Designing of optimal double loop networks*

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Abstract The double loop network $G(N;r,s)$ has N vertices and $2N$ directed edges. A natural question is how to choose r and s such that $G(N;r,s)$ has diameter as short as possible for a given N. In 1993, Li, Xu and Zhang proposed a method of constructing double loop networks with the minimum diameter for the case of $r=1$. The method is developed to construct such networks that none of their minimum diameters can be reached at $r=1$. As a by-product, a flaw in an assertion by Esqué et al. is pointed out.

Keywords: double loop networks, computer interconnection networks, optimal designing, circulant digraphs, diameters.

It is well known that the topological structure of a computer interconnection network or a communication system can be modeled by a digraph G in which the vertex set represents the set of processors or switch elements and the directed edge set represents the set of unilateral communication links connecting one processor with another. An important parameter to measure the efficiency of a network is transmission delay of information, which can be measured by the diameter of a corresponding digraph. Double loop networks have been widely used in the topological structures of computer interconnection networks and communication systems because of their symmetry, simplicity and extensionality. The graphical model of a double loop network is a digraph $G(N;r,s)$ with N vertices $\{0,1,2,\cdots,N-1\}$ and $2N$ directed edges $\{i\to i+r(\bmod N),\ i\to i+s(\bmod N):\ i=0,1,2,\cdots,N-1\}$, where r and s are two given integers with $1\leqslant r\neq s<N$. From the definition, it is clear that a double loop network $G(N;r,s)$ can only be determined by N, r and s, and so can its diameter. A natural question is as follows. For a given positive integer N, how can we choose r and s such that $G(N;r,s)$ has diameter as short as possible?

It is well known that $G(N;r,s)$ has a finite diameter if and only if $G(N;r,s)$ is strongly connected if and only if N,r,s satisfy the following condition:

$$g.c.d.(N,r,s)=1. \tag{1}$$

We use $d(N;r,s)$ to denote the diameter of $G(N;r,s)$ if $G(N;r,s)$ has a finite diameter. Let

$$d_1(N)=\min\{d(N;1,s):\ 1<s<N\},$$

$$d(N)=\min\{d(N;r,s):\ 1\leqslant r\neq s<N\}.$$

Obviously, $d_1(N)\geqslant d(N)$. If $d_1(N)>d(N)$, then $d(N)$ cannot be reached at $r=1$. An N is said to be singular if $d_1(N)>d(N)$. $G(N;1,s)$ is said to be good if $d(N;1,s)=d_1(N)$. $G(N;r,s)$ is said to be optimal if $d(N;r,s)=d(N)$. For a positive real number m, we use

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$\lceil m \rceil$ to denote the least integer greater than or equal to m.

In ref. [1], Wong and Coppersmith proved that

$$d_1(N) \geqslant \lceil \sqrt{3N} \rceil - 2, \tag{2}$$

and proposed a combinatorial optimization problem as follows. For any given $N \geqslant 4$, determine the value of $d_1(N)$ and construct a good $G(N;1,s)$. This problem did not bring to reseachers' enough attention until 1987 due to the difficulties in itself and the limitations of its applications. With the wide applications of computers and the advent of VLSI circuits and fiber optics technology, great attention has been paid to it and 16 infinite families of good double loop networks were successively found[2—8]. However, these infinite families still cannot include every $N \leqslant 50$. A breakthrough occurred in 1993. Li et al.[7, 8] presented a systematic method of finding infinite families of good double loop networks. As illustrations, they exhibited 102 such infinite families constructed by their method, which contain 16 known infinite families and all positive integers within 300.

In ref. [6], Fiol et al. showed that the lower bound in (2) still holds for general double loop networks $G(N;r,s)$, i.e.

$$d(N) \geqslant \lceil \sqrt{3N} \rceil - 2. \tag{3}$$

Also, they found singular N's exist indeed with the aid of a computer, of which the minimum is 450. In fact, $d_1(450) = 36$ and so $G(450;1,59)$ is good. However, $d(450;2,185) = \lceil \sqrt{3 \cdot 450} \rceil - 2 = 35$ and so $G(450;2,185)$ is optimal according to (3).

