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Some Results on Arithmetic and Balanced Graphs*

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Abstract In this paper, three conjectures by Acharya and Hegde concerning arithmetic and balanced graphs are considered first. One of them has been proved by themselves. We will here present a very simple proof of it and another and point out that the other is false in general, but true under a stronger condition. Then, the relations between ours and some known results are discussed.

Key words graph, arithmetic numbering, balanced graphs

1 Introduction

For all the terminology and notation used here we follow [1].

Let $G=(V, E)$ be a (p, q)-graph, N be the set of nonegative integers and $N^{+}=N\setminus\{0\}$. For a vertex function $f: V(G)\to N$, define two edge functions $f^{+}: E(G)\to N$ and $g_f: E(G)\to N$ given respectively by

$$f^{+}(uv)=f(u)+f(v),\qquad \forall\in uv\in E(G)$$

and by

$$g_f(uv)=|f(u)-f(v)|,\qquad \forall\in uv\in E(G)$$

Let

$$f(G)=\{f(u): u\in V(G)\}$$
$$f^{+}(G)=\{f^{+}(e): e\in E(G)\}$$
$$g_f(G)=\{g_f(e): e\in E(G)\}$$

Let $k, d\in N^{+}$. The vertex function f is a (k, d)-arithmetic numbering of G if both f and f^{+} are injective and $f^{+}(G)=\{k, k+d, k+2d, \cdots, k+(q-1)d\}$. G is a (k, d)-arithmetic graph if G admits of a (k, d)-arithmetic numbering f. The vertex function f is a (k, d)-balanced numbering of G if both f and g_f are injective, $f(G)\subset\{0, 1, 2, \cdots, k+(q-1)d\}$, $g_f(G)=\{k, k+d, k+2d, \cdots, k+(q-1)d\}$ and there is an integer $m=m(f)$ with either

$$f(u)\le m< f(v)\qquad \text{or}\qquad f(u)>m\ge f(v)\qquad \forall\ uv\in E(G)$$

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where $m(f)$ is called the characteristic of f. Denote by $m_0(f)$ the minimum of the characteristic of f. G is a (k, d)-arithmetic (resp. balanced) graph if G admits of a (k, d)-arithmetic (resp. balanced) numbering f. G is arithmetic (resp. balanced) if G is (k, d)-arithmetic (resp. balanced) for some positive integers k and d. It is easily seen that if G is a balanced graph, then G necessarily is bipartite and $0 \in f(G)$. When speaking of a bipartite graph G in this paper, we can always suppose that G has the bipartition $\{A, B\}$ with $A = \{u_1, u_2, \cdots, u_a\}$ and $B = \{v_1, v_2, \cdots, v_b\}$, $a \leq b$.

In [1], Acharya and Hegde obtained several classes of arithmetic and balanced graphs and then proposed three conjectures. Two of them can now be stated as the following theorems:

Theorem 1 (Conjecture 2 in [1], p294) For any integer $n \geq 5$, the complete graph K_n is not arithmetic.

Theorem 2 (Conjecture 3 in [1], p297) If the odd cycle C_{2t+1} is (k, d)-arithmetic, then $k = td + 2r$ for some $r \in N$.

The former has been proved in [2] and the latter has not yet as far as we know. We will give very simple proofs of them in Section 2 and Section 3 respectively.

It is obvious that if f is a (k, d)-arithmetic numbering of G, then there is a partition (k_1, k_2), $0 \leq k_1 < k_2$, of k with $k_1, k_2 \in f(G)$. It has already been verified that for any partition (k_1, k_2), $0 \leq k_1 < k_2$, of k, the star $K_{1,b}$ has a (k, d)-arithmetic numbering f with $(k_1, k_2) \in f(K_{1,b})$ (cf. Theorem 13 in [1]). But this is not always so for any arbitrary (k, d)-arithmetic numbering of arithmetic graphs. Consequently the following problem naturally arises:

For a (k, d)-arithmetic graph G and a partition (k_1, k_2), $0 \leq k_1 < k_2$, of k, what conditions must be satisfied for k_1 and k_2 such that there exists a (k, d)-arithmetic numbering f of G with $k_1, k_2 \in f(G)$?

