ON CONDITIONAL EDGE-CONNECTIVITY OF GRAPHS*

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Abstract

Let k and h be two integers, $0 \le h < k$. Let G be a connected graph with minimum degree at least k. The conditional h-edge-connectivity of G, denoted by $\lambda^{(h)}(G)$, is defined as the minimum cardinality $|S|$ of a set S of edges in G such that $G-S$ is disconnected and is of minimum degree at least h. This type of edge-connectivity is a generalization of the traditional edge-connectivity and can more accurately measure the fault-tolerance of networks. In this paper, we will first show that $\lambda^{(2)}(G) \le g(k-2)$ for a $k(\ge 3)$-regular graph G provided G is neither K_4 and K_5 nor $K_{3,3}$, where g is the length of a shortest cycle of G, then show that $\lambda^{(h)}(Q_k)=(k-h)2^h$ for a k-dimensional cube Q_k.

Key words. Regular Graphs, connectivity, conditional connectivity, hypercubes

1. Introduction

In this paper, a graph $G = (V, E)$ always means a simple graph (without loops and multiple edges) with the vertex-set V and the edge-set E. We follow [1] for graph-theoretical terminology and notation not defined here.

It is well known that when the underlying topology of a computer interconnection network is modeled by a graph G, the edge-connectivity $\lambda(G)$ of G is an important measure for fault-tolerance of the network. However, it has many deficiencies (see [2]). Motivated by the shortcomings of the edge-connectivity measure, Harary[3] introduced the concept of conditional edge-connectivity by requiring some property for every connected component of $G - S$ for a minimum edge-cut S of G. In this paper we will specify a property that every connected component has minimum degree at least h. In fact, requiring this property for every connected component is particularly important for applications where parallel algorithms can run on subnetworks with a given topology[4].

Let G be a connected graph with minimum degree at least k, h be an integer, $0 \le h < k$. A set S of edges in G is called a C_h-cut if $G - S$ is disconnected and is of the minimum degree at least h. The conditional h-edge-connectivity of G, C_h-edge-connectivity briefly, denoted by $\lambda^{(h)}(G)$, is defined as the minimum cardinality $|S|$ of a C_h-cut S of G. A C_h-cut S of G is called a $\lambda^{(h)}$-cut if $|S| = \lambda^{(h)}(G) > 0$.

Observe that when $h = 0$, no conditions or restrictions are imposed on the connected components and we have the traditional edge-connectivity. In addition, in the special

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case of $h = 1$, this connectivity will be reduced to the restricted edge-connectivity introduced by Esfahanian and Hakimi in [2,5]. This connectivity for k-regular graph will be the R_h-edge-connectivity proposed by Latifi, Hegde and Naraghi-Pour in [4]. Thus the C_h-edge-connectivity can be regarded as a generalization of the above three types of edge-connectivities and can provide a more accurate fault-tolerance measure of networks and has received much attention recently. Esfahanian and Hakimi in [5] presented a polynomial-time algorithm for the computation of $\lambda^{(1)}$. However, one has not known yet if the problem of computing $\lambda^{(h)}$ is NP-hard as there is no known polynomial-time algorithm to find $\lambda^{(h)}(G)$ for a given graph G with $h \geq 2$. For the sake of convenience, we write λ' for $\lambda^{(1)}$ and write λ'' for $\lambda^{(2)}$. The λ' and λ'' for some special classes of graphs have been determined by several authors[2,4,6–11].

Use K_n to denote a complete graph with n vertices, and $K_{m,n}$ to denote a complete bipartite graph with $m+n$ vertices. Observe that some connected graphs do not have $\lambda^{(h)}$ for some $h \geq 1$. For instance, K_3 and $K_{1,n}$ do not have λ', and K_4, K_5 and $K_{3,3}$ do not have λ''. Thus for a given connected graph G with $h(\geq 1)$, the existence of $\lambda^{(h)}(G)$ is an important problem. The existence of λ' has been solved by Esfahanian and Hakimi[5]. However, the existence of $\lambda^{(h)}$ has not been solved yet for $h \geq 2$ even if G is regular.