For $N \geqslant 4$, let $lb(N) = \lceil \sqrt{3N} \rceil - 2$. $G(N;r,s)$ is said to be tight optimal if $d(N;r,s) = lb(N)$. It is clear that if $G(N;r,s)$ is tight optimal, then it must be optimal. The converse is false in general. A tight optimal $G(N;r,s)$ is said to be singular if N is singular. For instance, $G(450;2,185)$ is singularly tight optimal.

At the end of ref. [3], Esqué et al. claimed to find an infinite family of singularly tight optimal double loop networks $\{G(N(e);r(e),s(e)): e \in Z\}$, where $N(e) = 2\,700e^2 + 2\,220e + 450$, $r(e) = 30e + 2$, $s(e) = 420e + 185$, Z is the infinite set of all nonnegative integers. Let $Z' = \{157k + 136: k \in Z\}$, then $Z' \subset Z$ and $\{G(N(e);r(e),s(e)): e \in Z'\}$ is an infinite subfamily of $\{G(N(e);r(e),s(e)): e \in Z\}$. However, it is not difficult to show that

$$g.c.d.(N(e),r(e),s(e)) \geqslant 157, \quad \forall e \in Z'.$$

According to the condition in (1), none of the networks in $\{G(N(e);r(e),s(e)): e \in Z'\}$ is strongly connected. This fact means that Esqué et al.'s assertion is false.

In the present paper, we develop Li et al.'s method to construct an infinite family of tight optimal double loop networks $\{G(N(t);r(t),s(t)): t \in U \subseteq Z\}$, which contains a singularly infinite subfamily $\{G(N(t(e));r(t(e)),s(t(e))): e \in E \subseteq Z\}$ such that they take a given singularly tight optimal $G(N_0;r_0,s_0)$ as their initial element. This kind of property of double loop networks has not been revealed by the previous approaches. As an illustration, we further discuss and correct the above-mentioned assertion by Esqué et al.

1 Definitions and lemmas

To obtain the diameter of $G(N;r,s)$ we need only consider the distance from the vertex 0 to other vertices since $G(N;r,s)$ is vertex-transitive (see ref. [10]). For research purpose, let the unit on X axis be r (or s) and the unit on Y axis be s (or r) in Cartesian rectangular coordinates system. We put all lattice-point (x, y)'s in the first quadrant in order as follows. $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\cdots,(j,0),(j-1,1),\cdots,(j-i,i),\cdots,(1,j-1),(0,j),\cdots$. We then successively arrange a number $k \in \{0,1,2,\cdots,N-1\}$ in the unit square of the upper right-hand corner of the lattice-point (x, y) such that $k \equiv xr + ys \pmod{N}$. If k appeared in some square what we have considered, then let this square empty and consider the next square. Such a process of arrangement does not end until all $0,1,2,\cdots,N-1$ appear in the squares.

That shown in fig. 1 is the planar pattern of $G(16;3,5)$ constructed in the above way, where $N = 16$, $r = 3$, $s = 5$. It was already proved in refs. [1,6] that if N, r, s satisfy the condition in (1), then the planar pattern consisting of N unit squares determined by $G(N; r, s)$ forms an L-shape (maybe a rectangle or a square, for instance, $G(15;1,3)$ and $G(16;1,4)$) region as shown in fig. 2. We use $L(N; r, s)$ to denote such an L-shape region determined by $G(N; r, s)$. It is clear that if the number k is located in the unit square of the upper right-hand corner of the lattice-point (x, y), then the distance from the vertex 0 to the vertex k is equal to $x + y$ provided $G(N; r, s)$ is strongly connected.

