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# The Connectivity of Generalized de Bruijn Digraphs*

Xu Junming

( *Dept. of Mathematics, USTC*)

**Abstract** A generalized de Bruijn digraph $G_I(n, d)$ is such a digraph with the vertex-set $\{0, 1, 2, \cdots, n-1\}$ and the arc-set

$$i \to d(n-1-i) + r(\bmod n), \quad \forall 0 \le i \le n-1, 0 \le r \le d-1.$$

This paper presents the following result. If $G_I(n, d)$ has diameter at least five, then $G_I(n, d)$ has connectivity $d$ if and only if g. c. d. $(n, d) \ge 2$ and $d+1$ divides $n$.

**Key words** digraph, de Bruijn digraph, generalized de Bruijn digraphs, connectivity, interconnection network

**AMS Subject Classification (1991)** 05C40

## 1 Introduction

It is well-known that the underlying topology of a computer interconnection network and multiprocessor systems can be modeled by a digraph $G = (V, A)$, where the vertex-set $V$ represents processors and the arc-set $A$ represents one-way communication links. The transmissiondelay and fault-tolerance of the networks can be measured by diameters and connectivity of the corresponding digraphs, respectively.

An important class of digraphs, called de Bruijn digraphs, has been widely studied as models of the networks in literature mainly becaue they have short diameters and simple routing strategies (see [1,2,3,4]). Several variations and extensions of de Bruijn digraphs have been proposed by many authors. One of them proposed by Imase and Itoh[5], is called here a generalized de Bruijn digraph after Du and Hwang[6], denoted by $G_I(n, d)$, where $n$, $d$ are given integers, $1 < d < n-1$. It consists of the vertices $0, 1, 2, \cdots, n-1$ and the arcs

$$i \to d(n-1-i) + r(\bmod n), \quad \forall 0 \le i \le n-1, 0 \le r \le d-1.$$

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Xu Junming: male, born in October 1949, Associate Professor. Postcode: 230026, Hefei

Clearly $G_I(n,d)$ is a $d$-regular (strongly) connected digraph and maybe has self-loop arcs. What we are interested in is the connectivity of $G_I(n,d)$. Even though much attention has been paied by many authors and a number of results have been obtained, the connectivity of $G_I(n,d)$ has not been completely determined yet. In this paper, the following result is presented.

**Theorem** If $G_I(n,d)$ has diameter at least five, then $G_I(n,d)$ has connectivity $d$ if and only if g.c.d. $(n,d) \geq 2$ and $d+1$ divides $n$.

The proof of the theorem is in Section 3. In the next section, some lemmas used in the proof are given.

## 2 Several Lemmas

**Lemma 1** (Theorem 1 in [5]) $G_I(n,d)$ has diameter at most $\lceil \log_d n \rceil$, where $\lceil m \rceil$ is the minimum integer not smaller than $m$.

**Lemma 2** (Theorem 3 in [7] and Theorem 3.3 in [6]) If $G_I(n,d)$ has diameter at least four, then its connectivity at least $d-1$. If g.c.d. $(n,d)=1$, then $G_I(n,d)$ has connectivity exact $d-1$.

**Lemma 3** (Lemma 1 in [7]) For any vertex $x$ of $G_I(n,d)$, if $t$ is an integer less than the diameter of $G_I(n,d)$, then

$$|J_t^+(x)| = |J_t^-(x)| = d^t,$$

where $J_t^+(x)$ (resp. $J_t^-(x)$) denotes the set of vertices of $G_I(n,d)$ to which there exists a walk of length $t$ from (resp. to) $x$.

**Lemma 4** $G_I(n,d)$ is simple if and only if $d+1$ divides $n$.

**Proof** Clearly the arc of $G_I(n,d)$

$$i \to d(n-1-i)+r \pmod n, \quad \forall 0 \leq i \leq n-1, 0 \leq r \leq d-1.$$

is a self-loop arc if and only if $i$ satisfies the congruence relations

$$(d+1)i \equiv r-d \pmod n,$$

or equivalently, there exists some integer $p$ such that the equation in the unknown $i$

$$(d+1)i = pn + r - d \tag{1}$$

has an integral solution.

If $d+1$ divides $n$ then the equation (1) has no integral solution since $\frac{pn}{d+1}$ is an integer while $\frac{r-d}{d+1}$ is not for any $r(0 \leq r \leq d-1)$. And so $G_I(n,d)$ has no self-loop arcs, whereby $G_I(n,d)$ is simple.