In this paper, we will restrict ourselvies to k-regular graphs and first show that $\lambda''(G)$ certainly exists and $\lambda''(G) \leq g(k-2)$ for any connected k-regular graph G as long as G is neither K_4, K_5 nor $K_{3,3}$, where $g = g(G)$ is the length of a shortest cycle of G, then show that $\lambda^{(h)}(Q_k) = (k-h)2^h$ for any k-dimensional cube Q_k, and any h, $0 \leq h < k$.

Next, we give some notation to be used in this paper later. A vertex x of G is called singular if its degree is less than h for a given h. Let X and Y be two disjoint nonempty proper subsets of $V(G)$ and let $(X,Y) = \{e \in E(G)$: there are $x \in X$ and $y \in Y$ such that $e = xy \in E(G)\}$. If $Y = \overline{X} = V(G) \setminus X$, then we write $E(X)$ for $(X, \overline{X})$ and $d(X)$ for $|E(X)|$. X is called a $\lambda^{(h)}$-fragment of G if $E(X)$ is a $\lambda^{(h)}$-cut of G. It is clear that if X is a $\lambda^{(h)}$-fragment of G, then $G[X]$ and $G[\overline{X}]$ are connected. A $\lambda^{(h)}$-fragment X of G is called a $\lambda^{(h)}$-atom if X has a minimum cardinality.

The rest of this paper is organized as follows. In the next section we will show the existence of $\lambda''(G)$ for a regular graph G. In Section 3 we will determine the value of $\lambda^{(h)}$ for a k-dimensional cube. Some remarks will be given in Section 4.

2. The Existence of λ'' for Regular Graphs

In this section, we use g to denote the length of a shortest cycle in G.

Theorem 1. Let G be a connected $k(\geq 3)$-regular graph but neither K_4 and K_5 nor $K_{3,3}$. Let X be the vertex-set of a shortest cycle in G. If $E(X)$ is not a C_2-cut of G, then $\lambda''(G) < g(k-2)$ with $3 \leq g \leq 4$.

Proof. Suppose G is a connected k-regular graph but neither K_4 and K_5 nor $K_{3,3}$ and X is the vertex-set of a shortest cycle of G, $k \geq 3$ and $g = g(G) \geq 3$. Then $\overline{X} \neq \emptyset$ and $E(X)$ is an edge-cut of G since $k \geq 3$ and every vertex of X has only two neighbors in X. If $E(X)$ is not a C_2-cut of G, then all singular vertices in $G - E(X)$ are certainly contained in $\overline{X}$. Let y be a singular vertex with minimum degree in $G - E(X)$. Noting the minimality of X, it is clear that $|N_G(y) \cap X| \leq 2$ if $g = 4$ and $|N_G(y) \cap X| \leq 1$ if $g \geq 5$. If y is an isolated vertex in $G - E(X)$, then $3 \leq k = d_G(y) = |N_G(y) \cap X| \leq 3$ whereby $k = 3$, $g = 3$ and $\overline{X} = \{y\}$. And so $G = K_4$, which contradicts our assumption. It follows that y is an one-degree vertex in $G - E(X)$ whereby $3 \leq g \leq 4$. Let $Y = X \cup \{y\}$. Then $\overline{Y} \neq \emptyset$ since y is a one-degree vertex in $G[\overline{X}]$. Let $\zeta(G) = g(k-2)$.

Case 1. $g = 3$. In this case $k = 3$ or 4.

If $k=3$, then $\zeta(G)=3$ and $d(Y)=2$. If there is no singular vertex in $G-E(Y)$, then $E(Y)$ is a C_2-cut of G and $\lambda''(G)\le d(Y)=2<3=\zeta(G)$. If there is some singular vertex z in $G-E(Y)$, then z is only such a vertex and is of degree one in $G-E(Y)$. Thus z is adjacent to y. Let $Z=Y\cup\{z\}$. Then $\overline{Z}\ne\emptyset$, $d(Z)=1$, and $G-E(Z)$ contains no singular vertex. It follows that $E(Z)$ is a C_2-cut of G and so $\lambda''(G)\le d(Z)=1<3=\zeta(G)$.

If $k=4$, then $\zeta(G)=6$ and $d(Y)=4$. If there is no singular vertex in $G-E(Y)$, then $E(Y)$ is a C_2-cut of G, and $\lambda''(G)\le d(Y)=4<6=\zeta(G)$. Suppose that there is some singular vertex z in $G-E(Y)$. If z is an isolated vertex in $G-E(Y)$, then $G=K_5$, which contradicts our assumption. Thus z is a one-degree vertex in $G-E(Y)$. Let $Z=Y\cup\{z\}$. Then $\overline{Z}\ne\emptyset$, $d(Z)=2$ and $G-E(Z)$ contains no singular vertex. It follows that $E(Z)$ is a C_2-cut of G. So $\lambda''(G)\le d(Z)=2<6=\zeta(G)$.

