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On $(d,2)$-dominating numbers of binary undirected de Bruijn graphs ☆

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Abstract

In this paper, we show that: (i) For n-dimensional undirected binary de Bruijn graphs, $UB(n)$, $n \geqslant 4$, there is a vertex $x = x_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1 x_1$ $(x_1 = 1$ or $0)$ such that for any other vertex t there exist at least two internally disjoint paths of length at most $n-1$ between x and t in $UB(n)$, i.e., the $(n-1,2)$-dominating number of $UB(n)$ is equal to one. (ii) For $n \geqslant 5$, let $S = \{100\cdots 01, 011\cdots 10\}$. For any other vertex t there exist at least two internally disjoint paths of length at most $n-2$ between t and S in $UB(n)$, i.e., the $(n-2,2)$-dominating number of $UB(n)$ is no more than 2.

Keywords: De Bruijn graph; Diameter; Dominating number

1. Introduction and notation

The *binary directed de Bruijn graph* of the dimension n, denoted $B(n)$, has 2^n vertices, which are labeled with the binary strings of length n. There is an arc from any vertex $x_1x_2\cdots x_n$ to the vertices $x_2x_3\cdots x_n0$ and $x_2x_3\cdots x_n1$. We say that the ith coordinate of x is x_i, being equal to 0 or 1, and $\bar{x}_i = 1 - x_i$.

The unidirected binary de Bruijn graph $UB(n)$ is obtained from $B(n)$ by deleting the orientation of the arcs and omitting multiple edges and loops. It is well known that $UB(n)$ is 2-connected and that its diameter (maximum of the distances between all pairs of vertices) is equal to n. Due to their bounded maximum degree equal to

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4 and their low diameter, de Bruijn graphs have been proposed as a possible good interconnection network for a parallel architecture [1,10].

In order to characterize the reliability of transmission delay in a network, Hsu and Lyuu [4], and Flandrin and Li [2] independently introduced the concept of m-diameter (i.e. wide-diameter): For any pair (x, y) of vertices in a graph G, the m-distance of x and y, denoted by $D_m(x, y)_G$, is defined as the minimum integer d for which there are at least m internally disjoint path of length at most d between x and y. The m-diameter of G, denoted by $D_m(G)$, is the maximum of $D_m(x, y)_G$ over all pairs (x, y) of vertices of G. General results on the m-diameters of m-connected graphs can be found in [2,4,5]. Results for some particular classes of graphs can be also found in [3,6,7]. In particular, for the undirected binary de Bruijn graphs of dimension n, its 2-diameter is n (see [7]).

Recently, Li and Xu [8] define a new parameter (d,m)-*dominating number* in m-connected graphs, in some sense, which can more accurately characterize the reliability of networks than the wide-diameter can.

Definition. Let G be an m-connected graph, S a nonempty and proper subset of $V(G)$, y a vertex in $G - S$. For a given positive integer d, y is (d,m)-dominated by S in G if there are at least m internally disjoint (y,S)-paths in G, each of them is of length at most d. S is said to be a (d,m)-dominating set of G, denoted by $S_{d,m}(G)$ if either $S = V(G)$ or S can (d,m)-dominate every vertex in $G - S$. The parameter

$$s_{d,m}(G) = \min\{|S_{d,m}(G)|: S_{d,m}(G) \text{ is a } (d,m)\text{-dominating set of } G\}$$

will be called the (d,m)-dominating number of G.

Li and Xu [8] have shown some general properties of the (d,m)-dominating sets and the (d,m)-dominating numbers of m-connected graphs. In particular, they prove that for any $m \geqslant 2$, the (d,m)-dominating numbers $(m - 1 \leqslant d \leqslant m)$ of the m-dimensional hypercube Q_m are 2. In [9], we prove that the (d,m)-dominating numbers of the m-dimensional hypercube Q_m $(m \geqslant 4)$ are also 2 for any integer $d, (\lfloor m/2 \rfloor + 2 \leqslant d \leqslant m)$. Since 2-diameter of $UB(n)$ is n, which implies that $s_{n,2}(UB(n)) = 1$. An interesting problem is what the value of $s_{d,2}(UB(n))$ is when $d \leqslant n - 1$. The aim of this paper is to prove that $s_{n-1,2}(UB(n)) = 1$ and $s_{n-2,2}(UB(n)) \leqslant 2$.

Let us first introduce some notation and recall some properties of de Bruijn graph $B(n)$.