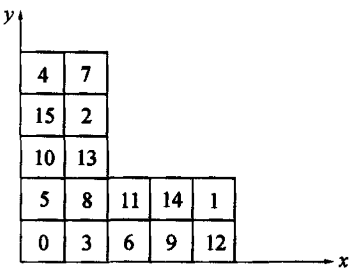

Fig. 1

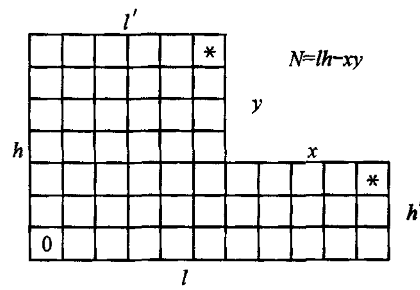

Fig. 2

As shown in fig. 2, an L-shape region with area N is said to be an L-tile, denoted by $L(N; l, h, x, y)$, if it is determined by integers l, h, x, y, where $l, h \geqslant 2$, $0 \leqslant x < l$, $1 \leqslant y < h$, $y < l$, $x \leqslant h$. Let

$$D(L(N; l, h, x, y)) = \max\{h + l' - 2, l + h' - 2\}$$

and say it to be the diameter of $L(N; l, h, x, y)$. Use $D(N)$ to denote the minimum value of the diameters of all L-tiles with area N. Use $D(N; r, s)$ to denote the diameter of the L-tile determined by $G(N; r, s)$. Let $L = L(N; l, h, x, y)$ be an L-tile. L is said to be tight if $D(L) = lb(N)$, and be (r, s)-implementable if there exists a $G(N; r, s)$ such that $L(N; r, s) = L(N; l, h, x, y)$.

According to the above notation and definitions, it is clear that the following facts hold. For any $G(N; r, s)$, $d(N; r, s) = D(N; r, s)$, and $d(N) \geqslant D(N)$. If $G(N; r, s)$ is tight optimal, then the corresponding L-tile must be tight.

Lemma 1[6,7,9]. *Let $L = L(N; l, h, x, y)$ be an L-tile. Then*

(a) *L is $(1, s)$-implementable if and only if g.c.d.$(y, h - y) = 1$;*

(b) *L is (r, s)-implementable if and only if g.c.d.$(l, h, x, y) = 1$, where r (or s) $\equiv h\alpha + y\beta \pmod{N}$, s (or r) $\equiv x\alpha + l\beta \pmod{N}$, and N, r, s satisfy the condition in (1), α and β are some integers that satisfy both $lr - ys = \alpha N$ and $-xr + hs = \beta N$.*

Lemma 2[7]. *For a positive integer T, the interval of integer $[4, 3T^2 + 6T + 3]$ can be parti-*

tioned into the union of $3T$ *intervals of integers*:

$$\bigcup_{t=1}^{T}\bigcup_{i=1}^{3} I_i(t),$$

where $I_1(t)=[3t^2+1, 3t^2+2t]$, $I_2(t)=[3t^2+2t+1, 3t^2+4t+1]$, $I_3(t)=[3t^2+4t+2, 3t^2+6t+3]$. *Furthermore*, $N\in I_i(t)$ *if and only if* $lb(N)=3t-2+i$ *for every* $i=1,2,3$.

By the same way as one used in the proofs of Lemma 1 and Lemma 5(a) in ref. [7], we can prove the following two Lemmas.

Lemma 3. *Let* $L=L(N;l,h,x,y)$ *be an L-tile. If* $|y-x|\geqslant 1$, *then*

$$D(L)\geqslant\left\lceil\sqrt{3N-\frac{3}{4}}+\frac{1}{2}\right\rceil-2.$$

Lemma 4. *Let* $L=L(N;l,h,x,y)$ *be an L-tile, where* $N=3t^2+At+B\in I_i(t)$, $l=2t+a$, $h=2t+b$. *If* $z=y-x\geqslant 0$, *then* L *is tight if and only if*

$$(a+b-i)(a+b-i+z)-ab+(A+z-2i)t+B=0. \tag{4}$$

2 Method and illustrations

Our method presented here aims at constructing an infinite family of tight optimal double loop networks that contains a singular infinite subfamily. There are two cases to consider. In the first case, some singularly tight optimal double loop network $G(N_0;r_0,s_0)$ is known (it can be found by a computer, for instance, $G(450;2,185)$). Let Z be the infinite set of all nonnegative integers. We attempt to construct an infinite family of tight optimal double loop networks $\{G(N(t);r(t),s(t)):\ t\in U\subseteq Z\}$ with initial element $G(N_0;r_0,s_0)$ such that it contains a singularly infinite subfamily $\{G(N(t(e));r(t(e)),s(t(e))):\ e\in E\subseteq Z\}$, and $t(0)=t_0$, $N(t_0)=N_0$, $r(t_0)=r_0$, $s(t_0)=s_0$. In the second case, we construct such infinite families under the condition that an initial element is unknown.