For some special classes of bipartite graphs, such as the complete bipartite graph $K_{a,b}$, the caterpillar $C_{a,b}$ and the cycle C_{4t} of order $4t$, $t \geq 1$, $a = b = 2t$, Acharya and Hegde found that for any partition (k_1, k_2), $0 \leq k_1 < k_2$, of k satisfying one of the following conditions:

$$d \mid (k_2 - k_1) \tag{1}$$

and

$$k_2 - k_1 = rd \quad \text{for some integer} \quad r \geq a \tag{2}$$

any G in the above-mentioned classes has a (k, d)-arithmetic numbering f with $k_1, k_2 \in f(G)$ (cf. Theorem 14, Theorem 15, Theorem 17(A) in [1], respectively).

At the same time they pointed out (cf. [1], p. 289) that each of Condition (1) and Condition (2) is also necessary for $K_{a,b}$, $2 \leq a \leq b$, to have a (k, d)-arithmetic numbering f with $k_1, k_2 \in f(K_{a,b})$ and the proof, which is rather tedious, just as they say, essentially made use of the following:

"**Lemma**" For any (k, d)-arithmetic numbering of $K_{a,b}$ either

$\{f^+(u_i u_j): 1 \leq j \leq b\} = \{k + ((i-1)b + j - 1)d: 1 \leq j \leq b\}$ for each i, $1 \leq i \leq a$

or

$\{f^+(u_iu_j): 1 \le i \le a\} = \{k + ((j-1)a + i - 1)d: 1 \le i \le a\}$ for each j, $1 \le j \le b$

We will, in Section 4, point out by a counterexample that this "Lemma" is not true in general. However, we can still prove the following result:

Theorem 3 The complete bipartite graph $K_{a,b}$, $2 \le a \le b$, is arithmetic if and only if there is a partition (k_1, k_2), $0 \le k_1 < k_2$, of k satisfying either Condition (1) or Condition (2).

Prompted by the fact that every connected balanced graph is bipartite and arithmetic (cf. Theorem 12 in [1]), and that each of the class of bipartite graphs shown to be arithmetic for various values of k and d is a class of balanced graphs, Acharya and Hegde proposed the following

Conjecture (Conjecture 1 in [1], p. 293) For any quadruple (a, b, k, d) of positive integers and a partition (k_1, k_2), $0 \le k_1 < k_2$, of k satisfying either Condition (1) or Condition (2), any balanced bipartite graph G with the bipartition $\{A, B\}$, $|A| = a \le b = |B|$, has a (k, d)-arithmetic numbering f with $k_1, k_2 \in f(G)$.

We will, in Section 5, point out by two counterexamples that this conjecture is not true if G is disconnected and that Condition (2) is not sufficient for a balanced and connected bipartite graph to have a required arithmetic numbering. However, if Condition (2) is modified as the following condition

$$k_1 - k_2 = rd, \qquad \text{for some integer } r > \frac{m_0(f')}{d'} \tag{3}$$

where f' is some (k', d')-balanced numbering of G, then the following positive result can be obtained:

Theorem 4 Let f' be a (k', d')-balanced numbering of G, k and d be positive integers and (k_1, k_2) be a partition of k, $0 \le k_1 < k_2$. Then G admits of a (k, d)-arithmetic numbering f with $k_1, k_2 \in f(G)$ if either

(i) G is connected and k_1, k_2 satisfy either Condition (1) or Condition (3), or

(ii) d' divides d and k_1, k_2 satisfy Condition (3).

The proof of Theorem 4 is in Section 6. In Section 7, we will futher discuss Theorem 4 and its relations to some known results.

2 The Proof of Theorem 1

Suppose that $K_n (n \ge 5)$ is an arithmetic graph and f a (k, d)-arithmetic numbering of K_n. Our aim is to arrive at a contradiction. Let $V(K_n) = \{u_1, u_2, \cdots, u_n\}$, $n \ge 5$ and (k_1, k_2), $0 \le k_1 < k_2$, a partition of k. We can, without loss of generality, assume that

$$f(u_n) = k_1, \qquad f(u_i) = k_2 + a_i d$$

where $a_i \in N$, $1 \le i \le n-1$ and $0 = a_1 < a_2 < \cdots < a_{n-1}$. Hence

$$f^+(u_n u_l) = k + a_l d, \qquad 1 \le l \le n-1,$$

$$f^+(u_i u_j) = 2k_2 + (a_i + a_j)d, \ 1 \le i \ne j \le n-1$$

Noting that f^+ is injective and $f^+(K_n)=\{k+sd: 0\le s\le \frac{1}{2}n(n-1)-1\}$, we have that

