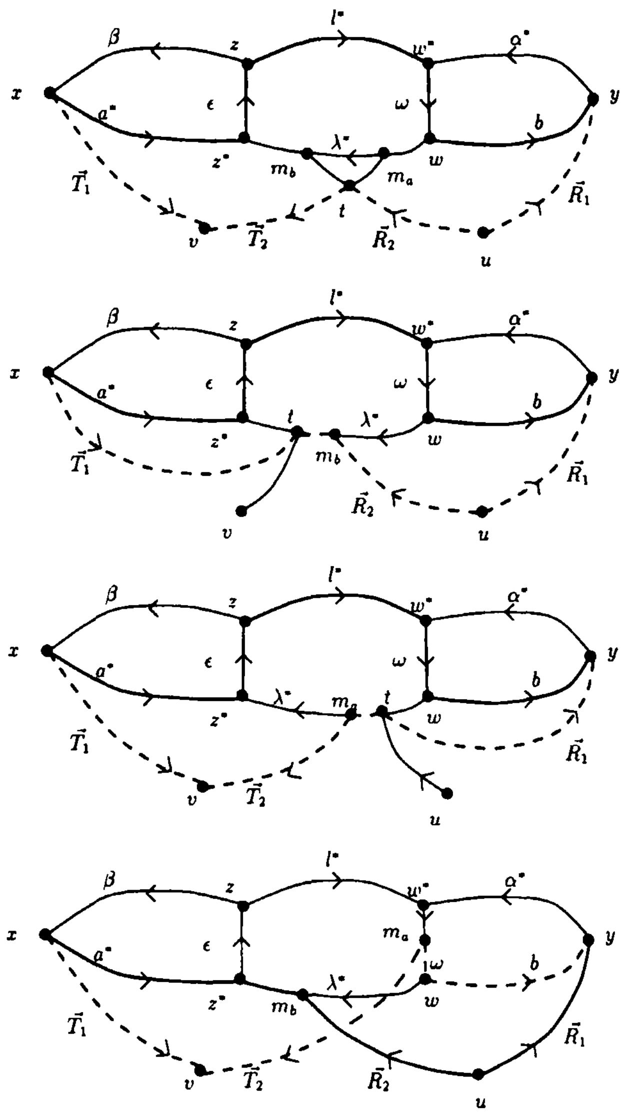

Fig. 12. Illustration of Case 2.2.3(a)–(d).

(b) There exists a vertex t in $T_1 \cap R_2$.

Then, necessarily, $t \in R_2[u, z^*]$, since $T_1 \cap P(x, y) = \emptyset$. Take $P_1 = P$ and $P_2 = T_1[x, t] \cup R_2[u, t] \cup R_1$. We have $|P_2| < a + \lambda + b < n$ from (5).

(c) There exists a vertex t in $R_1 \cap T_2[w^*, v]$.

Then, t is not in $Q[w^*, w]$ since $R_1 \cap P(x, y) = \emptyset$, and, thus, we also have $T_2[t, v] \cap P(x, y) = \emptyset$. Consider the undirected paths $P_1 = P$ and $P_2 = T_1 \cup T_2[t, v] \cup R_1[t, y]$. They are internally disjoint, and $|P_2| \le a + |T_2[w, v]| + b \le a + \lambda + b < n$ by (5).

(d) We are in Subcase 2.2.3 but in none of the above (a)-(c).

Note that we always have $T_1 \cap R_1 = \emptyset$; otherwise, $P[x, y]$ would not be a shortest path. Take $P_1 = R_1 \cup R_2[u, z^*] \cup P[x, z^*]$ and $P_2 = T_1 \cup T_2[m_a, v] \cup P[w, y] \cup \Phi$, where $\Phi = P[m_a, w]$ if $m_a \in Q[w^*, w]$ and $\Phi = Q[w, m_a]$ if $m_a \in Q[w, m_b]$. These paths are internally vertex disjoint, and $|P_1| \le b + \lambda + a < n$ and, in both cases, $|P_2| \le a + (\lambda + 2\omega) + b < n$ by (5). ∎

4. CONCLUSION

We have proved that the 2-diameter of the binary de Bruijn graph $UB(2, n)$ is equal to its diameter n. It is known that the undirected de Bruijn graph $UB(d, n)$, whose vertices are words of length n on an alphabet of size d, is $2d - 2$ connected [3]. It would be interesting to find the value of the $2d - 2$-diameter of $UB(d, n)$. It would be also interesting to study the connectivity and the k-diameters of the generalized de Bruijn graphs (see [4] for definitions and more problems).

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REFERENCES

[1] J. Bermond and C. Peyrat, de Bruijn and Kautz networks: a competitor for the hypercube? *Proceedings of the 1st European Workshop on Hypercube and Distributed Computers*, North-Holland, Amsterdam (1989) 279–294.

[2] D. K. Pradhan, Fault tolerant VLSI architectures based on de Bruijn graphs (Galileo in the mid nineties). *Reliability of Computer and Communication Networks, DIMACS Series in Discr. Math. Theor. Comput. Sci.* **5** (1991) 183–196.

[3] A. Esfahanian and L. Hakimi, Fault tolerant routing in de bruijn networks. *IEEE Trans. Comput.* **34**(9) (1985) 777–788.

[4] F. Hsu, On container width and length in graphs, groups, and networks. *IEICE Trans. Fundam.* **E(77A)** (1994) 668–680.

[5] F. Hsu and T. Luczak, Note on the k-diameter of k-regular k-connected graphs. *Discr. Math.*, **132** (1994) 291–296.

[6] F. Hsu and Y. Lyun, A graph theoretical study of transmission delay and fault tolerance. *Proceedings of the 4th ISSM International Conference on Parallel and Distributed Computing and Systems* (1991) 20–24.

[7] D. Pradhan and S. Reddy, A fault tolerant communication architecture for distributed systems. *IEEE Trans. Comput.* **31**(9) (1982) 863–870.

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