We first sketch out our strategy to deal with the former case. Let $G(N_0;r_0,s_0)$ be a singularly tight optimal double loop network, and take it as an initial element.

Step 1. Appropriately choose A, B and t_0 such that for any $t\geqslant t_0$, $N(t)=3t^2+At+B\in I_i(t)$, $1\leqslant i\leqslant 3$, and $N(t_0)=N_0$.

Step 2. Find out all tight L-tiles $L(t)=L(N(t);l(t),h(t),x(t),y(t))$ with area $N(t)=3t^2+At+B$ by making good use of Lemma 3 and Lemma 4, and then decide whether they are $(r(t),s(t))$-implementable or not by making good use of Lemma 1(b). If there exists some $(r(t),s(t))$-implementable $L(t)$, then appropriately choose α and β in order to determine $r(t)$ and $s(t)$ such that $r(t_0)=r_0$ and $s(t_0)=s_0$. Whereby an infinite subset U of Z is determined, and an infinite family $\{G(N(t);r(t),s(t)):\ t\in U\}$ of tight optimal double loop networks with the initial element $G(N_0;r_0,s_0)$ is obtained. Otherwise, go to step 1 and rechoose A, B and t_0.

Step 3. Find out the sufficient conditions that t must satisfy such that all L-tile $L(t)$'s found in step 2 are $(r(t),s(t))$-implementable but not $(1,s(t))$-implementable by making good use of Lemma 1(a). Otherwise, go to step 1 and rechoose A, B and t_0.

Step 4. Appropriately choose a function $t=t(e)$ with $t(0)=t_0$ such that it satisfies the sufficient conditions found out in step 3. Determine an infinite subset E of Z such that $t(e)\in U$ for all $e\in E$. Whereby, a singularly infinite subfamily $\{G(N(t(e));r(t(e)),s(t(e))):\ e\in E\}$ of

$\{G(N(t);r(t),s(t)):\ t\in U\}$ is constructed. Otherwise, go to step 1 and rechoose A, B and t_0.

Next, we cite a concrete illustration of this method. For instance, take the singularly tight optimal double loop network $G(450;2,185)$ as an initial element of the infinite family which we want to construct.

Step 1. Take $A=2$, $B=-6$ and $t_0=12$. Then $N(t)=3t^2+2t-6$, $N(12)=450$. Also $lb(N(t))=3t-1$ since $N(t)=3t^2+2t-6\in I_1(t)$ provided $t\geqslant 12=t_0$.

Step 2. Let $L(t)=L(N(t);l(t),h(t),x(t),y(t))$ be a tight L-tile with area $N(t)=3t^2+2t-6$. Then $D(L(t))=lb(N(t))=3t-1$. Let $z=|x-y|$. If $z\geqslant 1$, then when $t\geqslant 7$, we have

$$3N(t)-\frac{3}{4}=8t^2+6t-18-\frac{3}{4}=\left(3t+\frac{1}{2}\right)^2+3t-19>\left(3t+\frac{1}{2}\right)^2.$$

Thus by Lemma 3 when $t\geqslant 7$, we have

$$D(L(t))\geqslant\left\lceil\sqrt{3N(t)-\frac{3}{4}}+\frac{1}{2}\right\rceil-2\geqslant 3t-2+2=3t.$$