Case 2. $g\ge 4$. In this case $|N_G(y)\cap X|=2$, $k=3$, $g=4$ and $\zeta(G)=4$.

If there is no sigular vertex in $G-E(Y)$, then $E(Y)$ is a C_2-cut of G, and $\lambda''(G)\le d(Y)=3<4=\zeta(G)$. Suppose that there is some singular vertex z in $G-E(Y)$. If z is a isolated vertex in $G-E(Y)$, then z is certainly adjacent to y and so $G=K_{3,3}$, which contradicts our assumption. It follows that z is an one-degree vertex in $G-E(Y)$. Let $Z=Y\cup\{z\}$.

If z is a one-degree vertex in $G-E(X)$, then z is not adjacent to y and so $\overline{Z}\ne\emptyset$, $d(Z)=2$ and $G-E(Z)$ contains no singular vertex. It follows that $E(Z)$ is a C_2-cut of G, and $\lambda''(G)\le d(Z)=2<4=\zeta(G)$.

If z is a two-degree vertex in $G-E(X)$, then z is certainly adjacent to y and so $\overline{Z}\ne\emptyset$ and $d(Z)=2$. If $G-E(Z)$ contains no singular vertex, then $E(Z)$ is a C_2-cut of G, and so $\lambda''(G)\le d(Z)=2<4=\zeta(G)$. If $G-E(Z)$ contains some singular vertex u, then u is only such a vertex. u is adjacent to z and has one neighbor in $G-E(Z)$. Let $U=Z\cup\{u\}$. Then $\overline{U}\ne\emptyset$, $d(U)=1$ and $G-E(U)$ contains no singular vertex. It follows that $E(U)$ is a C_2-cut of G and $\lambda''(G)\le d(U)=1<4=\zeta(G)$.

The proof of Theorem 1 is completed.

Theorem 2. Let G be a connected $k(\ge 3)$-regular graph. If G is neither K_4 and K_5 nor $K_{3,3}$, then $\lambda''(G)$ exists and $\lambda''(G)\le g(k-2)$.

Proof. We want only to show $\lambda''(G)\le g(k-2)$. To this purpose, let X be the vertex-set of a shortest cycle of G. Then $\overline{X}\ne\emptyset$ and $E(X)$ is an edge-cut of G since $k\ge 3$ and each vertex of X has only two neighbors in X. If $E(X)$ is a C_2-cut of G, then $\lambda''(G)\le d(X)=g(k-2)$. If $E(X)$ is not a C_2-cut of G, then $\lambda''(G)<g(k-2)$ by Theorem 1.

Theorem 3. Let G be a connected $k(\ge 3)$-regular graph but neither K_4 and K_5 nor $K_{3,3}$. Then $\lambda''(G)=g(k-2)$ if and only if $G[X]$ is a shortest cycle of G for any λ''-atom X of G.

Proof. Clearly $\lambda''(G)$ exists by Theorem 2 and $d(X)=\lambda''(G)$ and $\delta(G[X])\ge 2$ for any λ''-atom X of G since $G[X]$ does not contain any singular vertex in $G-E(X)$. Thus $G[X]$ certainly contains a cycle C.

Suppose $G[X]=C$ is a shortest cycle of G. Then clearly $\lambda''(G)=d(X)=g(k-2)$.

Conversely suppose that $\lambda''(G)=g(k-2)$ and C is a proper subgraph of $G[X]$. Let X' be the vertex-set of C. Then $E(X')$ is not a C_2-cut of G since X is a λ''-atom of G. $\lambda''(G)<g(k-2)$ by Theorem 1, which contradicts our assumption. Thus $G[X]$ is a shortest cycle of G.

