Appl. Math.-JCU
13B(1998), 179-187

CONNECTIVITY OF CARTESIAN PRODUCT DIGRAPHS AND FAULT-TOLERANT ROUTINGS OF GENERALIZED HYPERCUBE

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Abstract In this paper, the problem of fault-tolerant routings in fault-tolerant networks is considered. A routing in a network assigns to each ordered pair of nodes a fixed path. All communication among nodes must go on this routing. When either a node or a link in a fault-tolerant network fails, the communication from one node to another using this faulty element must be sent via one or more intermediate nodes along a sequence of paths determined by this routing. An important and practical problem is how to choose a routing in the network such that intermediate nodes to ensure communication are small for any fault-set. Let C_d be a directed cycle of order d. In this paper. The author first discusses connectivity of Cartesian product digraphs, then proves that the Cartesian product digraph $C_{d_1}\times C_{d_2}\times\ldots\times C_{d_n}$ $(d_i\geq 2, 1\leq i\leq n)$ has a routing such that at most one intermediate node is needed to ensure transmission of messages among all non-faulty nodes so long as the number of faults is less than n. This is a generalization of Dolev et al's result for the n-dimensional cube.

§1 Introduction

We consider in the present paper the problem of fault-tolerant routings in highly fault-tolerant telecommunication or interconnection networks. A routing ρ assigns to each ordered pair (x, y) of nodes in a network a fixed (directed) path from x to y, denoted by $\rho(x, y)$. All communication among nodes must go on this fixed routing. Let F be the set of faulty nodes and/or links in the fault-tolerant network, x and y be two non-faulty nodes. We say that $\rho(x, y)$ avoids F if it does not include any element of F. If $\rho(x, y)$ avoids F, then the node x sends messages to the node y along $\rho(x, y)$. The question is that if $\rho(x, y)$ can not avoid F, then x can no longer use this fixed path $\rho(x, y)$ to send

Received February 3, 1997.
1991 *MR Subject Classification*: 05C12, 05C40.
Keywords: Fault-tolerant networks, routings, digraphs, hypercube, connectivity, diameter.
Supported partly by the National Natural Science Foundation of China (19671057).

messages to y. However, since the network is fault-tolerant there must exist several intermediate nodes, say, $x_1, x_2, \ldots, x_n$ such that $\rho(x, x_1), \rho(x_1, x_2), \ldots, \rho(x_n, y)$ all avoid F. Then x sends messages to y via intermediate nodes $x_1, x_2, \ldots, x_n$ along the fixed paths $\rho(x, x_1), \rho(x_1, x_2), \ldots, \rho(x_n, y)$. In order that the communication does not become too long, an important and practical problem is how to choose a routing in the network such that the number of intermediate nodes used is as small as possible.

It is well known that any network can be modelled by a graph or digraph $G=(V, A)$, where the vertex-set $V=V(G)$ represents the node-set of the network and the edge-set or the arc-set $A=A(G)$ represents the link-set of the network. With this model, the transmission delay and the fault-tolerance of the network are often measured by, respectively, the diameter and the connectivity of the corresponding graph or digraph G. Following Dolev et al[1], we can formalize the foregoing problem as follows.

Given a (strongly) connected simple (without loops and paralled edges or arcs) graph (digraph) G, a routing ρ and a fault-set F, $|F| < \kappa(G)$, the (strong) connectivity of G, the surviving routing graph (digraph) $R(G, \rho)/F$ is such a simple graph (digraph) with the same vertices as $G-F$, and a vertex x being adjacent to a vertex y if and only if $\rho(x, y)$ avoids F. If we assume that the time to send a message along a fixed path is independent of its length, then the diameter of $R(G, \rho)/F$, denoted by $\mathrm{Diam}(R(G, \rho)/F)$, also gives a good estimate of the time required to complete transmission of messages in the presence of faults in G. As the fault-set F is not known in advance, the parameter

$$\mathrm{Diam}(G, \rho) = \max_{F:\, |F| < \kappa(G)} \mathrm{Diam}(R(G, \rho)/F)$$

is an important measure to estimate if the routing ρ is efficient and reliable. Thus, our problem will be how to choose a routing ρ in G such that $\mathrm{Diam}(G, \rho)$ is as small as possible.

