SOME RESULTS ON $R_2$-EDGE-CONNECTIVITY OF EVEN REGULAR GRAPHS

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Abstract Let $G$ be a connected $k$ ($\geq 3$)-regular graph with girth $g$. A set $S$ of the edges in $G$ is called an $R_2$-edge-cut if $G - S$ is disconnected and contains neither an isolated vertex nor a one-degree vertex. The $R_2$-edge-connectivity of $G$, denoted by $\lambda'(G)$, is the minimum cardinality over all $R_2$-edge-cuts, which is an important measure for fault-tolerance of computer interconnection networks. In this paper, $\lambda'(G) = g(2k - 2)$ for any $2k$-regular connected graph $G(\neq K_5)$ that is either edge-transitive or vertex-transitive and $g \geq 5$ is given.

§ 1 Introduction

In this paper, a graph $G = (V, E)$ always means a simple graph (without loops and multiple edges) with the vertex-set $V$ and the edge-set $E$. We follow [1] for graph-theoretical terminology and notation not defined here.

It is well-known that when the underlying topology of a computer interconnection network is modeled by a graph $G$, the connectivity of $G$ is an important measure for fault-tolerance of the network. However, it has many deficiencies (see [2]). To compensate for the shortcomings of the traditional connectivity measure, one can make use of several generalized measures of connectedness. One of them is referred to as an $R_h$-edge-connectivity proposed by Latifi et al. [3].

Let $G$ be a connected $k$-regular graph, $k \geq h + 1$. A set $S$ of the edges in $G$ is called an $R_h$-edge-cut if $G - S$ is disconnected and contains neither an isolated vertex nor a one-degree vertex. The $R_h$-edge-connectivity of $G$, denoted by $\lambda^{(h)}(G)$, is the minimum cardinality over all $R_h$-edge-cuts of $G$.

Observe that when $h = 0$, there will be no restriction on connected components and we have the traditional edge-connectivity. In addition, in the special case of $h = 1$, this connectivity will be reduced to the restricted edge-connectivity given in [2, 4]. Thus this con-
nectivity can be regarded as a generalization of the traditional edge-connectivity, which could provide a more accurate fault-tolerance measure of networks and has received much attention recently (for example, see [4] 6)). In this paper we restrict our attention to \( h = 2 \) and even regular graphs. For the sake of convenience, we write \( \lambda'' \) for \( \lambda(2) \).

\( G \) is called vertex-transitive if for any two vertices \( x \) and \( y \) in \( G \), there is an element \( \pi \in \Gamma(G) \), the automorphism group of \( G \), such that \( \pi(x) = y \). It is well-known that any vertex-transitive graph is regular\(^7\). \( G \) is called edge-transitive if for any two edges \( e = xy \) and \( e' = uv \) in \( G \), there is an element \( \sigma \in \Gamma(G) \) such that \( \sigma(\{x, y\}) = \{u, v\} \). For a special class of vertex-transitive graphs referred to as circulant graphs, Li Qiaoliang\(^5\) has obtained their \( R_2 \)-edge-connectivity in his Ph.D thesis. Motivated by Li's work, we will, in the present paper, show that for a connected \( 2k \)-regular graph \( G \neq K_2 \), \( \lambda''(G) = g(2k - 2) \) if \( G \) is either edge-transitive or vertex-transitive and \( g \geq 5 \).

The rest of this paper is organized as follows. The next section contains several notations and preliminary results used in this paper later. In § 3, we present two lemmas used in the proofs of our main results in § 4.

### § 2 Notations and Preliminary Results

Let \( G \) be a \( k \)-regular graph. If \( k \geq 2 \), then \( G \) certainly contains a cycle. We use \( g(G) \) to denote the girth of \( G \), the length of a shortest cycle in \( G \). It is known in [8, Problem 10.11] that if \( G \) is a \( k \)-regular graph with girth \( g \), then

\[
|V(G)| \geq f(k, g) = \begin{cases} 
1 + k + k(k - 1) + \ldots + k(k - 1)(g - 3)/2, & \text{if } g \text{ is odd;} \\
2[1 + (k - 1) + \ldots + (k - 1)(g - 2)/2], & \text{if } g \text{ is even.}
\end{cases}
\]