3. C_h-Edge-Connectivity of k-Dimensional Cubes

The k-dimensional cube, denoted by Q_k, is widely used in the desingn and analysis of computer interconnection networks and is defined and characterized in a number of ways[5]. By a common definition, Q_k is such a graph with the vertex-set

$$V(Q_k) = \{(x_1x_2\cdots x_k) : x_i \in \{0,1\}, 1 \le i \le k\},$$

where one vertex $x = (x_1x_2\cdots x_k)$ and another vertex $y = (y_1y_2\cdots y_k)$ are linked if and only if they differ in exactly one coordinate. By the operation of the Cartesian products of graphs, an equivalent definition of Q_k can be expressed as the Cartesian product $Q_k = K_2 \times K_2 \times \cdots \times K_2$ of k K_2 (see [12]). Using this definition, we can express Q_k as $K_2 \times Q_{k-1}$, denoted by (see [5])

$$Q_1 = K_2, \qquad Q_k = Q_{k-1} \odot Q_{k-1}, \qquad k \ge 2.$$

It is well known that Q_k is a k-regular and k-connected bipartite graph with 2^k vertices.

Lemma 4. Let X be the vertex-set of any subgraph of Q_k isomorphic to Q_h. Then $|X| = 2^h$, $E(X)$ is a C_h-cut of Q_k and $\lambda^{(h)}(Q_k) \le (k-h)2^h$ for any k and h, $0 \le h < k$.

Proof. It is clear that $|X| = 2^h$ and $Q_k - E(X)$ is disconnected. We want to prove that $E(X)$ is a C_h-cut of Q_k. To this purpose we want only to verify that $Q_k - E(X)$ contains no singular vertex. Since $Q_k[X]$ is isomorphic to Q_h, every vertex of X has h neighbors in X and so X contains no singular vertex of $Q_k - E(X)$.

Let $\overline{X} = V(Q_k) \setminus X$. Then $\overline{X} \ne \emptyset$ for $h < k$. Since $Q_k[X] \cong Q_h$, all vertices of X have the same values of $k-h$ coordinates. We can, without loss of generality, suppose that all the first $k-h$ coordinates are 0. Let $x = (0\cdots 0x_{k-h+1}\cdots x_k)$ and $y = (0\cdots 0y_{k-h+1}\cdots y_k)$ be any two vertices of X. If x and y have a common neighbor z in $\overline{X}$, then there is certainly 1 in the first $k-h$ coordinates of z. Without loss of generality, let $z = (0\cdots 01z_{k-h+1}\cdots z_k)$. Then from $xz \in E(Q_k)$ we have that $z_i = x_i, i = k-h+1, \cdots, k$ and from $yz \in E(Q_k)$ we have that $z_i = y_i, i = k-h+1, \cdots, k$, which imply $x = y$. Therefore any two vertices of X have no common neighbor in $\overline{X}$. Every vertex of $\overline{X}$ has at least $k-1(\ge h)$ neighbors in $\overline{X}$ whereby $\overline{X}$ contains no singular vertex of $Q_k - E(X)$. Therefore, $E(X)$ is a C_h-cut of Q_k and $\lambda^{(h)}(Q_k) \le |E(X)| = (k-h)2^h$.

Lemma 5. Let X be a proper subset of $V(Q_k)$. If $|X| \ge 2$, $Q_k[X]$ is connected and $E(X)$ is a C_h-cut of Q_k, then there are two subgraphs Q^L_{k-1} and Q^R_{k-1} of Q_k, of which each is isomorphic to Q_{k-1}, such that $Q_k = Q^L_{k-1} \odot Q^R_{k-1}$ and $X_L = X \cap V(Q^L_{k-1}) \ne \emptyset$ and $X_R = X \cap V(Q^R_{k-1}) \ne \emptyset$. Furthermore, $E(X_L)$ and $E(X_R)$ are C_{h-1}-cuts of Q^L_{k-1} and Q^R_{k-1}, respectively.

Proof. Since $|X| \ge 2$ and $Q_k[X]$ is connected, there are two vertices x and y of X such that $xy \in E(Q_k)$. By the common definition of Q_k, just one coordinate of x and y is different. Without loss of generality, suppose the first coordinate of x is 0 and the first coordinate of y is 1. Let Q^L_{k-1} be the subgraph of Q_k induced by the set of vertices whose first coordinates are 0 and let Q^R_{k-1} be the subgraph of Q_k induced by the set of vertices whose first coordinates are 1. Then Q^L_{k-1} and Q^R_{k-1} are isomorphic to Q_{k-1} and there is an edge between the two vertices whose coordinates are of the same value except for the first in Q_k. It follows that $Q_k = Q^L_{k-1} \odot Q^R_{k-1}$.