Property 1.1. *Given any two vertices* $x = x_1 x_2 \cdots x_n$ *and* $y = y_1 y_2 \cdots y_n$ *of* $B(n)$, *there is a unique shortest path from* x *to* y, *and the distance* $d(x, y)$ *is equal to the smallest* $i \leqslant n - 1$ *such that* $x_{i+1} \cdots x_n = y_1 \cdots y_{n-i}$ *if it exists and to* n *otherwise.*

Property 1.2. *If* C *is a closed walk of length* $l < n$ *in* $B(n)$ *and* $z_1 z_2 \cdots z_n$ *is a vertex on* C, *then* $z_i = z_{i+l}$ *for all* $1 \leqslant i \leqslant n - l$.

For two given vertices x and y in $B(n)$, we will denote $P[x, y]$ as the shortest path P from x to y. The length of this path denoted by $|P[x, y]|$, is the number of edges in the path and is also the distance $d(x, y)$ from x to y. $P[x, y]$ also represents the set of vertices of the path, including its extremities. $P(x, y)$ will denote the set of vertices of the path excluding the extremities x and y. $P(x, y]$ is the set of vertices including y, and excluding x (and similarly for $P[x, y)$).

2. Preliminary results

Let us first give the following two lemmas which can be found in [7].

Lemma 2.1 (Li, Sotteau and Xu [7]). *For any two vertices x and y of $B(n)$, if the shortest path from x to y intersects the shortest path from y to x in a vertex other than x and y, then, necessarily, the sum of the lengths of the two paths is strictly more than n.*

Lemma 2.2 (Li, Sotteau and Xu [7]). *For any two vertices x and y of $B(n)$, the union of the shortest path from x to y and the shortest path from y to x consists of at most three circuits*

Lemma 2.3. *In $B(n)$, the vertex $x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ ($x_1 = 1$ or 0) cannot be on any closed walk of length l ($0 < l < n-1$).*

Proof. If not, we assume that the vertex $x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ is on a closed walk C of length l ($0 < l < n-1$) in $B(n)$. By Property 1.2, the $(l+1)$th coordinate of $x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ is x_1. Then $l = n-1$, a contradiction. □

Lemma 2.4. *Let $x = x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ with $x_1 = 1$ or 0. Then for any other vertex $t \neq t_1\bar{x}_1\cdots\bar{x}_1t_n$ in $B(n)$ we have*

(a) *$|P[x,t]| \leqslant n-1$ and $|Q[t,x]| \leqslant n-1$.*

(b) *The union of the shortest path $P[x,t]$ and the shortest path $Q[t,x]$ in $B(n)$ consists of at most two circuits.*

Proof. (a) If $t_1 = x_1$ it is easy to know $|P[x,t]| \leqslant n-1$. Since $t \neq t_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1t_n$, there exists at least one coordinate of t except the first and the last coordinates of t, which is x_1. If $t_1 = \bar{x}_1$, then $t = \bar{x}_1\cdots\bar{x}_1x_1t_{i+1}\cdots t_n$, where $1 < i < n$. By Property 1.1, we know $|P[x,t]| < n-1$. Similarly, we have $|Q[t,x]| \leqslant n-1$.

(b) Suppose that the union of the $P[x,t]$ and $Q[t,x]$ consists of three circuits by Lemma 2.2 as shown in Fig. 1, we use the following notation: let z (resp. w^*) be the first (resp., last) vertex that $P(x,t)$ has in common with $Q(t,x)$. And let w (resp., z^*) be the first (resp., last) vertex that $Q(t,x)$ has in common with $P(x,t)$. By Lemma 2.1, $|P[z,t]| + |Q[t,z]| > n$, and $|P[x,z]| + |Q[z,x]| \geqslant n-1$ by Lemma 2.3. Therefore, $|P[x,t]| + |Q[t,x]| = |P[x,z]| + |P[z,t]| + |Q[t,z]| + |Q[z,x]| > 2n-1$. But $|P[x,t]| \leqslant n-1$ and $|Q[t,x]| \leqslant n-1$ by (a), which leads to a contradiction. □

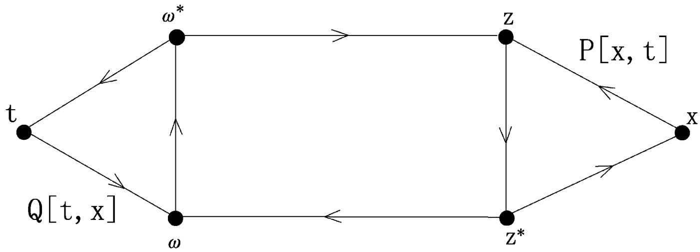

Fig. 1.