Conversely, let $G_I(n,d)$ be simple, i.e. $G_I(n,d)$ has no self-loop arcs. Suppose to the contrary that $d+1$ does not divide $n$. Then there exist integers $x$ and $y$ such that $n = x(d+1)+y$, $0 < y \leq d$. Let $r = d-y$. Then $0 \leq r \leq d-1$ and $y+r-d=0$. Thus, for $p=1$ and $r=$

$d - y$, the equation (1) has an integral solution $i = x + \frac{y + r - d}{d + 1} = x$. This implies that $G_I(n, d)$ has a self-loop arc at the vertex $i$, a contradiction.

## 3 Proof of the Theorem

($\Rightarrow$) Suppose that the connectivity of $G_I(n, d)$ is equal to $d$. Then $G_I(n, d)$ has vertices at least $d + 1$. If $G_I(n, d)$ has self-loop arcs, then there is a vertex $x$ such that $|J_1^+(x)| = d - 1$ since $G_I(n, d)$ is $d$-regular. And so $J_1^+(x)$ is a vertex cut. Thus the connectivity of $G_I(n, d)$ is at most $d - 1$, a contradiction. It follows that $G_I(n, d)$ has no self-loop arcs and so $G_I(n, d)$ is simple. From Lemma 4, $d + 1$ divides $n$. If g.c.d. $(n, d) = 1$, then the connectivity of $G_I(n, d)$ is equal to $d - 1$ by Lemma 2, which contradicts our assumption. Thus g.c.d. $(n, d) \geq 2$.

($\Leftarrow$) Suppose that $G_I(n, d)$ has diameter at least five and g.c.d. $(n, d) \geq 2$. Our aim is to show that $G_I(n, d)$ has connectivity $d$. Suppose to the contrary that $G_I(n, d)$ has connectivity at most $d - 1$. We will derive contradictions.

From Lemma 2 $G_I(n, d)$ has connectivity $d - 1$. There is a vertex cut $Z$ of $G_I(n, d)$ such that $|Z| = d - 1$. Then $V \setminus Z$ can be partitioned into two disjoint nonempty sets $X$ and $Y$ such that there is no arc from $X$ to $Y$ in $G_I(n, d)$. Let

$$p = \max\{d(x, Z) : x \in X\}, \quad q = \max\{d(Z, y) : y \in Y\},$$

where $d(x, Z)$ denotes the minimum distance from $x$ to any vertex in $Z$, and $d(Z, y)$ denotes the minimum distance from any vertex in $Z$ to $y$. It is easy to observe that $0 < p < k$ and $0 < q < k$ since $X \cap Z = \emptyset$ and $Y \cap Z = \emptyset$.

Let the diameter of $G_I(n, d)$ be $k$. By our assumption $k \geq 5$. Let $x$ be a vertex in $X$ such that $d(x, Z) = p$ and $y$ be a vertex in $Y$ such that $d(Z, y) = q$. Since any directed path from $x$ to $y$ certainly goes through $Z$, there exists a vertex $z$ in $Z$ such that $d(x, z) + d(z, y) = d(x, y) \leq k$. By our choice of $x$, $d(x, z) \geq p$, and so

$$q = d(Z, y) \leq d(z, y) \leq k - d(x, z) \leq k - p \tag{2}$$

Let $X_i = \{x \in X : d(x, Z) = i\}$, $1 \leq i \leq p$. Then $|X_1| \leq |Z| d = (d - 1) d$, $|X_i| \leq |X_{i-1}| d \leq (d - 1) d^i$ $(i = 2, 3, \cdots, p)$ since $G_I(n, d)$ is $d$-regular. Notice that $X \subseteq X_1 \cup X_2 \cup \cdots \cup X_p$. It follows that

$$|X| \leq \sum_{i=1}^{p} |X_i| \leq (d - 1) \sum_{i=1}^{p} d^i = d^{p+1} - d. \tag{3}$$

Similarly, let $Y_j = \{y \in Y : d(Z, y) = j\}$, $1 \leq j \leq q \leq k - p$. Then

$$|Y| \leq d^{k-p+1} - d. \tag{4}$$

It follows from (3) and (4) that

$$n = |X| + |Z| + |Y| \leq d^{p+1} + d^{k-p+1} - d - 1. \tag{5}$$

Since the function $f(p) = d^{p+1} + d^{k-p+1}$ is concave upward on the interval $[3, k - 3]$,

$$f(p) \le f(3) = f(k-3) = d^4 + d^{k-2}. \tag{6}$$

If $3 \le p \le k-3$, then $k \ge 6$. It follows from (5) and (6) that

$$n \le f(p) - d - 1 \le d^{k-2} + d^4 - d - 1 < d^{k-1},$$

which contradicts the fact that $n \ge d^{k-1}$ by Lemma 1.

Hence in the rest of the proof, we need only consider the case of $p = 1$ or $p = k-1$ and the case of $p = 2$ or $p = k-2$. Notice from (2) that $q \le 1$ if $p = k-1$ and $q \le 2$ if $p = k-2$. By the symmetric relation of $p$ and $q$ in the problem, we need only consider the cases of $p = 1$ and $p = 2$.