$$a_l\ne a_i+a_j,\ 3\le l\le n-1,\ 1\le i\ne j<l \tag{4}$$

and there exists $r\in N^+$ such that $2k_2=k+rd$. Now,

$$f^+(u_iu_j)=k+(a_i+a_j+r)d,\qquad 1\le i\ne j\le n-1$$

Let

$$X=\{a_i: 1\le i\le n-1\},$$

$$Y=\{a_i+a_j+r: a_i,a_j\in X,\ a_i\ne a_j, a_i+a_j+r\le \frac{1}{2}n(n-1)-1\}$$

Clearly,

$$X\cup Y=\{0,1,2,\cdots,\frac{1}{2}n(n-1)-1\},\ X\cap Y=\emptyset$$

and $x\ge r+1$ for any $x\in Y$. This implies $\{0,1,2,\cdots,r\}\subset X$ and

$$a_1=0,\ a_2=1,\ a_3=2,\cdots,a_{r+1}=r.$$

If $r\ge 3$, then $a_4=3=1+2=a_2+a_3$, which contradicts (4).

If $r=2$, then $\{3,4,5\}\subset Y$ and $a_4=6$. Hence $10\in Y, n\ge 6$, $a_5=7$ and so $a_5=1+6=a_2+a_4$, which contradicts (4).

If $r=1$, then $2\in Y$, $a_3=3$, $a_4=6$, $10\in Y$, $n\ge 6$ and so $a_5=9=3+6=a_3+a_4$, which contradicts (4).

Therefore, Theorem 1 follows

3 The Proof of Theorem 2

Let $V(C_{2t+1})=\{u_1,u_2,\cdots,u_{2t+1}\}$, and f be a (k,d)-arithmetic numbering of C_{2t+1}. By Theorem 2 of [1]

$$\sum_{i=1}^{2t+1}2f(u_i)=\sum_{e\in E(C_{2t+1})}f^+(e)=(2t+1)(k+td) \tag{5}$$

It follows that

$$k-td\equiv 0\ (\mathrm{mod}\ 2) \tag{6}$$

On the other hand, $k\in f^+(C_{2t+1})$ and there exists $e\in E(C_{2t+1})$, say, $e=u_1u_2$ such that $f^+(u_1u_2)=k$ and a partition (k_1,k_2), $0\le k_1<k_2$, of k such that $f(u_1)=k_1$ and $f(u_2)=k_2$. Hence

$$f(u_{2i-1})=k_1+x_id,\ x_i\in N^+,\ 2\le i\le t+1$$

$$f(u_{2j})=k_2+y_jd,\ y_j\in N^+,\ 2\le j\le t$$

Noting that $f^+(C_{2t+1})=\{k,k+d,k+2d,\cdots,k+2td\}$, we have

$$f^+(u_1u_{2t+1})=2k_1+x_{t+1}d=k+md,\quad \text{for some}\ \ m\in N^+$$

i e ,

$$k_2-k_1=sd,\quad \text{for some}\quad s\in N^+. \tag{7}$$

We can prove that

$$|f(u_l) - f(u_{l'})| \geq d \quad \text{for} \quad \forall u_l, u_{l'} \in V(C_{2t+1}), u_l \neq u_{l'} \tag{8}$$

In fact, it is clear that (8) holds if $l - l' \equiv 0 \pmod 2$. Next, we suppose $l = 2i - 1$ and $l' = 2j$. Noting (7) and the injectivity of f, we have

$$|f(u_{2i-1}) - f(u_{2j})| = |k_2 - k_1 + (y_j - x_i)d| = nd \quad \text{for some} \quad n \in N^+$$

From (8) and the injectivity of f, we have

$$\sum_{i=1}^{2t+1} f(u_i) \geq \sum_{i=1}^{2t+1} (i-1)d = dt(2t+1) \tag{9}$$

It follows from (5) and (9) that $k \geq dt$. So from (7) there exists $r \in N$ such that $k = dt + 2r$, and Theorem 2 holds.

4 The Proof of Theorem 3

The proof of the sufficiency has been given in [1, Theorem 14], but it can be reduced to a by-product of our theorem 4 in Section 6 and Section 7. Next, we need only to prove the necessity.