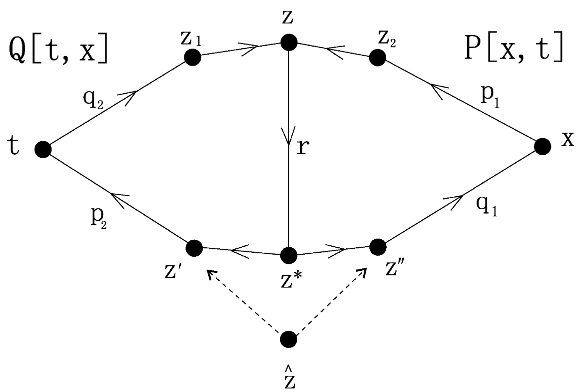

Fig. 2.

Let $x = x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ and $t \neq t_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1t_n$. Suppose that the union of $P[x,t]$ and $Q[t,x]$ consists of two circuits as shown in Fig. 2. Let z (resp., z^*) be the first (resp., last) vertex that $P(x,t)$ has in common with $Q(t,x)$. Let $|P[x,z]| = p_1$, $|P[z,z^*]| = |Q[z,z^*]| = r$, $|P[z^*,t]| = p_2$, $|Q[t,z]| = q_2$, $|Q[z^*,x]| = q_1$, thus $p_1+r+p_2 = |P[x,t]| \leqslant n-1$ and $q_2 + r + q_1 = |Q[t,x]| \leqslant n-1$ by Lemma 2.4(a). All these integers are strictly positive except for r which may equal 0. z_1 and z_2 are inneighbors of z; z' and z'' are outneighbors of z^*.

Lemma 2.5. $p_1 = q_1$.

Proof. Clearly, $|P[z,t]| + |Q[t,z]| = p_2 + r + q_2 \leqslant n-1$ since $|P[x,z]| + |Q[z,x]| = p_1 + r + q_1 \geqslant n-1$ by Lemma 2.3. By Property 1.2, we assume

$$t = t_1t_2\cdots t_{q_2+r+p_2}t_1t_2\cdots t_{q_2+r+p_2}\cdots t_1t_2\cdots t_{q_2+r+p_2}t_1\cdots t_k \tag{1}$$

with $n \equiv k (\mathrm{mod}\,(q_2 + r + p_2))$, $1 \leqslant k \leqslant q_2 + r + p_2$.

Let us consider the vertex z. With the notation introduced above, since z can be reached in q_2 steps from t on Q, it can be written as

$$z = t_{q_2+1} \cdots t_{q_2+r+p_2} t_1 t_2 \cdots t_{q_2+r+p_2} \cdots t_1 t_2 \cdots t_{q_2+r+p_2} t_1 \cdots t_k \underbrace{\bar{x}_1 \cdots \bar{x}_1}_{q_2} \tag{2}$$

Since z can be reached in p_1 steps from $x = x_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1 x_1$ on $P[x,t]$,

$$t_{q_2+1} = \cdots = t_{q_2+r+p_2} = \bar{x}_1 \tag{3}$$

Noting $p_1 + r + q_1 \geqslant n-1 \geqslant q_2 + r + q_1$, we have $p_1 \geqslant q_2$. Similarly, we have $q_1 \geqslant p_2$. Note that $p_1 \neq 1$ and $q_1 \neq 1$. If $p_1 = 1$, then $q_2 = 1$. So $t = \bar{x}_1\bar{x}_1 \cdots \bar{x}_1 x_1$, a contraction to $t \neq t_1\bar{x}_1 \cdots \bar{x}_1 t_n$. Similarly, we also have $q_1 \neq 1$.

Let us consider z_1 and z_2 in Fig. 2. The first coordinate of z_2 is $\bar{x}_1$ since $p_1 > 1$. So, the first coordinate of z_1 is x_1. Since z_1 can be reached in $q_2 - 1$ steps from t, its first coordinate is t_{q_2}. Hence $t_{q_2} = x_1$.