A vertex \( x \) in \( G \) is called singular if it is of degree either zero or one. Let \( X \) and \( Y \) be two disjoint nonempty proper subsets of \( V \). \( (X, Y) = \{e \in E(G) : \text{there are } x \in X \text{ and } y \in Y \text{ such that } e = xy \in E(G) \} \). If \( Y = \bar{X} = V \setminus X \), then we write \( E(X) \) for \( (X, \bar{X}) \) and \( d(X) \) for \( |E(X)| \). The following inequality is well-known (see [8], Problem 6.48).

\[
d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y).
\]

A \( R_2 \)-edge-cut \( S \) of \( G \) is called a \( \lambda'' \)-cut if \( |S| \geq \lambda''(G) > 0 \). Let \( X \) be a proper subset of \( V \). If \( E(X) \) is a \( \lambda'' \)-cut of \( G \), then \( X \) is called a \( \lambda'' \)-fragment of \( G \). It is clear that if \( X \) is a \( \lambda'' \)-fragment of \( G \), then so is \( \bar{X} \) and both \( G[X] \) and \( G[\bar{X}] \) are connected. Let

\[
r(G) = \min\{|X : X \text{ is a } \lambda'' \text{-fragment of } G\}.
\]

A \( \lambda'' \)-fragment \( X \) is called a \( \lambda'' \)-atom of \( G \) if \( |X| = r(G) \). Since \( G[X] \) is connected and contains no singular vertices for a given \( \lambda'' \)-atom \( X \) of \( G \), \( G[X] \) certainly contains a cycle. Thus

\[
r(G) = \min\{|X : |X| \geq g(G)\}.
\]

**Theorem 1.** Let \( G \) be a connected \( 2k \)-regular graph, \( k \geq 2 \). If \( G \neq K_2 \), then \( \lambda''(G) \) exists and \( \lambda''(G) \leq g(2k - 2) \).

**Proof.** Let \( G \) be a connected \( 2k \)-regular graph, \( G \neq K_2 \) and \( k \geq 2 \). We want to show
\[ \lambda'(G) \leq g(2k - 2) \]. For this purpose, let \( X \) be the vertex-set of a shortest cycle \( C_s \) in \( G \). Then \( X \neq \emptyset \) and \( E(X) \) is an edge-cut of \( G \) since \( k \geq 2 \) and each in \( X \) is a two-degree vertex in \( C_s \). If \( E(X) \) is an \( R \)-edge-cut of \( G \), then \( \lambda'(G) \leq d(X) = g(2k - 2) \).

Suppose that \( E(X) \) is not an \( R \)-edge-cut of \( G \). Note the minimality of \( C_s \), it is clear that for any \( y \in X \), \( |N_G(y) \cap X| \leq 2 \) if \( g \geq 4 \), by which \( g = 3 \) and \( k = 2 \). Let \( y \) be a singular vertex in \( G - E(X) \). Then, obviously, \( y \notin X \) and \( y \) is a one-degree vertex in \( G - E(X) \). Let \( Y = X \cup \{ y \} \). Then \( d(X) = 6 \) and \( d(Y) = 4 \). If there are no singular vertices in \( G - E(Y) \), then \( E(Y) \) is an \( R \)-edge-cut of \( G \) and \( \lambda'(G) \leq d(Y) = 4 < 6 \). Suppose that there is some singular vertex \( z \) in \( G - E(Y) \). If \( z \) is an isolated vertex in \( G - E(Y) \), then \( G = K_3 \), which contradicts our assumption. Thus \( z \) is a one-degree vertex in \( G - E(Y) \). Let \( Z = Y \cup \{ z \} \), then \( Z \neq \emptyset \), \( d(Z) = 2 \) and \( G - E(Z) \) contains no singular vertices. It follows that \( E(Z) \) is an \( R \)-edge-cut of \( G \). So \( \lambda'(G) \leq d(Z) = 2 < 6 \).

\section*{§ 3 Two Lemmas}

\textbf{Lemma 1} \[ \emptyset \subseteq \emptyset \subseteq K_3 \] and \( k \geq 2 \). Let \( R \) be a proper subset of \( V(G) \) and \( U \) be the set of the singular vertices in \( G - E(R) \). If \( \emptyset \neq U \subseteq R \) and \( d(R) \leq \lambda'(G) + 1 \), then \( |R| \leq g(G) \).