Let $\overline{X}_L = \overline{X} \cap V(Q^L_{k-1})$. Since X is a C_h-cut of Q_k, every vertex of X_L has at least h neighbors in X and every vertex of $\overline{X}_L$ has at least h neighbors in $\overline{X}$. Note that every vertex of Q^L_{k-1} has just one neighbor in Q^R_{k-1}, every vertex of X_L and $\overline{X}_L$ has at least $h-1$ neighbors in $Q^L_{k-1} - E(X_L)$. This implies $E(X_L)$ is a C_{h-1}-cut of Q^L_{k-1}. We can, similarly, show that $E(X_R)$ is a C_{h-1}-cut of Q^R_{k-1}.

Theorem 6. $\lambda^{(h)}(Q_k) = (k-h)2^h$ for any k and h, $0 \le h < k$.

Proof. By Lemma 4 we want only to show that $\lambda^{(h)}(Q_k) \ge (k-h)2^h$. By induction on $k \ge 1$, clearly the conclusion holds for $k=1, h=0$ and $k=2, h=0,1$. Suppose the conclusion is true for all $n<k$, and $h<n$. Let X be a $\lambda^{(h)}$-atom of Q_k. Then $d(X)=\lambda^{(h)}(Q_k)$, $E(X)$ is a $\lambda^{(h)}$-cut of Q_k and so $E(X)$ is a C_h-cut of Q_k. If $|X|=1$, then $h=0$ and so $d(X)=k$; the conclusion holds. Suppose $|X|\ge 2$. Then $Q_k[X]$ is connected. So by Lemma 5 Q_k can be expressed as $Q_k=Q_{k-1}^L \odot Q_{k-1}^R$, $E(X_L)$ and $E(X_R)$ are C_{h-1}-cuts of Q_{k-1}^L and Q_{k-1}^R, respectively, where $X_L = X\cap V(Q_{k-1}^L)$ and $X_R = X\cap V(Q_{k-1}^R)$. Let $\overline{X}_L=\overline{X}\cap V(Q_{k-1}^L)$ and $\overline{X}_R=\overline{X}\cap V(Q_{k-1}^R)$. By the induction hypothesis we have

$$\big|(X_L,\overline{X}_L)\big| \ge (k-h)2^{h-1}, \qquad \big|(X_R,\overline{X}_R)\big| \ge (k-h)2^{h-1}. \tag{1}$$

It follows from (1) that $\lambda^{(h)}(Q_k)=d(X)\ge \big|(X_L,\overline{X}_L)\big| + \big|(X_R,\overline{X}_R)\big| \ge (k-h)2^h$.

Corollary 7[2]**.** $\lambda(Q_k)=k$ and $\lambda'(Q_k)=2(k-1)$.

Theorem 8. Let X be a $\lambda^{(h)}$-fragment of Q_k. Then $|X|\ge 2^h$ and the equality holds if and only if $Q_k[X]\cong Q_h$ for any k and h, $0\le h<k$.

Proof. If $Q_k[X]\cong Q_h$, then the theorem holds obviously. We want only to prove that $|X|\ge 2^h$ and $Q_k[X]\cong Q_h$ if $|X|=2^h$. By induction on $k\ge 1$, clearly the conclusion holds for $k=1$, $h=0$ and $k=2$, $h=0,1$. Suppose the conclusion is true for all $n<k$, and $h<n$.

Since X is a $\lambda^{(h)}$-fragment of Q_k, $Q_k[X]$ is connected. If $|X|$=1, then $h=0$ and the conclusion holds obviously. Suppose that $|X|\ge 2$. Then by Lemma 5 Q_k can be expressed as $Q_k=Q_{k-1}^L\odot Q_{k-1}^R$, $E(X_L)$ and $E(X_R)$ are C_{h-1}-cuts of Q_{k-1}^L and Q_{k-1}^R, respectively, where $X_L=X\cap V(Q_{k-1}^L)$ and $X_R=X\cap V(Q_{k-1}^R)$. Thus (1) still holds by Theorem 6. It follows from Theorem 6 and (1) that

$$\big|(X_L,\overline{X}_L)\big| = (k-h)2^{h-1}, \qquad \big|(X_R,\overline{X}_R)\big| = (k-h)2^{h-1}. \tag{2}$$