Let us now consider z' and z'' in Fig. 2. The latest coordinate of z'' is $\bar{x}_1$ since $q_1 > 1$. So, the latest coordinate of z' is x_1. Since z' can be reached in $q_2 + r + 1$ steps from t, it can be written as

$$z' = t_{q_2+r+2} \cdots t_{q_2+r+p_2} t_1 t_2 \cdots t_{q_2+r+p_2} \cdots t_1 t_2 \cdots t_{q_2+r+p_2} t_1 t_2 \cdots t_k \underbrace{\bar{x}_1 \cdots \bar{x}_1}_{q_2+r} x_1 \tag{4}$$

Since t can be reached in $p_2 - 1$ steps from z', we can also write z' as

$$t = t_1 t_2 \cdots t_{q_2+r+p_2} \cdots t_1 \cdots t_{q_2+r+p_2} t_1 \cdots t_k \underbrace{\bar{x}_1 \cdots \bar{x}_1}_{q_2+r} x_1 \underbrace{* * \cdots *}_{p_2-1} \tag{5}$$

Note that we always have $q_2 \leqslant k$; otherwise, we have $t_{q_2} = \bar{x}_1$ if we compare expression (5) with expression (1) of t, this leads to a contradiction with $t_{q_2} = x_1$. Hence, by (3), we have

$$t_{q_2+1} = \cdots = t_k = \bar{x}_1 \quad \text{and} \quad t_{q_2} = x_1 \tag{6}$$

If $p_2 > k$, then $t_{k+q_2+r+1} = x_1$ from (5); Noting $q_2 + r + 1 < k + q_2 + r + 1 \leqslant p_2 + q_2 + r$, we have $t_{k+q_2+r+1} = \bar{x}_1$ from (3), a contradiction. Hence $p_2 \leqslant k$. Comparing expression (1) with expression (5) of t, we have

$$t_1 = t_2 = \cdots = t_{k-p_2} = \bar{x}_1 \quad \text{and} \quad t_{k-p_2+1} = x_1. \tag{7}$$

Now, from (6) and (7), we have

$$t = \underbrace{\bar{x}_1\bar{x}_1 \cdots \bar{x}_1}_{k-p_2} x_1 * * \cdots * x_1 \underbrace{\bar{x}_1\bar{x}_1 \cdots \bar{x}_1}_{k-q_2}. \tag{8}$$

Thus by Property 1.1, we have

$$n - (p_1 + r + p_2) = k - p_2 + 1,$$

$$n - (q_2 + r + q_1) = k - q_2 + 1.$$

So, we have $p_1 = q_1$. □

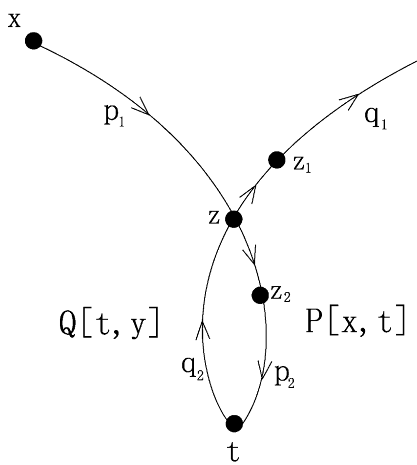

Fig. 3.

Lemma 2.6. *Let* $x = x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ *and* $y = \bar{x} = \bar{x}_1x_1x_1\cdots x_1\bar{x}_1$. $t = t_1t_2\cdots t_n$ *is a vertex other than* x *and* y *in* $B(n)$. *If* $|P[x,t]| \leqslant n-2$ *and* $|Q[t,y]| \leqslant n-2$, *then* $P(x,t)\cap Q(t,y) = \emptyset$.

Proof. By Property 1.1, we first have $t_1 = \bar{x}_1$ and $t_n = x_1$ since $1 \leqslant |P[x,t]| \leqslant n-2$ and $1 \leqslant |Q[t,y]| \leqslant n-2$. If $P(x,t)\cap Q(t,y) \neq \phi$, and then if $z = z_1z_2\cdots z_n$ is a vertex in the intersection such that its outneighbors z_1 and z_2, respectively, on P and Q are distinct (see Fig. 3). We denote $|P[x,z]| = p_1$, $|P[z,t]| = p_2$ and $|Q[t,z]| = q_2$, $|Q[z,y]| = q_1$. It is clear that $q_2 + p_2 \leqslant n-2$ since $p_1 + q_1 \geqslant d(x,y) = n-2$ and $|P[x,t]| + |Q[t,y]| = p_1 + p_2 + q_1 + q_2 \leqslant 2n-4$. Using Property 1.2, we assume

$$t = t_1t_2\cdots t_{q_2+p_2}t_1t_2\cdots t_{q_2+p_2}\cdots t_1t_2\cdots t_{q_2+p_2}t_1\cdots t_k \tag{9}$$

with $n \equiv k \pmod{(q_2+p_2)}, 1 \leqslant k \leqslant p_2 + q_2$ and $t_1 = \bar{x}_1, t_k = x_1$.