Let f be a (k, d)-arithmetic numbering of $K_{a,b}$. Then there exist two vertices u_i and v_j in $K_{a,b}$ such that $f^+(u_i v_j) = k$. Let

$$k_1 = \min(f(u_i), f(v_j)) \quad \text{and} \quad k_2 = \max(f(u_i), f(v_j)).$$

We can, without loss of generality, suppose

$$f(A) = \{k_1, k_1 + x_1 d, k_1 + x_2 d, \cdots, k_1 + x_{a-1} d\},$$
$$f(B) = \{k_2, k_2 + y_1 d, k_2 + y_2 d, \cdots, k_2 + y_{b-1} d\}$$

where $x_i, y_j \in N^+$, $1 \leq x_1 < x_2 < \cdots < x_{a-1}$ and $1 \leq y_1 < y_2 < \cdots < y_{b-1}$. Let

$$X = \{0, x_1, x_2, \cdots, x_{a-1}\},$$
$$Y = \{0, y_1, y_2, \cdots, y_{b-1}\},$$
$$X + Y = \{\alpha + \beta: \alpha \in X, \beta \in Y\},$$
$$f(A) + f(B) = \{\alpha' + \beta': \alpha' \in f(A), \beta' \in f(B)\}$$

It is clear that

$$f(A) + f(B) = f^+(K_{a,b}) = \{k + id: 0 \leq i \leq ab - 1\},$$

and

$$X + Y = \{0, 1, 2, \cdots, ab - 1\}.$$

Next, we prove that k_1 and k_2 neccessarily satisfy both Condition (1) and Condition (2).

By contradiction. Suppose that both k_1 and k_2 satisfy neither Condition (1) nor Condition (2). Then there exists $r \in N^+$ such that $r < a$ and $k_2 - k_1 = rd$. Consider

$$X - Y = \{\alpha - \beta: \alpha \in X, \beta \in Y\}.$$

Let $x_i - y_j$, $x_{i'} - y_{j'} \in X - Y$. Note that no two of ab elements in $X + Y$ are identical. If $x_i - y_j = x_{i'} - y_{j'}$, then $x_i + y_{j'} = x_{i'} + y_j \in X + Y$, i.e., $x_i = x_{i'}$ and $y_j = y_{j'}$. Hence, no two of ab elements in $X - Y$ are identical. In $X - Y$, clearly, the maximal element is x_{a-1}, the minimal element is $(-y_{b-1})$, and $x_{a-1} + y_{b-1} = ab - 1$. We therefore have

$$X - Y = \{i \in Z: -y_{b-1} \leq i \leq x_{a-1}\},$$

where Z is the set of integers. Noting that $1 \le x_1 < x_2 < \cdots < x_{a-1}$, and $0 < r < a$, we have $a-1 \le x_{a-1}$ and $0 < r \le x_{a-1}$, i.e., $r \in X - Y$. Hence, there exists $x_i \in X$, $y_j \in Y$ such that $x_i - y_j = r$. Noting that $k_2 - k_1 = rd$, we have

$$k_1 + x_i d = k_2 + y_j d$$

But $k_1 + x_i d \in f(A)$ and $k_2 + y_j d \in f(B)$, which contradicts the injectivity of f. The proof of Theorem 3 is completed.

Next, we point out by the following counterexample that "lemma" mentioned in Section 1 is not true in general. To this aim we consider the complete bipartite graph $K_{2,4}$, where $A = \{u_1, u_2\}$ and $B = \{v_1, v_2, v_3, v_4\}$. For any given $k, d \in N^+$ and a partition (k_1, k_2) of k, the vertex function $f: V(K_{2,4}) \to N$ defined by:

$$f(u_1) = k_1, f(u_2) = k_1 + 2d,$$

$$f(v_1) = k_2, f(v_2) = k_2 + d, f(v_3) = k_2 + 4d, f(v_4) = k_2 + 5d$$

is a (k, d)-arithmetic numbering of $K_{2,4}$. It is easily verified that f does not satisfy "Lemma".

5 Counterexamples to Conjecture

We give two counterexamples to Conjecture mentioned in Section 1.

The first example is the graph $4K_2$ (see figure 1). It is disconnected and $(2, 2)$-balanced, a $(2, 2)$-balanced numbering f is shown in figure 1(a), where $m(f) = 3$ or 4. Let $(a, b, k, d) = (4, 4, 3, 2)$. Then $(0, 3)$ is a partition of $k = 3$ and $d \nmid k_2 - k_1$, where $d = 2$, $k_1 = 0 < 3 = k_2$. So Condition (1) holds, but it is easily verified that $4K_2$ has no $(3, 2)$-arithmetic numbering f with $0, 3 \in f(G)$. This shows that Conjecture is not true if G is disconnected.