This problem was first introduced by Dolev et al[1]. They proved that for any $(k+1)$-connected graph G, there is a routing ρ such that $\mathrm{Diam}(G, \rho) \leq \max\{2k, 4\}$. In particular, they claimed that the n-dimensional cube Q_n has $\mathrm{Diam}(Q_n, \rho) \leq 3$ for any minimal length routing ρ (i.e. every fixed path between any pair of nodes in ρ is a minimal length path between them in Q_n). However, their proof seems to have a flaw.

For a (strongly) connected (di)graph G, it is clear that $\mathrm{Diam}(G, \rho) = 1$ if $F = \varnothing$ and $\mathrm{Diam}(G, \rho) \geq 2$ if $F \neq \varnothing$ for any routing ρ in G. G is good if there is a routing ρ in G such that $\mathrm{Diam}(G, \rho) \leq 2$. The network modelled by a good (di) graph is called a good fault-tolerant network. We are interested in obtaining good fault-tolerant networks. But only a few of such networks are known as far as we know.

For instance, it is easy to check that the complete graph (or digraph) is good. Dolev et al proved that the n-dimensional cube Q_n is good. Homobono and Peyrat [3] proved that the de Bruijn digraphs and the Kautz digraphs are good. Escudero et al [2] proved that the double loop networks are good.

In this paper, we will generalize Dolev et al's results for the cube to a class of more general digraphs, which are called the generalized hypercube. Namely, we will prove that the Cartesian product digraphs $C_{d_1}\times C_{d_2}\times\ldots\times C_{d_n}$, where C_{d_i} is a directed cycle of order $d_i \geq 2$, $(i=1,2,\ldots,n)$ and $n\geq 1$, are good and Diam $(C_{d_1}\times C_{d_2}\times\ldots\times C_{d_n},\rho)\leq 3$ for any minimal length routing ρ. Our proof makes up for a flaw in Dolev et al's proof.

The rest of the paper is organized as follows. In Section 2 the definition and some fundamental properties of Cartesian product digraphs are given. The fault-tolerant routings in the generalized hypercube are considered in Section 3.

§2 Some Properties of Cartesian Product Digraphs

In this section we first give the definition of the Cartesian product digraph and then discuss their several properties. The Cartesian product digraph G of n simple digraphs G_1, $G_2,\ldots,G_n$, denoted by $G=G_1\times G_2\times\ldots\times G_n$ is the simple digraph with the vertex-set $V(G)=V(G_1)\times V(G_2)\times\ldots\times V(G_n)$, and an arc from a vetex $x=x_1x_2\ldots x_n$ to another vertex $y=y_1y_2\ldots y_n$ $(x_j,y_j\in V(G_j),\ j=1,2,\ldots,n)$ if and only if they differ in exactly one coordinate, and for this coordinate, say jth, there is an arc from the vertex x_j to the vertex y_j in G_j. C_d $(d\geq 2)$ denotes a directed cycle of order d and $C(d_1,d_2,\ldots,d_n)$ denotes the Cartesian product digraph $C_{d_1}\times C_{d_2}\times\ldots\times C_{d_n}$. The digraphs shown in the Figure 1 are the dicycles C_3 and C_4, and their Cartesian product digraph $C(3,4)$, respectively. Hsu and Lyuu[4] have considered $C(d_1,d_2,\ldots,d_n)$ and studied its wide-diameter.