Let us consider the vertex z. Since z can be reached in q_2 steps from t on Q, it can be written as

$$z = t_{q_2+1}\cdots t_{q_2+p_2}t_1t_2\cdots t_{q_2+p_2}\cdots t_1t_2\cdots t_{q_2+p_2}t_1t_2\cdots t_k\underbrace{x_1\cdots x_1}_{q_2}. \tag{10}$$

Since z can be also reached in p_1 steps from $x = x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ on P,

$$t_{q_2+1} = \cdots = t_{q_2+p_2} = \bar{x}_1. \tag{11}$$

Since $p_1+q_1 \geqslant n-2 \geqslant p_1+p_2$, we have $q_1 \geqslant p_2 \geqslant 1$. If $q_1 = 1$, then $z = \bar{x}_1\bar{x}_1x_1x_1\cdots x_1$ and $t = \bar{x}_1x_1x_1\cdots x_1$ by $p_2 = 1$. Clearly, $|Q[t,y]| = n$, which leads to a contradiction.

We now consider z_1 and z_2. Note that $q_1 > 1$. We also have that the last coordinate of z_1 is $\bar{x}_1$, so

$$z_1 = t_{q_2+2}\cdots t_{q_2+p_2}t_1t_2\cdots t_{q_2+p_2}\cdots t_1t_2\cdots t_{q_2+p_2}t_1\cdots t_k\underbrace{x_1\cdots x_1}_{q_2}\bar{x}_1. \tag{12}$$

Since t can be reached in $p_2 - 1$ steps from z_1, it can also be written as

$$t = t_1 t_2 \cdots t_{q_2+p_2} \cdots t_1 t_2 \cdots t_{q_2+p_2} t_1 t_2 \cdots t_k \underbrace{x_1 \cdots x_1}_{q_2} \bar{x}_1 \underbrace{* * \cdots *}_{p_2-1}. \tag{13}$$

Comparing expression (9) with expression (13) of t, we have $k \leqslant p_2$; otherwise, we have $t_1 = x_1$, it leads a contradiction. Thus, from expression (13) of t, we have

$$t_{k+1} = \cdots = t_{k+q_2} = x_1 \tag{14}$$

which leads to a contradiction with (11).

Thus $P(x,t) \cap Q(t,y) = \phi$. $\square$

3. The main results

Theorem 3.1. *For $n \geqslant 4$, there is a vertex $x = x_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1 x_1$ ($x_1 = 1$ or 0) in $UB(n)$ such that for any other vertex t there exist at least two internally disjoint paths of length at most $n-1$ between x and t, i.e., $s_{n-1,2}(UB(n)) = 1$.*

Proof. For any vertex t other than x, we will exhibit the two undirected paths P_1 and P_2 between x and t in $UB(n)$ which are internally disjoint and of lengths at most $n-1$.

Case 1. $t \neq t_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1 t_n$. If $P[x,t]$ and $Q[t,x]$ are internally disjoint in $B(n)$, it can be directedly verified since $|P[x,t]| \leqslant n-1$ and $|Q[t,x]| \leqslant n-1$ by Lemma 2.4(a). We take $P_1 = P$ and $P_2 = Q$.

If $P = P[t,x]$ and $Q = Q[y,t]$ are not internally disjoint in $B(n)$, by Lemma 2.4, the union of $P[x,t]$ and $Q[t,x]$ consists of two circuits as shown in Fig. 2, and by Lemma 2.5, $|P[x,z]| = |Q[z^*,x]| = p_1 = q_1$.

If $r \neq 0$, we take $P_1 = P[x,z] \cup Q[t,z]$ and $P_2 = Q[z^*,x] \cup P[z^*,t]$ since $|P_1| = q_2 + p_1 = q_2 + q_1 < q_2 + q_1 + r = |Q| \leqslant n-1$ and $|P_2| = p_2 + q_1 = p_2 + p_1 < p_2 + r + p_1 = |P| \leqslant n-1$.