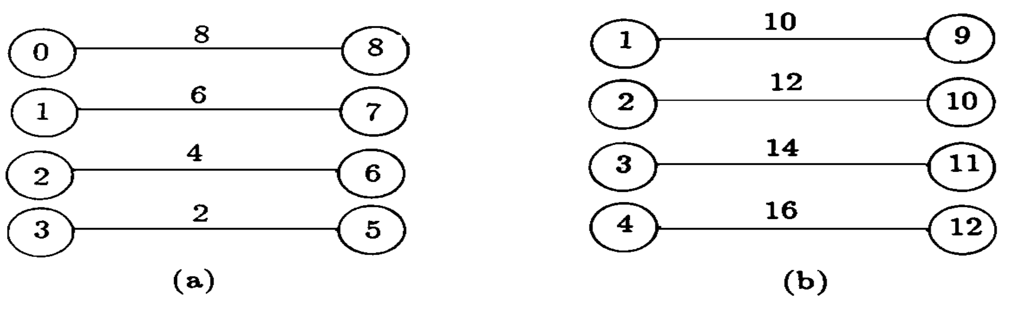

Fig. 1

The second example is the complete lattice grid $L_{3,4} = P_3 \times P_4$ (see fig. 2). $L_{3,4}$ is a connected and balanced bipartite graph with the bipartition $\{A, B\}$, where $A = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, $|A| = a = 6 = b = |B|$ (see fig. 2(a)), a $(1, 1)$-balanced numbering of $L_{3,4}$ is exhibited in fig. 2(b). For the quadruple $(a, b, k, d) = (6, 6, 31, 3)$ and the partition $(k_1, k_2) = (5, 26)$ of $k = 31$, we have $k_2 - k_1 = rd$ and $r = 7 > 6 = a$. So Condition (2) holds, but it has already been verified (cf. [1] p. 292) that there is no $(31, 3)$-arithmetic numbering f with $5, 26 \in f(L_{3,4})$.

6 Proof of Theorem 4

Let f' be a (k', d')-balanced numbering of G, then we have $f'(G) \subset \{0, 1, 2, \cdots, k' + (q-1)d'\}$, $g_{f'}(G) = \{k', k' + d', k' + 2d', \cdots, k' + (q-1)d'\}$. And so there are two adjacent vertices u and v with $f'(u) = 0$ and $f'(v) = k' + (q-1)d'$. Let $A = \{u: f'(u) \le m(f')\}$, $B = V \setminus A$. Then the $\{A, B\}$ is a bipartition of G, and $u \in A$, $v \in B$. We can, without loss of generality, suppose $A = \{u_1, u_2, \cdots, u_a\}$ with $0 = f'(u) = f'(u_1) < f'(u_2) < \cdots < f'(u_a)$ and $B = \{v_1, v_2, \cdots, v_b\}$ with $k' + (q-1)d' = f'(v) = f'(v_1) > f'(v_2) > \cdots > f'(v_b)$. Then $f'(u_a) \le m_0(f') < f'(v_b)$. Noting when G is connected, we have

$$\begin{cases} f'(u_i) = i'd', & \text{for some } i' \in N \\ f'(v_j) = k' + j'd', & \text{for some } j' \in N \end{cases} \tag{10}$$

Thus, if either G is connected or d' divides d, then a vertex function $f: A \cup B \to N$ is defined by

$$\begin{cases} f(u_i) = \dfrac{f'(u_i)}{d'}d + k_1 & 1 \le i \le a \\ f(v_j) = \dfrac{k' + (q-1)d' - f'(v_j)}{d'}d + k_2 & 1 \le j \le b \end{cases} \tag{11}$$

It is clear that $f(u_1) = k_1$ and $f(v_1) = k_2$. In order to complete the proof, we need only to prove that f is a (k, d)-arithmetic numbering of G.

For any arbitrary edge u_iv_j of G, we have from (11)

$$\begin{aligned} f^+(u_iv_j) &= k + \frac{d}{d'}(k' + (q-1)d' - (f'(v_j) - f'(u_i))) \\ &= k + \frac{d}{d'}(k' + (q-1)d' - (k' + j'd')), \quad 0 \le j' \le q-1 \\ &= k + \frac{d}{d'}(q - j' - 1)d', \quad 0 \le j' \le q-1 \\ &= k + md, \quad 0 \le m \le q-1 \end{aligned}$$

It follows that $f^+(G) = \{k, k+d, k+2d, \cdots, k+(q-1)d\}$, therefor f^+ is injective. So, the remaining task is proved that f is injective. In other words, we need only to prove that $f(u_i) \ne f(v_j)$ for each pair of distinct vertices u_i and v_j.