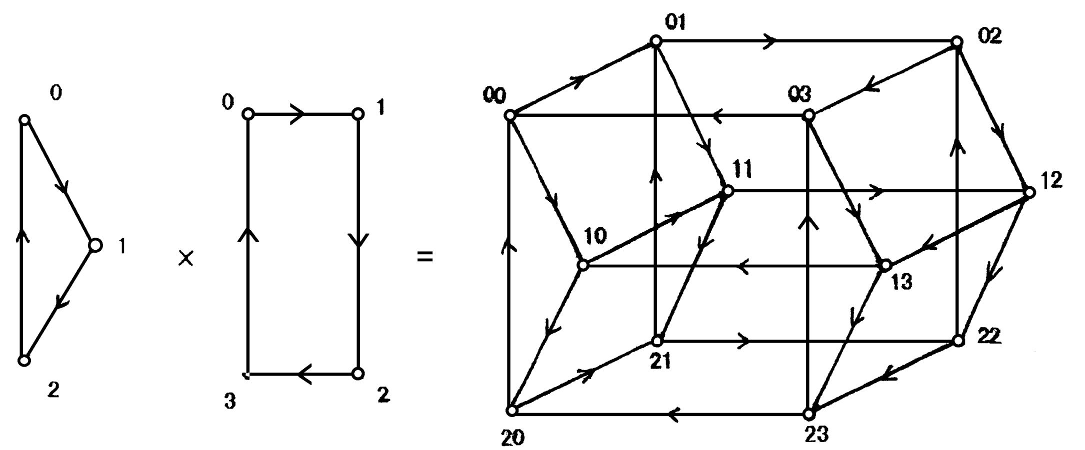

Fig. 1

We shall be content to point out that although graphs and digraphs are essentially different objects, a graph can in circumstance be thought of as a digraph in which there are two arcs, one in each direction, corresponding to each edge. In view of this fact, $C(2,$

2, ..., 2) can be thought to be the n-dimensional cube Q_n. As a result, we call $C(d_1, d_2, \ldots, d_n)$ the generalized hypercube.

If we identify isomorphic digraphs, the product thus defined is clearly associative and commutative. We will in the sequel see it is such a simple observation that can make us greatly simplify the proof of some results concerning Cartesian product digraphs.

Theorem 2.1 *For $G = G_1 \times G_2 \times \ldots \times G_n$ and for $i = 1, 2, \ldots, n$, let u_i be a vertex of G_i, $d^+_{G_i}(u_i)$ and $d^-_{G_i}(u_i)$ be the outdegree and the indegree of u_i in G_i, respectively. If $u = u_1u_2\ldots u_n \in V(G)$, then the outdegree $d^+_G(u)$ and the indegree $d^-_G(u)$ of u in G, respectively, are*

$$d^+_G(u) = d^+_{G_1}(u_1) + d^+_{G_2}(u_2) + \ldots + d^+_{G_n}(u_n)$$

and

$$d^-_G(u) = d^-_{G_1}(u_1) + d^-_{G_2}(u_2) + \ldots + d^-_{G_n}(u_n).$$

Proof. In order to show the theorem, it suffices to show that the theorem holds for $n = 2$ by associativity of the product.

Let V^+_x, V^+_y, V^+_{xy} be the sets of those vertices of G_1, G_2 and $G_1 \times G_2$ which are adjacent from $x \in V(G_1), y \in V(G_2)$, and $xy \in V(G_1 \times G_2)$, respectively. Then the definition of the digraph product implies that $V^+_{xy} = (V^+_x \times \{y\}) \cup (\{x\} \times V^+_y)$. Hence $d^+_G(xy) = d^+_{G_1}(x) + d^+_{G_2}(y)$. Similarly, we can prove $d^-_G(xy) = d^-_{G_1}(x) + d^-_{G_2}(y)$.

Theorem 2.2 *Let $G = G_1 \times G_2 \times \ldots \times G_n$ and $\kappa(G_i) = \kappa_i > 0$ be the strong connectivity of G_i, $i = 1, 2, \ldots, n$. Then the strong connectivity of G*

$$\kappa(G) \geq \kappa_1 + \kappa_2 + \ldots + \kappa_n.$$

Proof. By associativity of Cartesian product, it suffices to prove that $\kappa(G_1 \times G_2) \geq \kappa_1 + \kappa_2$. Let $G = G_1 \times G_2$.

Note that if $P = (v_1, v_2, \ldots, v_m)$ is a directed path from v_1 to v_m in G_1, then for any $b \in V(G_2)$, $(v_1b, v_2b, \ldots, v_mb)$, denoted by Pb, is a directed path from the vertex v_1b to the vertex v_mb in G. Similarly, if $W = (u_1, u_2, \ldots, u_l)$ is a directed path from u_1 to u_l in G_2, then for any $a \in V(G_1)$, $(au_1, au_2, \ldots, au_l)$, denoted by aW, is a directed path from the vertex au_1 to the vertex au_l in G.