If $r = 0$, i.e. $z = z^*$, we consider the vertex $\hat{z} = \bar{z}_1 z_2 \cdots z_n$ which has the same outneighbors as z (see Fig. 2). Then, clearly, since every vertex of $B(n)$ has out-degree at most 2, $\hat{z}$ is not on P and not on Q. Thus, the undirected path $P_1 = P[x,z] \cup Q[t,z]$ of length $q_2 + p_1 = q_2 + q_1$ and the undirected path $P_2 = P[z',t] \cup [\hat{z},z'] \cup [\hat{z},z''] \cup Q[z'',x]$ of length $p_2 + q_1 = p_2 + p_1$ are internally vertex-disjoint and of length at most $n-1$.

Case 2. $t = t_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1 t_n$ *and* $t \neq x$. If $t = x_1\bar{x}_1\bar{x}_1 \cdots \bar{x}_1$, we easily find two internally disjoint paths in $B(n)$, each of which has length not more than 3:

$$P_1 \colon t \leftarrow x_1 x_1 \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 \rightarrow x,$$

$$P_2 \colon t \rightarrow \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 x_1 \rightarrow \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 x_1 x_1 \leftarrow x.$$

If $t = \bar{x}_1\bar{x}_1 \cdots \bar{x}_1 x_1$, we can similarly construct P_1 and P_2 as follows:

$$P_1 \colon t \leftarrow x_1 \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 \leftarrow x_1 x_1 \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 \rightarrow x,$$

$$P_2 \colon t \rightarrow \bar{x}_1 \bar{x}_1 \cdots \bar{x}_1 x_1 x_1 \leftarrow x.$$

If $t=\bar{x}_1\bar{x}_1\cdots\bar{x}_1$, we construct P_1 and P_2 as follows:

$$P_1: t \leftarrow x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1 \leftarrow x_1x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1 \rightarrow x,$$

$$P_2: t \rightarrow \bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1 \rightarrow \bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1x_1 \leftarrow x.$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *For $n\geqslant 5$, let $S=\{100\cdots01, 011\cdots10\}$. For any other vertex t there exist at least two internally disjoint paths of length at most $n-2$ between t and S in $UB(n)$, i.e., $s_{n-2,2}(UB(n))\leqslant 2$.*

Proof. We will prove that S is a $(n-2,2)$-dominating set of undirected de Bruijn graph $UB(n)$. We will divide the proof into two cases by considering any $t=t_1t_2\cdots t_n\in V-S$. In every case, we exhibit the two undirected paths P_1 and P_2 which are internally disjoint and of lengths at most $n-2$. Let $x=x_1\bar{x}_1\bar{x}_1\cdots\bar{x}_1x_1$ and $y=\bar{x}=\bar{x}_1x_1x_1\cdots x_1\bar{x}_1$ ($x_1=1$ or 0). We will prove that S is a $(n-2,2)$-dominating set of $UB(n)$.

Case 1. $t_1\neq t_2$. Without loss of generality, we assume that $t_1=\bar{x}_1$ and $t_2=x_1$. If $t=\bar{x}_1x_1**\cdots*x_1\bar{x}_1$ or $\bar{x}_1x_1**\cdots*\bar{x}_1\bar{x}_1$, then $|P[x,t]|\leqslant n-2$ and $|Q[t,x]|\leqslant n-2$ in $B(n)$. By Lemma 2.5, we can take P_1 and P_2 as similarly as that in case 1 of Theorem 3.1.

If $t=\bar{x}_1x_1**\cdots*\bar{x}_1x_1$, we know that $|P[x,t]|\leqslant n-2$ and $|Q[t,y]|\leqslant n-2$ in $B(n)$. By Lemma 2.6, we can take $P_1=P[x,t]$ and $P_2=Q[t,y]$.

