第29卷第2期 中国科学技术大学学报 Vol. 29, No. 2
1999年4月 JOURNAL OF CHINA UNIVERSITY OF SCIENCE AND TECHNOLOGY Apr. 1999

A Very Short Proof of Vizing's Theorem*

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Abstract The classical Vizing's edge-colouring theorem states that for a loopless multigraph G of multiplicity μ and of maximum degree Δ, $\Delta + \mu$ colours suffice to colour the edges of G such that adjacent edges have got different colours. A very short proof of the theorem is presented.

Key words multigraphs, edge-colourings, vizing's theorem

AMS Classification(1991) 05C15

For all the terminologies and notations used and not defined here we follow [1]. Let $G = (V(G), E(G), \psi_G)$ be a loopless multigraph of the multiplicity $\mu(G)$, and of the maximum degree $\Delta(G)$ and the edge chromatic number $\chi'(G)$. The classical Vizing's theorem can be stated as follows.

Vizing's theorem[2] If G is loopless, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$.

The lower bound is clear. There are (for example [2], [3] and [4]) proofs for the upper bound. However, every one of them contains so many examinations for several cases that it is not included in any graph theory textbook. A very short proof is presented here.

Proof By contradiction. Suppose that there exist a graph G of the edge chromatic number $\chi'(G) = k > \Delta(G) + \mu(G)$ and an edge $e_0 \in E(G)$ such that $G - e_0$ has a proper $(k-1)$-edge-colouring $\sigma = (E_1, E_2, \cdots, E_{k-1})$.

For $u \in V(G)$, denote by $C_\sigma(u)$ (resp. $C'_\sigma(u)$) the set of the colours appearing (resp. not appearing) at u under σ. Then $C'_\sigma(u) \neq \emptyset$ because $d_{G-e_0}(u) \leq \Delta(G) < k - \mu(G)$. Let $\psi_G(e_0) = xy_0$. A subset $F_x(n, \sigma)$ of the edges incident with x is constructed as follows.

$$F_x(n, \sigma) = \{e_0, e_1, \cdots, e_n\},$$

where $n \geq 1$, and

$$\psi_G(e_i) = xy_i, \ i = 0, 1, 2, \cdots, n,$$

* 收稿日期:1997-10-13

*国家自然科学基金资助项目(19671057)

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$$\sigma(e_i) \in C'_\sigma(y_{i-1}),\ i = 1, 2, \cdots, n.$$

$F_x(n, \sigma)$ is said to be a σ-(e_0, e_n)-fan. A σ-(e_0, e_n)-fan $F_x(n, \sigma)$ is said to be a fundamental fan if $y_0, y_1, y_2, \cdots, y_n$ are distinct. Recolouring a fundamental fan $F_x(n, \sigma)$ implies such a colouring procedure that colours e_{i-1} the colour $\sigma(e_i)$ for $i = 0, 1, 2, \cdots, n$ and makes e_n uncoloured. Notice that recolouring $F_x(n, \sigma)$ gives $G - e_n$ a proper $(k-1)$-edge colouring. Let

$$F_x(\sigma) = \{e \in E_G(x) : \text{there exists a } \sigma\text{-}(e_0, e)\text{-fan}\},$$

where $E_G(x)$ is a set of the edges incident with x in G. Let

$$A(x) = \{y \in N_G(x) : \text{there exists } e \in F_x(\sigma) \text{ with } \psi_G(e) = xy\}.$$

We have the following two claims.

ⅰ $C'_\sigma(x) \cap C'_\sigma(y) = \varnothing$ for $\forall y \in A(x)$.

Suppose that there are a colour α and some vertex $y \in A(x)$ such that $\alpha \in C'_\sigma(x) \cap C'_\sigma(y)$. Then there are an edge $e \in F_x(\sigma)$ such that $\psi_G(e) = xy$ and a fundamental fan $F_x(n, \sigma) = \{e_0, e_1, \cdots, e_n (= e)\}$. A proper $(k-1)$-edge colouring of G can be obtained by recolouring $F_x(n, \sigma)$ and by colouring e_n the colour α, which contradicts the hypothesis that $\chi'(G) = k$.

ⅱ $C'_\sigma(y) \cap C'_\sigma(y') = \varnothing$ for $\forall y, y' \in A(x),\ y' \neq y$.

Suppose that there are a colour β and two vertices y and y' in $A(x)$ such that $\beta \in C'_\sigma(y) \cap C'_\sigma(y')$. Then there are two edges e and e' in $F_x(\sigma)$ such that $\psi_G(e) = xy$ and $\psi_G(e') = xy'$, and there are two fundamental fans $F_x(l, \sigma) = \{e_0, e_1, \cdots, e_l (= e)\}$ and $F_x(t, \sigma) = \{e_0, e'_1, \cdots, e'_t (= e')\}$. Take y and y' such that both l and t are as small as possible. Without loss of generality, suppose that $l \leq t$.

Let $\alpha \in C'_\sigma(x)$. Then $\alpha \neq \beta$, $\beta \in C_\sigma(x)$, $\alpha \in C_\sigma(y) \cap C_\sigma(y')$ by ⅰ. Let $H = G[E_\alpha \cup E_\beta]$. Then $d_H(x) = d_H(y) = d_H(y') = 1$. Let H' be the connected component containing x in H. Then H' is a path and at least one of y and y' is not in H'. A proper $(k-1)$-edge colouring of G can be obtained either by interchanging the colours α and β in the component containing y in H if y is not in H' and recolouring $F_x(l, \sigma)$ and colouring e_l the colour α or by interchanging the colours α and β in the component containing y' in H if y is in H' and recolouring $F_x(t, \sigma)$ and giving e_l the colour α. This contradicts the hypothesis that $\chi'(G) = k$.

Take $F_x(\sigma)$ such that $|F_x(\sigma)|$ is as large as possible. Let $A(x) = \{y_0, y_1, y_2, \cdots, y_n\}$. By ⅱ each colour in $C'(y_i),\ i = 0, 1, \cdots, n$, must be used on an edge from x to $A(x)$. Thus there are $|C'(y_0)| + |C'(y_1)| + \cdots + |C'(y_n)| \geq (n+1)(k-1-\Delta) + 1$ edges from x to $A(x)$. At least $k - \Delta$ of them must go to the same y_i, which is a contradiction, since $k - \Delta > \mu$.

The proof is completed.

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Vizing 定理的简单证明

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摘要 经典的 Vizing 边染色定理断言:对于任何一个重数为 μ 且最大度为 Δ 的重图 G,只须用 $\mu+\Delta$ 种颜色就可以将 G 中的边进行染色,使得相邻边的颜色不同. 该文给出它的一个简单证明.

关键词 重图,边染色,Vizing 定理

中图法分类号 O157.5