\textbf{Proof} \[ \emptyset \subseteq \emptyset \subseteq K_3 \] Let \( g = g(G) \). Since \( \lambda'(G) \leq g(2k - 2) \) by Theorem 1. Suppose to the contrary that \( |R| \geq g \). We want to derive contradictions. 

If \( G[R] \) contains no cycles, then \( |E(G[R])| \leq |R| - 1 \). So we can deduce a contradiction as follows

\[ g(2k - 2) + 1 \geq \lambda'(G) + 1 \geq d(R) = 2k |R| - 2 |E(G[R])| \geq 2k |R| - 2(2 |R| - 1) = |R| - 2k + 2 \geq g(2k - 2) + 2 \]

If \( G[R] \) contains cycles, then let \( R' \) be the vertex-set of the union of all blocks that contain a cycle in \( G[R] \). Thus \( U \subseteq R \setminus R' \). Note that \( |N_G(u) \cap R'| \leq 1 \) for any \( u \in R \setminus R' \) and \( k \geq 2 \). \( G - E(R') \) contains no singular vertices. This implies that \( E(R') \) is an \( R \)-edge-cut of \( G \), by which \( d(R') \geq \lambda'(G) \). By the choice of \( R' \) we have that for any two distinct vertices in \( R' \), their neighbors in \( R \setminus R' \) are disconnected in \( G[R] \) and that for any neighbor \( z \) of \( R' \) in \( R \setminus R' \), either \( z \in U \) or there is a path in \( G[R \setminus R'] \) connecting \( z \) to some vertex in \( U \). Thus \( |R[R', R']| \leq |U| \), \( d(R') \geq |U| \geq (2k - 1) \) since \( U \subseteq R \setminus R' \). We can deduce a contradiction as follows

\[ \lambda'(G) \leq d(R') = d(R) - |(R \setminus R', R)| + |(R', R \setminus R')| \leq d(R) - |U| |(2k - 1) + |U| = d(R) - |U| |(2k - 2) \leq \lambda'(G) - 1 \]

The proof is complete.

\textbf{Lemma 2} \[ \emptyset \subseteq \emptyset \subseteq K_3 \] Let \( G \) be a connected \( 2k \)-regular graph. If \( \lambda'(G) < g(2k - 2) \), then \( X \cap X' = \emptyset \) for any two distinct \( \lambda' \)-atoms \( X \) and \( X' \) of \( G \).

\textbf{Proof} \[ \emptyset \subseteq \emptyset \subseteq K_3 \] Suppose that \( \lambda'(G) < g(2k - 2) \) and \( X \) and \( X' \) are two distinct \( \lambda' \)-atoms of \( G \).
Note that $|X| = |X'| = r(G) \geq g$. If $r(G) = g$, then $G[X]$ is a cycle of length $g$. Thus $g(2k-2) = d(X) = \lambda'(G) < g(2k-2)$. This contradiction implies that $|X| > g$. We want to show that $X \cap X' = \emptyset$. Suppose to the contrary that $X \cap X' \neq \emptyset$. Let 

$$A = X \cap X', \, B = X \cap X', \, C = X \cap X', \, D = X \cap X'.$$

Then $|B| \geq |A| \geq 1$, $|B| = |C| = r(G) - |A| \geq 1$ since $X$ and $X'$ are two distinct $\lambda'$-atoms of $G$. To derive contradictions, we consider two cases separately.

**Case 1**: $G- E(A)$ contains no singular vertices

It is clear that $E(A)$ is an $R$-edge-cut of $G$ and $G[A]$ certainly contains cycles since $G - E(A)$ does not contain any singular vertex. It follows that

$$d(X \cap X') = d(A) > \lambda'(G), \, \square \square \, |D| \geq |A| \geq g. \quad (3)$$

Noting $d(X) = d(X') = \lambda'(G)$, by (2) and the left inequality in (3), we have

$$d(D) = d(X \cup X') \leq d(X) + d(X') - d(X \cap X') < \lambda'(G).$$

This implies that $G - E(D)$ does certainly contain some singular vertices otherwise $E(D)$ is an $R$-edge-cut of $G$ whose cardinality is less than $\lambda'(G)$. These singular vertices are contained in $D$ obviously. Thus $|D| < g$ by Lemma 1. This contradicts (3).