This contradicts the fact that $D(L(t))=3t-1$. Therefore, $z=0$. Considering the equality (4) for $A=2, B=-6$, $i=1$, $z=0$, we have

$$b^2+(a-2)b+a^2-2a-5=0. \tag{5}$$

The Diophantine equation (5) in the unknowns a and b has the solutions

$$(a,b)=(-2,1),\ (-2,3),\ (1,3),\ (1,-2),\ (3,-2),\ (3,1).$$

All tight L-tiles $L(N(t);l(t),h(t),x(t),y(t))$ corresponding to these solutions are $L_1(t)=L(N(t);2t-2,2t+1,t-2,t-2)$, $L_2(t)=L(N(t);2t-2,2t+3,t,t)$ and $L_3(t)=L(N(t);2t+1,2t+3,t+3,t+3)$ as well as their transposes. It is easy to check whichever of these six L-tiles satisfies

$$g.c.d.(l(t),h(t),x(t),y(t))=1,\quad \forall\, t\in Z,\ t\geqslant 7.$$

By Lemma 1(b), each of them is $(r(t),s(t))$-implementable for any $t\in Z$ provided $t\geqslant 7$.

Consider the L-tile $L_1(t)=L(N(t);l(t),h(t),x(t),y(t))$, where $N(t)=3t^2+2t-6$, $l(t)=2t-2$, $h(t)=2t+1$, $x(t)=y(t)=t-2$. Appropriately choose α and β in Lemma 1(b). For instance, let $\alpha=9$, $\beta=-4$. Then we have

$$r(t)=t-10,\ s(t)=14t+17,\quad t\in Z, t\geqslant 12,$$

and $r(12)=2$, $s(12)=185$.

Next, we determine the range of the variable t by making good use of the condition in (1). Suppose that there exists some t, $t\geqslant 12$ such that

$$g.c.d.(3t^2+2t-6,t-10,14t+17)=m\geqslant 2.$$

Then there exist two nonzero integers k and g such that $t-10=km$ and $14t+17=gm$. As a result, we have $(g-14k)m=157$. Since 157 is a prime number and $m\geqslant 2$, we have $m=157$, $t=157k+10$ for any $k\in Z$, $k\geqslant 1$. Let

$$U=\{t:t\in Z,t\geqslant 12\}\setminus\{157k+167:k\in Z\}.$$

Then U is an infinite subset of Z, and

$$g.c.d.(N(t),r(t),s(t))=1,\quad \forall\, t\in U.$$

So $G(N(t);r(t),s(t))$ is tight optimal and is of diameter $lb(N(t))=3t-1$ for any $t\in U$, where $N(t) = 3t^2 + 2t - 6$, $r(t) = t - 10$ and $s(t) = 14t + 17$.

Step 3. Making good use of Lemma 1(a), we can easily vertify that if t simultaneously satisfies the following three conditions:

$$t \equiv 0(\mathrm{mod}2), \quad t \equiv 0(\mathrm{mod}3), \quad t \equiv 2(\mathrm{mod}5), \tag{6}$$

then for any $t \in Z$ and $t \geqslant 12$, none of these six L-tiles with area $N(t) = 3t^2 + 2t - 6$ is $(1, s(t))$-implementable; namely, the conditions in (6) are the sufficient conditions that t must satisfy for which any L-tile with area $N(t) = 3t^2 + 2t - 6$ is not $(1, s(t))$-implementable.

Step 4. To determine a function $t = t(e)$ that satisfies the conditions in (6), let us take $t = 2g$, $g = 3f$, $f = 5e$. Then $t = 30e + 12$, $t(0) = 12 = t_0$ and t satisfies the three conditions in (6). To determine the range of the variable e such that $t(e) \in U$, we first determine the range of such an e that $30e + 12 \equiv 10(\mathrm{mod}157)$. For this purpose, we suppose that there exist some $e \in Z$ and some $m \in Z$ such that $30e + 12 = 157m + 10$. As a result, we have $e = 5m + (7m - 2)/30$. Since $e \in Z$ there exists some $k \in Z$ such that $m = 30k + 26$. Thus, $e = 157k + 136$, $k \in Z$. Let

$$E = Z \setminus \{157k + 136 : k \in Z\}.$$

Noting that the minimum value of t that simultaneously satisfies the three conditions in (6) is 12, we can take

$$t = t(e) = 30e + 12, \quad \forall e \in E.$$

Then t satisfies the three conditions in (6), $t(0) = 12 = t_0$ and $t(e) \in U$ for any $e \in E$. In this case, we have

$$N(t(e)) = 2\,700e^2 + 2\,220e + 450, \quad \forall e \in E;$$
$$r(t(e)) = 30e + 2,\ s(t(e)) = 420e + 185, \quad \forall e \in E;$$
$$lb(N(t(e))) = 3t(e) - 1 = 90e + 35, \quad \forall e \in E.$$