If $t=\bar{x}_1x_1**\cdots*x_1x_1$, we first assume that $t\neq\bar{x}_1x_1x_1\cdots x_1$, so, there must exist some $t_i=\bar{x}_1$ for $3\leqslant i\leqslant n-3$. Suppose that t_j is the last coordinate of t which is equal to $\bar{x}_1$ ($3\leqslant j\leqslant n-3$). So, $|Q[t,y]|\leqslant n-3$ in $B(n)$. Now, we can take $P_1=P[x,t]$ and $P_2=Q[t,y]$ by Lemma 2.6. When $t=\bar{x}_1x_1x_1\cdots x_1$, we can take P_1 and P_2 as follows:

$$P_1: t \rightarrow x_1x_1\cdots x_1\bar{x}_1 \rightarrow x_1x_1\cdots x_1\bar{x}_1\bar{x}_1 \leftarrow y,$$

$$P_2: t \leftarrow \bar{x}_1\bar{x}_1x_1x_1\cdots x_1 \rightarrow y.$$

Case 2. $t_1=t_2$. Without loss of generality, we assume that $t_1=t_2=x_1$. If $t=x_1x_1**\cdots*\bar{x}_1x_1$, we know $|Q[t,y]|\leqslant n-2$ and $|P[y,t]|\leqslant n-3$ in $B(n)$ by Property 1.1. Note that Lemma 2.5, we can take P_1 and P_2 similar to case 1 of Theorem 3.1.

If $t=x_1x_1**\cdots*x_1\bar{x}_1$, we first assume that $t\neq x_1x_1\cdots x_1\bar{x}_1$. So, there must exist some $t_i=\bar{x}_1$ for $3\leqslant i\leqslant n-3$. Suppose that t_j is the first coordinate of t which is $\bar{x}_1$ ($3\leqslant j\leqslant n-3$). By Property 1.1, $|P[y,t]|\leqslant n-3$ and $|Q[t,x]|\leqslant n-2$ in $B(n)$. By Lemma 2.6, we can take $P_1=P[y,t]$ and $P_2=Q[t,x]$. When $t=x_1x_1\cdots x_1\bar{x}_1$, we construct P_1 and P_2 in $UB(n)$ as follows:

$$P_1: t \rightarrow x_1x_1\cdots x_1\bar{x}_1\bar{x}_1 \leftarrow y,$$

$$P_2: t \leftarrow \bar{x}_1x_1x_1\cdots x_1 \leftarrow \bar{x}_1\bar{x}_1x_1x_1\cdots x_1 \rightarrow y.$$

If $t=x_1x_1**\cdots*\bar{x}_1\bar{x}_1$, we easily know $|P[y,t]|\leqslant n-3$ and $|Q[t,x]|\leqslant n-3$ in $B(n)$. By Lemma 2.6, we take $P_1=P[y,t]$ and $P_2=Q[t,x]$ in $UB(n)$.

If $t = x_1x_1 * * \cdots * x_1x_1$, we first assume that $t \neq x_1x_1 \cdots x_1$. So, there must exist some $t_i = \bar{x}_1$ for $3 \leqslant i \leqslant n-3$. Suppose that t_j is the first coordinate of t which is $\bar{x}_1$ $(3 \leqslant j \leqslant n-3)$ and t_k is the last coordinate of t which is $\bar{x}_1$ $(3 \leqslant k \leqslant n-3)$. By Property 1.1, $|P[y,t]| \leqslant n-3$ and $|Q[t,y]| \leqslant n-3$ in $B(n)$. Note that Lemma 2.5, we can take P_1 and P_2 similar to case 1 of Theorem 3.1. When $t = x_1x_1 \cdots x_1$, we construct P_1 and P_2 in $UB(n)$ as follows:

$$P_1: t \rightarrow x_1x_1 \cdots x_1\bar{x}_1 \rightarrow x_1x_1 \cdots x_1\bar{x}_1\bar{x}_1 \leftarrow y,$$

$$P_2: t \leftarrow \bar{x}_1x_1x_1 \cdots x_1 \leftarrow \bar{x}_1\bar{x}_1x_1x_1 \cdots x_1 \rightarrow y.$$

Theorem 3.2 is proved. □

4. Conclusions and problems

For the undirected binary de Bruijn graphs of the dimension n, $UB(n)$, we prove that $s_{n-1,2}(UB(n)) = 1$ when $n \geqslant 4$. Another result in this paper is $s_{n-2,2}(UB(n)) \leqslant 2$ when $n \geqslant 5$. But we do not know if $s_{n-2,2}(UB(n))$ is equal to 2. For the undirected d-nary de Bruin graphs of the dimension n, $UB(d,n)$, $d \geqslant 3$, we know that they have connectivity $2d-2$ and diameter n. A more difficult problem is to determine the value of $s_{n,2d-2}(UB(d,n))$.

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