Let $x = x_1x_2$ and $y = y_1y_2$ be arbitrary two distinct vertices of G, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$. We claim that there are at least $\kappa_1 + \kappa_2$ internally-disjoint directed paths from x to y in G.

First assume $x_1 \neq y_1$ and $x_2 \neq y_2$. Since $\kappa(G_1) = \kappa_1$, by Menger's theorem there are κ_1 internally-disjoint directed paths P_i, $i = 1, 2, \ldots, \kappa_1$, from x_1 to y_1 in G_1. Let v_i be an internal vertex in P_i, $i = 2, \ldots, \kappa_1$. Then for $i = 2, 3, \ldots, \kappa_1$, v_i cuts the path P_i into two subpaths $P_i^{(1)}$ and $P_i^{(2)}$, where $P_i^{(1)}$ is the section of P_i from x_1 to v_i and $P_i^{(2)}$ is the section of P_i from v_i to y_1. And so the directed path P_i can be represented as

$$P_i = x_1 \xrightarrow{P_i^{(1)}} v_i \xrightarrow{P_i^{(2)}} y_1, \quad i = 2, 3, \ldots, \kappa_1.$$

Similarly, there are κ_2 internally-disjoint directed paths W_j, $j = 1, 2, \ldots, \kappa_2$, from x_2 to y_2

in G_2, and for $j=2,3,\ldots,\kappa_2$, there is an internal vertex u_j in W_j such that

$$W_j = x_2 \xrightarrow{W_j^{(1)}} u_j \xrightarrow{W_j^{(2)}} y_2, \quad j=2,3,\ldots,\kappa_2.$$

where $W_j^{(1)}$ is the section of W_j from x_2 to u_j and $W_j^{(2)}$ is the section of W_j from u_j to y_2. So the following $\kappa_1+\kappa_2$ directed paths from $x=x_1x_2$ to $y=y_1y_2$

$$\begin{cases} R_1 = x_1x_2 \xrightarrow{P_1x_2} y_1x_2 \xrightarrow{y_1W_1} y_1y_2, \\ R_i = x_1x_2 \xrightarrow{P_i^{(1)}x_2} v_ix_2 \xrightarrow{v_iW_1} v_iy_2 \xrightarrow{P_i^{(2)}y_2} y_1y_2, \quad i=2,\ldots,\kappa_1 \end{cases}$$

and

$$\begin{cases} R_{\kappa_1+1} = x_1x_2 \xrightarrow{x_1W_1} x_1y_2 \xrightarrow{P_1y_2} y_1y_2, \\ R_{\kappa_1+j} = x_1x_2 \xrightarrow{x_1W_j^{(1)}} x_1u_j \xrightarrow{P_1u_j} y_1u_j \xrightarrow{y_1W_j^{(2)}} y_1y_2, \quad j=2,\ldots,\kappa_2 \end{cases}$$

are internally-disjoint in G.

Secondly, we consider either $x_1=y_1$ or $x_2=y_2$. Let $x_1=y_1$ and $x_2\neq y_2$. We still use W_j, $j=1,2,\ldots,\kappa_2$, to stand for κ_2 internally-disjoint directed paths from x_2 to y_2 in G_2, and use P_i, $i=1,2,\ldots,\kappa_1$, to stand for κ_1 internally-disjoint directed paths from x_1 to some vertex, say v_1 in G_1. For any $i=2,3,\ldots,\kappa_1$, let v_i be an internal vertex in P_i. For the set of vertices $\{v_1,v_2,\ldots,v_{\kappa_1}\}$, by Menger's theorem, there are κ_1 internally-disjoint directed paths T_i, $i=1,2,\ldots,\kappa_1$, from v_i to x_1 in G_1. Now the $\kappa_1+\kappa_2$ directed paths

$$R_i = x_1x_2 \xrightarrow{P_i^{(1)}x_2} v_ix_2 \xrightarrow{v_iW_1} v_iy_2 \xrightarrow{T_iy_2} x_1y_2, \quad i=1,2,\ldots,\kappa_1$$

and

$$R_{\kappa_1+j} = x_1x_2 \xrightarrow{x_1W_j} y_1y_2, \quad j=1,2,\ldots,\kappa_2$$

are internally-disjoint in G.