By the choice of $t(e)$, we have $d_1(N(t(e))) > lb(N(t(e))) = 90e + 35$ for any $e \in E$. Thus $N(t(e))$ is singular and $\{G(N(t(e)); r(t(e)), s(t(e))) : e \in E\}$ is a singularly infinite subfamily of $\{G(N(t); r(t), s(t)) : t \in U\}$ with the initial element $G(450;2,185)$.

Summing up the above statements, we prove the following theorem.

Theorem 1. *Let Z be the infinite set of all nonnegative integers, $U = \{t : t \in Z, t \geqslant 12\} \setminus \{157k + 167 : k \in Z\}$, $E = Z \setminus \{157k + 136 : k \in Z\}$. Then $\{G(3t^2 + 2t - 6; t - 10, 14t + 17) : t \in U\}$ is an infinite family of tight optimal double loop networks with diameter $3t - 1$; whereas $\{G(2\,700e^2 + 2\,220e + 450; 30e + 2, 420e + 185) : e \in E\}$ is its singularly infinite subfamily with diameter $90e + 35$, their initial elements are $G(450; 2,185)$ with diameter 35.*

Remark. The infinite family of singularly tight optimal double loop networks in Theorem 1 corrects a flaw in the assertation by Esqué et al. mentioned at the beginning of this paper.

Next, we deal with the second case by the same strategy as one used in the first case except that an initial element $G(N_0; r_0, s_0)$ is undetermined. We simply illustrate our strategy as follows.

Consider $N(t) = 3t^2 + 4t - 5$. Then $N(t) \in I_2(t)$ and so $lb(t) = 3t$ if $t \geqslant 3$. For $t \geqslant 7$, all tight L-tiles with area $N(t) = 3t^2 + 4t - 5$ are $L_1(t) = L(N(t); 2t + 1, 2t - 1, t - 2, t - 2)$, $L_2(t) = L(N(t); 2t + 4, 2t + 1, t + 3, t + 3)$ and $L_3(t) = L(N(t); 2t + 4, 2t - 1, t + 1, t + 1)$ as well as their transposes.

Consider the L-tile $L_2(t)$. It is easy to check that $L_2(t)$ is $(r(t), s(t))$-implementable for any $t \in Z$ and $t \geqslant 7$. Let $\alpha = -1$, $\beta = 2$ in Lemma 1(b). Then we have $r(t) = 5$ and

$s(t)=3t+5$. It is easy to check that $N(t)$, $r(t)$, $s(t)$ satisfy the condition in (1) if $t \neq 5k+5$, $k \in Z$.

By Lemma 1(a), if t simultaneously satisfies the following three conditions:

$$\begin{aligned} t &\equiv 1(\mathrm{mod}2), \\ t &\equiv 2(\mathrm{mod}3), \\ t &\equiv 2(\mathrm{mod}5), \end{aligned} \tag{7}$$

then none of these six L-tiles is $(1, s(t))$-implementable. Noting that the minimum value of t that satisfies the three conditions in (7) is 17, we can take

$$V = \{t: t \in Z, t \geqslant 17\} \setminus \{5t+20: t \in Z\},$$
$$t = t(e) = 30e+17, \quad \forall e \in Z.$$

Then t simultaneously satisfies the three conditions in (7) and $t(e) \in V$ for any $e \in Z$. Also $t(0)=17=t_0$, $N(17)=930$ and

$$N(t(e)) = 2\,700e^2 + 3\,180e + 930, \quad \forall e \in Z;$$
$$r(t(e)) = 5, \ s(t(e)) = 90e+56, \quad \forall e \in Z;$$
$$lb(N(t(e))) = 3t(e) = 90e+51, \quad \forall e \in Z.$$

By the choice of $t(e)$, we have $d_1(N(t(e))) > lb(N(t(e))) = 90e+51$ for any $e \in Z$. Thus $N(t(e))$ is singular and $\{G(N(t(e)); r(t(e)), s(t(e))): e \in Z\}$ is a singularly infinite subfamily of $\{G(N(t); r(t), s(t)): t \in V\}$. For $e=0$, $G(930;5,56)$ is their initial element with diameter 51. We state this result as the following theorem.