Case 1. Soppose that G is connected and Condition (1) holds. If $f(u_i) = f(v_j)$ for some $u_i \in A$ and $v_j \in B$, then by the definition of f and (10), we have

$$k_2 - k_1 = \frac{d}{d'}(f'(u_i) + f'(v_j) - k' - (q-1)d') = nd \qquad \text{for some } n \in N$$

it is a contradiction.

Case 2. Suppose that either G is connected or d' divides d. If Condition (3) holds and $f(u_i) = f(v_j)$ for some $u_i \in A$ and $v_j \in B$, then from (11)

$$\begin{aligned} rd = k_2 - k_1 &= \frac{d}{d'}(f'(u_i) + f'(v_j) - k' - (q-1)d') \\ &\le \frac{d}{d'}(f'(u_a) + f'(v_1) - k' - (q-1)d') \end{aligned}$$

$$\leq \frac{d}{d'}(m_0(f') + k' + (q-1)d' - k' - (q-1)d)$$
$$< rd$$

It is a contradiction too.

Therefore, f is a required (k, d)-arithmetic numbering of G and the proof of Theorem 4 is completed.

Next, we give two examples to show Theorem 4. For the (1, 1)-balanced numbering f' of $L_{3,4}$ shown in figure 2(b), we have $m(f') = 8$. For $(k, d) = (31, 3)$, the partition $(k_1, k_2) = (2, 29)$ of $k = 31$ satisfies Condition (3). Thus, in view of Theorem 4, there is a (31, 3)-arithmetic numbering f of $L_{3,4}$ with $2, 29 \in f(L_{3,4})$ for it is defined by (11) as shown in figure 2(c), which is completely identical with the one shown in figure 9 of [1, p. 292]. Also $L_{3,4}$ is connected and the bipartition $(k_1, k_2) = (0, 31)$ of $k = 31$ satisfies Condition (1). By Theorem 4, there is a (31, 3)-arithmetic numbering f of $L_{3,4}$ with $0, 31 \in f(L_{3,4})$ for it is defined by (11) as shown in figure 2(d).

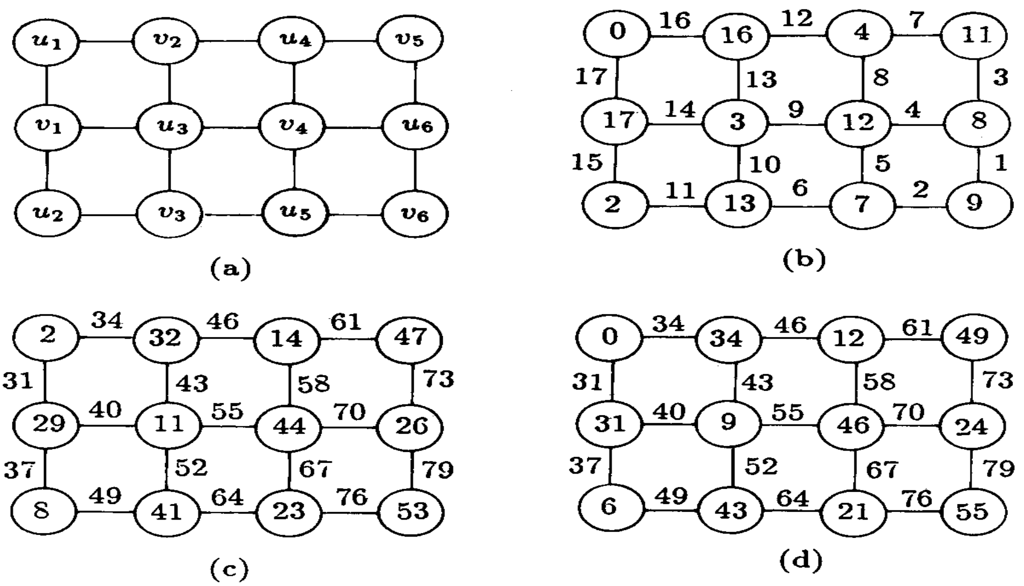

Fig. 2

Another example is $4K_2$, which is disconnected and balanced, a (2, 2)-balanced numbering f' is shown in figure 1(a). For $(k, d) = (10, 2)$, the partition $(k_1, k_2) = (1, 9)$ of $k = 10$ satisfies Condition (3) and d' divides d, where $m(f') = 3$. By Theorem 4, there is a (10, 2)-arithmetic numbering f of $4K_2$ with $1, 9 \in f(4K_2)$ for it is defined by (11) as shown in figure 1(b).