**Case 2**: $G - E(A)$ contains singular vertices

Let $y$ be a singular vertex in $G - E(A)$, then $y \in A$ obviously. Let $Y = X \setminus \{y\}$ if $|(y, C)| > |(y, B)|$ (or let $Y = X \setminus \{y\}$ if $|(y, C)| < |(y, B)|$, then $|Y| = |X| - 1$ and $d(Y) \leq d(X) - |(y, D)| - |(y, C)| + |(y, B)| + 1 \leq d(X) = \lambda'(G). \quad (4)$

Note that $X$ is a $\lambda'$-atom of $G$ and $Y \subseteq X$, there exist singular vertices in $G - E(Y)$, then they all are contained in $Y$. so $|Y| < g$ by Lemma 1, by which $|X| = |Y| + 1 \leq g$. This contradicts the fact that $|X| > g$.

We can similarly obtain a contradiction if we consider the case of $|(y, C)| < |(y, B)|$.

Next, we want to consider the case of $|(y, C)| = |(y, B)|$. Note that in this case the equality in (4) does not hold only when $|(y, D)| = 0$ and $y$ is a one-degree vertex in $G - E(A)$. It follows that $d_G(y) = 1 + |(y, C)| + |(y, B)|$, which is odd. This contradicts our assumption that the regularity of $G$ is even.

The proof of Lemma 2 is complete.

### § 4 Main Results

**Theorem 2**: Let $G$ be a connected $2k$-regular edge-transitive graph, $G \neq K_s$ and $k \geq 2$, then $\lambda'(G) = g(2k - 2)$.

**Proof**: By our assumption, $\lambda'(G)$ exists and $\lambda'(G) \leq g(2k - 2)$ by Theorem 1. Suppose that $\lambda'(G) < g(2k - 2)$. Let $X$ be a $\lambda'$-atom of $G$, then $|X| > g \geq 3$. Let $e = xy$ be an edge in $G[X]$ and $e' = yz$ be an edge in $E(X)$, $z \in \overline{X}$. Since $G$ is edge-transitive, there is $\sigma \in \Gamma(G)$ such that $\sigma(x, y) = (y, z)$. Hence $\sigma(X)$ is also a $\lambda'$-atom of $G$. Let $X' = \sigma(X)$, then $X \neq X'$ since $z \in X'$ and $z \notin X$. On the other hand, since $y \in X \cap X'$, $X = X'$ by Lemma-
Theorem 3 Let $G$ be a connected $2k$-regular vertex-transitive graph, $g \geq 5$ and $k \geq 2$, then $\lambda'(G) = g(2k - 2)$.

Proof It is clear that $\lambda'(G)$ exists and $\lambda'(G) \leq g(2k - 2)$ by Theorem 1. Suppose that $\lambda'(G) < g(2k - 2)$ and $X$ is a $\lambda'$-atom of $G$. We claim that $G[X]$ is vertex-transitive. To the end we let

$$\Pi = \{ \pi \in \Gamma(G) : \pi(x) = x \}, \quad \Psi = \{ \pi \in \Pi : x \in X \Rightarrow \pi(x) = x \}.$$  

It is clear from Lemma 2 that $\Pi$ is a subgroup of $\Gamma(G)$, and the constituent of $\Pi$ on $X$ acts transitively and $\Psi$ is a normal subgroup of $\Pi$. Thus there is an injective homomorphism from the quotient group $\Pi / \Psi$ to $\Gamma(G[X])$ where by each coset of $\Psi$ is associated with the restriction to $X$ of any representative. This proves that $G[X]$ is vertex-transitive.

Let the regularity of $G[X]$ be $t$, then $2 \leq t \leq 2k - 1$ and

$$g(2k - 2) > \lambda'(G) = d(X) = (2k - t) |X|$$  

Since $G[X]$ is $t$-regular and $t \geq 2$, $G[X]$ certainly contains a cycle of length at least $g$. It follows from (1) and (5) that

$$0 < g(2k - 2) - (2k - t)f(t, g).$$  

The right side of (6) is an increasing function with respect to $t$ and is a descending function with respect to $g$. It is not difficult to show that there exists no $t \in [2, 2k - 1]$ such that (6) holds if $g \geq 5$. This proves Theorem 3.

References


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