Theorem 2. *Let Z be the infinite set of all nonnegative integers, $V=\{t: t \in Z, t \geqslant 17\} \setminus \{5t+20: t \in Z\}$. Then $\{G(3t^2+4t-5; 5, 3t+5): t \in V\}$ is an infinite family of tight optimal double loop networks with diameter $3t$; whereas $\{G(2\,700e^2+3\,180e+930; 5, 90e+56): e \in Z\}$ is its singularly infinite subfamily with diameter $90e+51$, their initial elements are $G(930;5,56)$ with diameter 51.*

If we consider $N(t)=3t^2+4t-11 \in I_2(t)$, $t \geqslant 6$. Then when $t \geqslant 13$, all tight L-tiles with area $N(t)=3t^2+4t-11$ are $L_1(t)=L(N(t); 2t-2, 2t+1, t-3, t-3)$, $L_2(t)=L(N(t); 2t+1, 2t+5, t+4, t+4)$ and $L_3(t)=L(N(t); 2t-2, 2t+5, t+1, t+1)$ as well as their transposes. The sufficient conditions that none of these six L-tiles is $(1, s(t))$-implementable are

$$\begin{aligned} t &\equiv 2(\mathrm{mod}3), \\ t &\equiv 3(\mathrm{mod}4), \\ t &\equiv 3(\mathrm{mod}7). \end{aligned}$$

Consider the L-tile $L_2(t)$. It is $(3, 3t-2)$-implementable. Then when $t=t(e)=84e+59$, we can obtain the following result. The proof in detail is left to readers.

Theorem 3. *Let Z be the infinite set of all nonnegative integers. Then $\{G(3t^2+4t-11; 3, 3t-2): t \in Z, t \geqslant 59\}$ is an infinite family of tight optimal double loop networks with diameter $3t$; whereas $\{G(21\,168e^2+30\,072e+10\,688; 3, 252e+57): e \in Z\}$ is its singularly infinite subfamily with diameter $252e+59$, their initial elements are $G(10\,688;3,57)$ with diameter 59.*

References

1 Wong, G. K., Coppersmith, D., A combinatorial problem related to multimodule memory organization, *J. Assoc. Comput. Math.*, 1974, 21: 392.

2 Erdös, P., Hsu, D. F., Distributed loop networks with minimum transmission delay, *Theoretical Computer Sci.*, 1992, 100: 223.

3 Esqué, P., Aguiló, F., Fiol, M. A., Double commutative-step digraphs with minimum diameters, *Discrete Math.*, 1993, 114: 147.

4 Hwang, F. K., A survey of double loop networks, in *Reliability of Computer and Communication Networks* (eds. Roberts, F., Hwang, F. K., Monma, C.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science 5, American Methematical Society, Providence, RI, 1991, 143—152.

5 Hwang, F. K., Xu, Y. H., Double loop networks with minimum delay, *Discrete Math.*, 1987, 66: 109.

6 Fiol, M. A., Yebra, J. L., Alegre, I. et al., A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.*, 1987, 36: 702.

7 Li, Q., Xu, J. M., Zhang, Z. L., The infinite families of optimal double loop networks (in Chinese), *Science in China*, Series A, 1993, 23(9): 979.

8 Li, Q., Xu, J. M., Zhang, Z. L., The infinite families of optimal double loop networks, *Discrete Applied Math.*, 1993, 46 (2): 179.

9 Shen, J., Li, Q., Two theorems on double loop networks, *J. China Univ. Sci. Tech.*, 1995, 25(2): 127.

10 Xu Junming, *Graph Theory with Applications* (in Chinese), Hefei: University of Science and Technology of China Press, 1998, 340—342.