7 Some Remarks

First we explain that Condition (3) is stronger than Condition (2) in general

Proposition 1 Let f' be a (k, d)-balanced numbering of a bipartite graph G with the bipartition $\{A, B\}$ and $0 \in f'(B)$ (resp. $0 \in f'(A)$), then the vertex function $F': V(G) \to N$ defined by

$$F'(u) = k + (q - 1)d - f'(u), \qquad \forall u \in V(G) \tag{12}$$

is a (k, d)-balanced numbering of G and $0 \in F'(A)$ (resp. $0 \in F'(B)$).

The proof of Proposition 1 is a simple verification and is left to readers. The graph $S(K_{1,3})$ shown in figure 3 has the bipartition $\{A, B\}$, where $A = \{u_1, u_2, u_3\}$ and $B = \{v_1, v_2, v_3, v_4\}$ (see figure 3(a)). For the $(2, 1)$-balanced numbering f' of $S(K_{1,3})$ with $0 \in f'(B)$ shown in figure 3(b), we have the $(2, 1)$-balanced numbering F' of $S(K_{1,3})$ defined by (12) with $0 \in f'(A)$ for it is shown in figure 3(c).

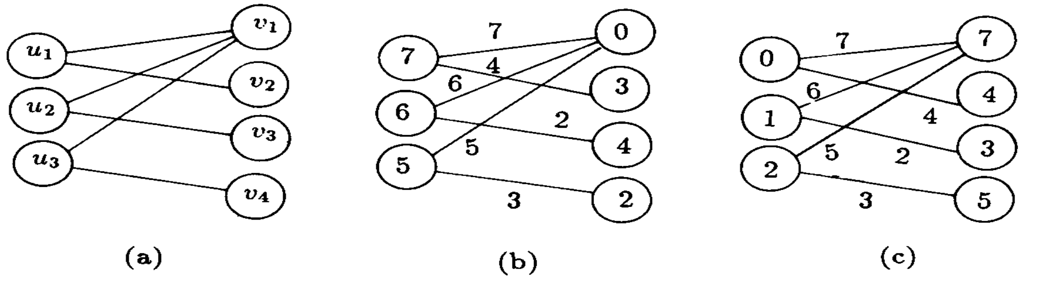

Fig 3

It follows from Proposition 1 that for any (k, d)-balanced bipartite graph G with a bipartition $\{A, B\}$, $|A| = a \leq b = |B|$, G has a (k, d)-balanced numbering f' with $0 \in f'(A)$. Note $m(f') \geq (a - 1)d$ for any (k, d)-balanced numbering f' of G with $0 \in f'(A)$. Thus Condition (3) is not weaker than Condition (2). Observe that the complete lattice grid $L_{3,4}$ has the $(1, 1)$-balanced numbering f' shown in figure 2(b), where $m(f') = 8 > 6 = a$. Therefore, Condition (3) is stronger than Condition (2).

Secondly, we show that Condition (3) is the same as Condition (2) for the above-mentioned classes of bipartite graphs, whence we can derive some known results from our theorem. Let f' be a balanced numbering of G.

Proposition 2 Let G be any one in the classes of bipartite graphs $K_{a,b}, C_{a,b}, C_{4t}$, then G admits of a $(1, 1)$-balanced numbering f' with $m_0(f') = a - 1$.

Proof We set a required $(1, 1)$-balanced numberings f' in the respective cases, as follows

(i) For $K_{a,b}$, define $f': V(K_{a,b}) \to N$ by letting

$$\begin{cases} f'(u_i) = i - 1 & 1 \leq i \leq a \\ f'(v_j) = (b + 1 - j)a & 1 \leq j \leq b \end{cases} \tag{13}$$

(ii) A $(1, 1)$-balanced numbering f' of $C_{a,b}$ is already displayed in figure 7 in [1, p. 290]. Namely, $f': V(C_{a,b}) \to N$ is defined by

$$\begin{cases} f'(u_i) = i - 1, & 1 \leq i \leq a \\ f'(v_j) = a + b - j, & 1 \leq j \leq b \end{cases} \tag{14}$$