This implies by Theorem 6 that X_L and X_R are $\lambda^{(h-1)}$-fragments of Q_{k-1}^L and Q_{k-1}^R, respectively. By the induction hypothesis we have

$$\begin{cases} |X_L|\ge 2^{h-1}, \text{ and } |X_L|=2^{h-1} \Longleftrightarrow Q_{k-1}^L[X_L]\cong Q_{h-1}, \\ |X_R|\ge 2^{h-1}, \text{ and } |X_R|=2^{h-1} \Longleftrightarrow Q_{k-1}^R[X_R]\cong Q_{h-1}. \end{cases} \tag{3}$$

It follows from (3) that $|X|=|X_L|+|X_R|\ge 2^h$.

If $|X|=2^h$, then by (3) we have $|X_L|=|X_R|=2^{h-1}$ and $Q_{k-1}^L[X_L]\cong Q_{h-1}\cong Q_{k-1}^R[X_R]$. From Theorem 6 and (2) we have $(X_L,\overline{X}_R)\ne\emptyset$ and $(X_R,\overline{X}_L)\ne\emptyset$. So $Q_k[X]=Q_k[X_L]\odot Q_k[X_R]\cong Q_{h-1}\odot Q_{h-1}=Q_h$.

Theorem 9. A subset X is a $\lambda^{(h)}$-atom of Q_k if and only if $Q_k[X]\cong Q_h$ for any k and h, $0\le h<k$.

Proof. $|X|\le 2^h$ by Lemma 4 and $|X|\ge 2^h$ by Theorem 8. Thus $|X|=2^h$ if and only if $Q_k[X]\cong Q_h$ by Theorem 8.

4. Some Remarks on the Existence of $\lambda^{(h)}$

Let G be a connected graph. For $e=xy\in E(G)$, $\xi_G(e)=d_G(x)+d_G(y)-2$ is defined as the degree of e. $\xi(G)=\min\{\xi_G(e): e\in E(G)\}$ is defined as the minimum edge-degree of G. Then $\xi(G)\le\Delta(G)+\delta(G)-2$, where $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degree of G, respectively. In particular, if G is k-regular, then $\xi(G)=2(k-1)$.

If $\delta(G)\ge 2$, then G certainly contains a cycle. We use $g=g(G)$ to denote the length of a shortest cycle in G. Let X be the vertex-set of a shortest cycle in G. Define $\zeta(G)=\sum\limits_{x\in X} d_G(x)-2g(G)$. In particular, if G is k-regular, then $\zeta(G)=g(k-2)$.

If $\delta(G) \geq k$ and for any h $(0 \leq h < k)$ there is an h-regular subgraph of G, then we call an h-regular subgraph H of G normal if the vertex-set of H is of minimum cardinality $a_h(G)$. Define $\tau(G) = \sum_{x \in V(H)} d_G(x) - ha_h$. In particular, if G is k-regular, then $\tau(G) = a_h(k-h)$.

Observe the three special cases of $h = 0, 1, 2$. Any normal 0-regular subgraph of G is a single vertex and so $a_0(G) = 1$, $\tau(G) = \delta(G)$. We have that $\lambda(G) \leq \delta(G) = \tau(G)$ for any connected graph G by Theorem 3.1 in [1]. Any normal 1-regular subgraph is a single edge and so $a_1(G) = 2$ and $\tau(G) = \xi(G)$. We have that $\lambda'(G) \leq \xi(G) = \tau(G)$ for every connected graph G except for K_3 and $K_{1,n}$ by Lemma 2.1 in [5]. Any normal 2-regular subgraph of G is a shortest cycle and so $a_2(G) = g(G)$ and $\tau(G) = \zeta(G)$. By Theorem 2 in the present paper we have that $\lambda''(G) \leq \zeta(G) = \tau(G)$ for any connected k-regular graph G provided G is neither K_4 and K_5 nor $K_{3,3}$. Motivated by these observations, we can propose the conjecture as follows.

Conjecture. If G has minimum degree at least k and $\lambda^{(h)}(G)$ exists, then $\lambda^{(h)}(G) \leq \tau(G)$; in particular, $\lambda^{(h)}(G) \leq a_h(G)(k-h)$ if G is k-regular.

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