If $x_1\neq y_1$ and $x_2=y_2$, the statement is similar.

For every pair (x,y) of vertices in G, we have contructed $\kappa_1+\kappa_2$ internally-disjoint directed paths from x to y in G, hence G is at least $(\kappa_1+\kappa_2)$-strongly connected i.e. $\kappa(G)\geq\kappa_1+\kappa_2$.

This result for undirected case is first obtained by Sabidussi[5].

Let G be a simple digraph and $\Gamma(G)$ be an automorphism group of G. G is said to be vertex-transitive if, for any two vertices x and y, there is an element $\sigma\in\Gamma(G)$ such that $\sigma(x)=y$.

Theorem 2.3 *If $G_1, G_2,\ldots, G_n$ are vertex-transitive, then $G=G_1\times G_2\times\ldots\times G_n$ is vertex-transitive.*

Proof. Let $x=x_1x_2\ldots x_n$ and $y=y_1y_2\ldots y_n$ be arbitrary two vertices of G, where $x_j,y_j\in V(G_j)$, $j=1,2,\ldots,n$. Since for each $j=1,2,\ldots,n$, G_j is vertex-transitive, there is $\sigma_j\in\Gamma(G_j)$ such that $\sigma_j(x_j)=y_j$. Hence the function $\phi: V(G)\to V(G)$ given by $\phi(x_1x_2\ldots x_n)=\sigma_1(x_1)\sigma_2(x_2)\ldots\sigma_n(x_n)$ is clearly an element in $\Gamma(G)$ and $\phi(x)=y$. Namely, G is vertex-transitive.

As a consequence of the above theorems, the following result is straightforward.

Theorem 2.4 *$C(d_1, d_2, \ldots, d_n)$ is vertex-transitive and $\kappa(C(d_1, d_2, \ldots, d_n)) = n$.*

It follows from the Theorem 2.4 that the network modelled by $C(d_1, d_2, \ldots, d_n)$ is a highly fault-tolerant network.

§3 Fault-Tolerant Routings of the Generalized Hypercube

For given positive integers $d_j(\geq 2)$ and $n(\geq 1)$, $j = 1, 2, \ldots, n$, we successively label the vertices of the directed cycle C_{d_j} by $0, 1, 2, \ldots, d_j - 1$. Then each vertex $x = x_1x_2\ldots x_n$ $(x_j \in V(C_{d_j}))$ of the generalized hypercube $C(d_1, d_2, \ldots, d_n)$ can be regarded as a vector x $= (x_1, x_2, \ldots, x_n)$ in the n-dimensional number space, where $x_j(0 \leq x_j \leq d_j - 1)$ is the jth coordinate.

Let $G_i = C_{d_i}$ and

$$G_j(i) = G_1 \times G_2 \times \ldots G_{j-1} \times \{i\} \times G_{j+1} \times \ldots \times G_n,$$
$$i = 0, 1, \ldots, d_j - 1; \quad j = 1, 2, \ldots, n.$$

Obviously, $G_j(i)$ is a sub-digraph of $C(d_1, d_2, \ldots, d_n)$ and is isomorphic to $G_1 \times \ldots \times G_{j-1} \times G_{j+1} \times \ldots \times G_n$. For any vertex x $\in V(C(d_1, d_2, \ldots, d_n))$, we always use x_j to stand for the jth coordinate of x, i.e. x $= (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$ unless it is especially defined. And we use $\mathrm{x}_j(i)$ to stand for the new vertex obtained by the jth coordinate x_j of x instead of i, i.e. $\mathrm{x}_j(i) = (x_1, \ldots, x_{j-1}, i, x_{j+1}, \ldots, x_n)$. For a non-empty subset F of $V(C(d_1, d_2, \ldots, d_n))$, let $F_j(0) = \{\mathrm{f}_j(0): \mathrm{f} \in F\}$.