(iii) Let the vertices on C_{4t}, $t \geq 1$, $a = 2t$, be consecutive in the order $u_1, u_2, \cdots, u_{4t}$. Define $f' : V(C_{4t}) \to N$ by letting

$$f'(u_i) = \begin{cases} \frac{1}{2}(i - 1), & \text{if } i \text{ is old,} \\ 4t - \frac{1}{2}(i - 2), & \text{if } i \text{ is even and } 2 \leq i \leq 2t \\ 4t - \frac{1}{2}i & \text{if } i \text{ is even and } 2t + 2 \leq i \leq 4t \end{cases} \tag{15}$$

The verification that the vertex functions defined by (13), (14) and (15) respectively, are required (1, 1)-balanced numberings is simple and is left to readers.

Substitute (13) for f' in (11) with $k' = d' = 1$ and $q = ab$. We have

$$\begin{cases} f(u_i) = (i - 1)d + k_1, & 1 \leq i \leq a \\ f(v_j) = (j - 1)ad + k_2 & 1 \leq j \leq b \end{cases} \tag{16}$$

Substitute (14) for f' in (11) with $k' = d' = 1$ and $q = a + b - 1$. We have

$$\begin{cases} f(u_i) = (i - 1)d + k_1, & 1 \leq i \leq a \\ f(v_j) = (j - 1)d + k_2 & 1 \leq j \leq b \end{cases} \tag{17}$$

Substitute (15) for f' in (11) with $k' = d' = 1$ and $q = 4t$. We have

$$f(u_i) = \begin{cases} \left(\frac{i - 1}{2}\right) d + k_1, & \text{if } i \text{ is old} \\ \left(\frac{i - 2}{2}\right) d + k_2, & \text{if } i \text{ is even and } 2 \leq i \leq 2t \\ \frac{i}{2} d + k_2, & \text{if } i \text{ is even and } 2t + 2 \leq i \leq 4t \end{cases} \tag{18}$$

From Proposition 2 and Theorem 4 we can immediately obtain Theorem 14, Theorem 15 and Theorem 17(A) of [1]. The (k, d)-arithmetic numberings defined by (11) are respectively shown in (16), (17) and (18). Interestingly, these expressions are completely identical with those as displayed in (13), (14) and (16) in [1] respectively.

Last, the following result can be obtained:

Proposition 3 Let k, d, k' and d' be any positve integers and f' be a (k', d')-balanced numbering of G. Suppose that either G is connected or d' divides d. If a partition (k_1, k_2) of k satisfies the condition:

$$k_2 - k_1 > \frac{m_0(f')}{d'} d \tag{19}$$

then G admits of a (k, d)-arithmetic numbering f with $k_1, k_2 \in f(G)$.

Proof Let $f: V(G) \to N$ be the vertex function defined by (11). From the proof of Theorem 4, we need only to prove that f is injective.

Suppose to the contrary that there exist vertices $u_i \in A$ and $v_j \in B$ such that $f(u_i) = f(v_j)$. Then by (11) we get

$$k_2 - k_1 = \frac{d}{d'}(f'(u_i) + f'(v_j) - k' - (q - 1)d') \leq \frac{d}{d'} f'(u_i) \leq \frac{d}{d'} m_0(f')$$

which contradicts Condition (19) and the proposition is proved.

If let $d' = d$, $k_1 = 0$, $k_2 = k = m(f') + 1$, then we immediately from Proposition 3 obtain Theorem 12 of [1], and the (k, d)-arithmetic numbering f defined by (11) is

$$\begin{cases} f(u_i) = f'(u_i), & 1 \le i \le a \\ f(v_j) = k' + (q - 1)d' - f'(v_j) + m(f') + 1 & 1 \le j \le b \end{cases}$$

which is the same as (11) of [1].

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References

[1] Acharya B D, Hegde S M. Arithmetic graphs J. Graph Theory, 1990, 14 (3): 275—299

[2] Acharya B D, Hegde S M. On certain vertex valuations of a graph-I Indian J. Pure Appl Math., 1991, 22: 553—560

关于算术平衡图的某些结果

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摘要　首先考虑 Acharya 和 Hegde 关于算术平衡图的三个猜想. 其中一个已由他们证明, 本文给出它和另一个猜想的简单证明, 并指出第三个猜想在一般情形下是不对的, 而在一个更强的条件下是正确的. 然后讨论本文结果与已知结果之间的关系.

关键词　图论, 算术标号, 平衡图

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