Let $x \in X$. Observe from Lemma 3 that the following fact is usefull.

$$\left| J_t^+(x) \cap X \right| \ge d \left| J_{t-1}^+(x) \cap X \right| - |Z|, \quad \forall 2 \le t < k. \tag{7}$$

**Case 1** $p = 2$.

In this case, $X = X_1 \cup X_2$. Let $x \in X_2$. If $k \ge 5$, then by (7) we have that $\left| J_1^+(x) \cap X \right| = d$, $\left| J_2^+(x) \cap X \right| \ge d^2 - d + 1$, $\left| J_3^+(x) \cap X \right| \ge d^3 - d^2 + 1$ and

$$\left| J_4^+(x) \cap X \right| \ge d^4 - d^3 + 1. \tag{8}$$

It follows from (3) and (8) that

$$d^4 - d^3 + 1 \le \left| J_4^+(x) \cap X \right| \le |X| \le d^3 - d.$$

This is impossible.

**Case 2** $p = 1$.

In this case, we use our assumption of g.c.d. $(n, d) = m \ge 2$. Let $n = sm$ and $d = lm$. Then the vertex-set $V$ of $G_I(n, d)$ can be partitioned into $s$ disjoint subsets $V_0, V_1, \cdots, V_{s-1}$, where

$$V_i = \{im, im+1, \cdots, im + (m-1)\}, \quad i = 0, 1, \cdots, s-1, \ m \ge 2.$$

By the definition of $G_I(n, d)$, for any vertex $x$ of $G_I(n, d)$, the out-neighbors of $x$ are

$$\begin{aligned} & d(n - 1 - x) + r (\mathrm{mod}\, n), \ \forall 0 \le r \le d-1 \\ & = ml(sm - 1 - x) + tm + r (\mathrm{mod}\, s), \ \forall 0 \le r \le m-1, 0 \le t \le l-1 \\ & = (l(sm - 1 - x) + t) (\mathrm{mod}\, s) m + r, \ \forall 0 \le r \le m-1, 0 \le t \le l-1. \end{aligned}$$

This implies that $J_1^+(x)$ is the union of $l$ successive elements $V_{i+1}, V_{i+2}, \cdots, V_{i+l}$ of $\{V_0, V_1, V_2, \cdots, V_{s-1}\}$, where the subscripts are modulo $s$ and $|V_i| \ge 2$ since $m \ge 2$. Suppose that there are two vertices $x$ and $y$ of $V_i$ such that $x$ is adjacent to $y$ in $G_I(n, d)$, then from the above statement $V_i \subset J_1^+(x)$ since $y \in V_i$. Thus there is a self-loop arc at $x$ since $x \in V_i$, which contradicts our assumption of $G_I(n, d)$ being simple. It follows that any two vertices of $V_i$ are not adjacent in $G_I(n, d)$.

Since $d = lm$ and $|Z| = d - 1 = lm - 1$, there exists a subset $V_i$ such that $V_i \cap X \ne \emptyset$ and $V_i \cap Z \ne \emptyset$. Let $x \in V_i \cap X$ and $z \in V_i \cap Z$. Then $x$ and $z$ are not adjacent in $G_I(n, d)$. Thus there are at least two vertices in $J_1^+(x)$ not in $Z$. In other words, $|J_1^+(x) \cap X| \ge 2$. Thus if $k \ge 4$, then by (7), we have that $|J_2^+(X) \cap X| \ge 2d - (d-1) = d+1$ and

$$|J_3^+(X) \cap X| \ge d(d+1) - (d-1) = d^2 + 1. \tag{9}$$

It follows from (3) and (9) that

$$d^2 + 1 \leq | J_3^+(x) \cap X | \leq | X | \leq d^2 - d.$$

This is impossiple. The proof of the theorem is completed.

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# 广义 de Bruijn 有向图的连通度


## 徐 俊 明

(中国科学技术大学数学系)



**摘要** **广义** de Bruijn **有向图** $G_I(n, d)$ **的顶点集为** $\{0, 1, ..., n-1\}$, **弧集为**

$$i \to d(n-1-i) + r (\bmod n), \quad \forall 0 \leq i \leq n-1, 0 \leq r \leq d-1.$$

**本文证明**: **如果** $G_I(n, d)$ **的直径不小于** 5, **那么它的连通度等于** $d$ **当且仅当** g. c. d. $(n, d) \geq 2$, **而且** $n$ **能被** $d+1$ **整除**.



**关键词** **有向图**, de Bruijn **有向图**, **广义** de Bruijn **有向图**, **连通度**, **互连网络**

**中图法分类号** O157.5, O157.9