Theorem 3.1 Diam $(C(d_1, d_2, \ldots, d_n), \rho) \leq 3$ *for any minimal length routing* ρ *in* $C(d_1, d_2, \ldots, d_n)$.

Proof. Let F be a fault-set of $C(d_1, d_2, \ldots, d_n)$, $|F| < n = \kappa(C(d_1, d_2, \ldots, d_n))$. Let x $= (x_1, x_2, \ldots, x_n)$ and y $= (y_1, y_2, \ldots, y_n)$ be arbitrary two non-faulty vertices of $C(d_1, d_2, \ldots, d_n)$. Let R(x, y) be the set of all shortest paths from x to y in $C(d_1, d_2, \ldots, d_n)$. R(x, y) avoids F if no element of F is contained in some paths of R(x, y). A sequence of vertices $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_s$ of $C(d_1, d_2, \ldots, d_n)$ is safe with respect to F if $R(u_i, u_{i+1})$ avoids F for each $i = 1, 2, \ldots, s - 1$. Hence in order to prove the theorem it is sufficient to show that the distance from x to y in $R(C(d_1, d_2, \ldots, d_n), \rho)/F$ is not greater than three, namely to show that there are at most two vertices u and v, different from x and y, such that the sequence of vertices x, u, v, y, is safe with respect to F.

We proceed by induction on n.

The argument for $n = 1$ is straightforward. Assume that the theorem holds for dimension $n - 1$ with $n > 1$. Since $C(d_1, d_2, \ldots, d_n)$ is vertex-transitive, we can, without loss of generality, suppose x $= (0, 0, \ldots, 0)$. If $F = \varnothing$, there is nothing to do and so below suppose $F \neq \varnothing$.

Let $F_V = F \cap V(C(d_1, d_2, \ldots, d_n))$, $F_A = F \cap A(C(d_1, d_2, \ldots, d_n))$. We can assume that $|F_V| \geq |F| - 1$, and if $|F_V| = |F| - 1$, then $(x, y) \in A(C(d_1, d_2, \ldots, d_n))$ and $F_A = \{(x, y)\}$.

So, if $F_A \neq \varnothing$, then $(x, y) \in A(C(d_1, d_2, \ldots, d_n))$ and $F_A = \{(x, y)\}$. We can, without loss of generality, suppose $y = (0, y_2, y_3, \ldots, y_n)$. If F_V is contained in the sub-digraph $G_1(0) = \{0\} \times C(d_2, d_3, \ldots, d_n)$, then the sequence of vertices $x, u = (1, 0, \ldots, 0), v = (1, y_2, \ldots, y_n), y$ is safe with respect to F. Otherwise, $G_1(0)$ includes at most $n - 2$ elements of F. We can apply our induction hypothesis to $G_1(0)$, there is a sequence of vertices x, u, v, y in $G_1(0)$, also in $C(d_1, d_2, \ldots, d_n)$ such that it is safe with respect to F. In the following we suppose $F = F_V$.

Let $F_j'(0) = F_j(0) \setminus \{x_j(0), y_j(0)\}$. If there is a $j\ (1 \leq j \leq n)$ such that $|F_j'(0)| < |F|$, then, to apply our induction hypothesis to $G_j(0)$, there exists a sequence of vertices $x, u_j(0), v_j(0), y_j(0)$ in $G_j(0)$ such that it is safe with respect to $F_j'(0)$. As a result, the sequence of vertices $x, u_j(0), v_j(y_j), y$ in $C(d_1, d_2, \ldots, d_n)$ is safe with respect to F. Consequently, we can suppose below that $|F_j'(0)| = |F|$. This assumption implies $x_j(0), y_j(0) \notin F_j(0)$ and $|F_j(0)| = |F|$.

If there are a $j\ (1 \leq j \leq n)$ and an $i\ (0 \leq i \leq d_j - 1)$ such that each element of F does not have i on the jth coordinate, then the sequence of vertices $x, x_j(i), y_j(i), y$ in $C(d_1, d_2, \ldots, d_n)$ is safe with respect to F.

Now we suppose that for each $j = 1, 2, \ldots, n$ and any $i = 1, 2, \ldots, d_j - 1$, there is at least an element of F with i on the jth coordinate. This assumption implies $d_j \leq |F|$ for any $j = 1, 2, \ldots, n$. Choose some $j\ (1 \leq j \leq n)$ such that d_j is as large as possible. Applying our induction hypothesis to $G_j(0)$, we can, without loss of generality, suppose $y_j \neq 0$.

Let $f \in F \cap V(G_j(y_j))$ such that the distance from $x_j(y_j)$ to f, denoted by $\| f \|$, is as small as possible. Let $F_j''(0) = F_j(0) - f_j(0)$. $|F_j''(0)| < n - 1$. So to apply our induction hypothesis to $G_j(0)$, there exists a sequence of vertices $x, u_j(0), v_j(0), y_j(0)$ in $G_j(0)$ safe with respect to $F_j''(0)$. If $v_j(0) = u_j(0)$, then the sequence $x, u_j(0), y_j(0), y$ is safe with respect to F in $C(d_1, d_2, \ldots, d_n)$. Below suppose $v_j(0) \neq u_j(0)$.

Note that the jth coordinate of f is y_j, the sequence of vertices $x, u_j(0), v_j(0), y$ is safe with respect to F in $C(d_1, d_2, \ldots, d_n)$ if $f \notin R(v_j(0), y)$ in $C(d_1, d_2, \ldots, d_n)$. So we only need consider the case of $f \in R(v_j(0), y)$. By the choice of f, we must have $\| f' \| \geq \| f \| \geq \| v_j(y_j) \|$ for all $f' \in F \cap V(G_j(y_j))$. Hence f is the only element of F in $R(x_j(y_j), y)$.

If there are at least two shortest paths from $x_j(y_j)$ to y in $G_j(y_j)$, then it is easy to find a vertex, for instance, one with the distance $\| f \|$ from $x_j(y_j)$, say $z_j(y_j)$, in $G_j(y_j)$ such that the sequence of vertices $x, x_j(y_j), z_j(y_j), y$ is safe with respect to F in $C(d_1, d_2, \ldots, d_n)$.

We suppose that there is unique shortest path, denoted by $P_j(y_j)$, from $x_j(y_j)$ to y in $G_j(y_j)$. Then there exists some $l\ (1 \leq l \neq j \leq n)$ such that $P_j(y_j)$ is isomorphic to some shortest path P in C_l. By the construction of $C(d_1, d_2, \ldots, d_n)$, for any $i=0, 1, \ldots, d_j-1$, there must be unique shortest path $P_j(i)$ from $x_j(i)$ to $y_j(i)$ in $G_j(i)$, and these shortest paths are all isomorphic to P. If for any $i=0, 1, \ldots, d_j-1$, $P_j(i)$ in $G_j(i)$ includes at least an element of F with noting $|F_j(0)|=|F|$, then the length of P, denoted by $l(P)$, satisfies that $d_j+2\leq l(P)\leq |C_l|=d_l$. This contradicts our choice of j. So there is an $i(0\leq i\neq y_j \leq d_j-1)$ such that the unique shortest path $P_j(i)$ from $x_j(i)$ to $y_j(i)$ in $G_j(i)$ does not include any element of F. Then the sequence of vertices $x, x_j(i), y_j(i), y$ is safe with respect to F in $C(d_1, d_2, \ldots, d_n)$.

The above discussion without omitting a single circumstance includes all cases that may happen, and so the proof of the theorem is commpleted.

Remark 1 Note that $C(2, 2, \ldots, 2)$ can be thought to be the n-dimensional cube Q_n, Theroem 3.1 includes Dolev *et al*'s Theorem 1 in [1]. But their proof of Lemma 2 has a flaw in the case (b2), to be exact, their proof is unavailable if $f_1=0$.

Remark 2 The upper bound of Diam $(C(d_1, d_2, \ldots, d_n), \rho)$ given in Theorem 3.1 can not be improved. For example, in $C(3, 4)$ shown in the Figure 1, $x=00, y=03, u=01$. Let $F=\{u\}$, then it is easy to choose a minimal length routing ρ in $C(3, 4)$ such that the distance from x to y in $R(C(3, 4), \rho)/F$ is three. However, the bound can be improved for some minimal length routing, i.e., we can prove the following result.

Theorem 3.2 *$C(d_1, d_2, \ldots, d_n)$ is good.*

Proof. Let ρ be such a minimal length routing in $C(d_1, d_2, \ldots, d_n)$ that for any ordered pair (x, y) of vertices, $\rho(x, y)$ proceeds from x to y by moving along the coordinates from the left to the right.

For example, let $x=00$ and $y=23$ be two vertices of $C(3, 4)$ shown as Figure 1. Then $\rho(x, y)=(00, 10, 20, 21, 22, 23)$.

Let x and y be arbitrary two non-faulty vertices of $C(d_1, d_2, \ldots, d_n)$. In order to prove the theorem, we only want to show that there exists a directed path from x to y with length not greater than two in $R(C(d_1, d_2, \ldots, d_n), \rho)/F$ for any non-empty fault-set F, $|F|<n$. We proceed by induction on n. The case $n=1$ is trivial. For $n>1$ there are two cases.

Without loss of generality, suppose $x=(0, 0, \ldots, 0)$ because $C(d_1, d_2, \ldots, d_n)$ is vertex-transitive and suppose $F_V=F$ if $(x, y)\notin F$ and $F_V=|F|-1$ and $F_A=\{(x, y)\}$ otherwise.

Case (a) y has 0 on some coordinate. We can, without loss of generality, suppose $y_1=0$ since Cartesian product is commutative.

If every element of F has 0 on the first coordinate, let $u=(d_1-1, y_2, \ldots, y_n)$, then (x, u, y) is a directed path from x to y in $R(C(d_1, d_2, \ldots, d_n), \rho)/F$. Otherwise, let $F \triangleq \{f\in F$: f has 0 on the first coordinate$\}$, then $|F\cap V(G_1(0))|<n-1$. So we can apply our induction hypothesis to $G_1(0)$. There is a directed path W of length one or two from x

to y in $R(G_1(0), \rho\cap G_1(0))/F'$. Then W is still a directed path from x to y in $R(C(d_1, d_2, \ldots, d_n), \rho)/F$ since all faults not in F' have non-zero on the first coordinate.

Case (b) y has no 0 on any coordinate. This case implies $F_V = F$. Let

$$\begin{cases} x_1 = (y_1, 0, 0, \ldots, 0, 0, 0), \\ x_2 = (0, y_2, y_3, \ldots, y_{n-2}, y_{n-1}, y_n), \\ x_3 = (0, 0, y_3, \ldots, y_{n-2}, y_{n-1}, y_n), \\ \quad\ldots \\ x_{n-1} = (0, 0, 0, \ldots, 0, y_{n-1}, y_n), \\ x_n = (0, 0, 0, \ldots, 0, 0, y_n). \end{cases}$$

Then n directed paths in $C(d_1, d_2, \ldots, d_n)$

$$P_j = \rho(x, x_j) \cup \rho(x_j, x), \qquad j = 1, 2, \ldots, n$$

are internally-disjoint and so one of them must avoid F because $|F| < n$.

References

[1] Dolev, D., Halpern, J., Simons, B. and Strong, R., A new look at fault tolerant network routing, in "Proceedings of the 16th Annual ACM Symposium on Theory of Computing", (1984), 526-535.

[2] Escudero, M., Fabrega, J. and Morillo, P., Fault-tolerant routings in double-loop networks, *Ars Combin.*, **25A** (1988), 187-198.

[3] Homobono, N. and Peyrat, C. Fault-tolerant routings in Kautz and de Bruijn networks, *Discrete Appl. Math.*, **24** (1989), 179-186.

[4] Hsu, D. F. and Lyuu, Y. D., A graph-theory study of transmission delay and fault tolerance, Proceedings of 4th ISMM International Conference on Paralled and Distributed Computing and Systems, 1991, 20-24.

[5] Sabidussi, G., Graphs with given group and given graph-theoretical properties, *Canad. J. Math.*, **9** (1957), 515-525.

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