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ON k-DIAMETER OF k-CONNECTED GRAPHS

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Abstract. Let G be a k-connected simple graph with order n. The k-diameter, combining connectivity with diameter, of G is the minimum integer $d_k(G)$ for which between any two vertices in G there are at least k internally vertex-disjoint paths of length at most $d_k(G)$. For a fixed positive integer d, some conditions to insure $d_k(G)\leqslant d$ are given in this paper. In particular, if $d\geqslant 3$ and the sum of degrees of any s ($s=2$ or 3) nonadjacent vertices is at least $n+(s-1)k+1-d$, then $d_k(G)\leqslant d$. Furthermore, these conditions are sharp and the upper bound d of k-diameter is best possible.

§ 1 Introduction

Throughout this paper the letter G always denotes a finite, connected, simple, and undirected graph (V,E) with order $n\geqslant 3$. We follow [1] for terminology and notation not defined here.

Consider a graph G that models a computer network with each vertex representing a processor and each edge representing a two-way communication link. To insure that the network is fault-tolerant with respect to processor failures, it is necessary that the number of internally vertex-disjoint paths between each pair of vertices of G exceed the number of possible failures. Connectivity is clearly a crucial graph parameter to measure fault-tolerance of the network. However, the length of time for the information to arrive is also important, so it is desirable that internally vertex-disjoint paths be short. This requires that between each pair of vertices of G there is a specified number of internally vertex-disjoint paths, each with a bound on the number of vertices. So the following concepts by combining connectivity with diameter emerge rather naturally.

Let G be k-connected, and x,y be two distinct vertices of G. The k-distance between x and y is the minimum integer $d_k(G;x,y)$ for which there are at least k internally vertex-disjoint paths of length at most $d_k(G;x,y)$ between x and y in G. The k-diameter of G, denoted by $d_k(G)$, is defined as the maximum k-distance $d_k(G;x,y)$ over all pairs (x,y)

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of vertices of G. It is clear that $d_1(G)$ is just the diameter of G and $d_k(G)\geqslant d_{k-1}(G)\geqslant\ldots\geqslant d_2(G)\geqslant d_1(G)$.

The k-diameter proposed in [2] is an important parameter to measure fault-tolerance and efficiency of parallel processing computer networks and has received much attention[3]. Menger's theorem solves the problem of existence of $d_k(G)$ if G is k-connected. Although Menger's theorem gives no information about length of the paths, many exact values of k-diameter for some classes of specially k-connected graphs have been obtained by various author (see, for example, [2~6]).

For a fixed positive integer d, we will in this paper present some conditions to insure $d_k(G)\leqslant d$. In particular, if $d\geqslant 3$ and the sum of degrees of any $s(=2$ or $3)$ nonadjacent vertices is at least $n+(s-1)k+1-d$, then $d_k(G)\leqslant d$. Furthermore, these conditions are sharp and the upper bound d of k-diameter is best possible.

§ 2 Notation and Propositions

Let B be a subset of $V(G)$. For $x\in V(G)$, $d_B(x)$ denotes the number of edges of $G[B\cup\{x\}]$ incidenct with x. Use a sequence of vertices $P=(x_0,x_1,x_2,\ldots,x_{l-1},x_l)$ to denote a path connecting x_0 and x_l, where $x_1,x_2,\ldots,x_{l-1}$ are called the internal vertices of P, $l=|E(P)|$ is the length of P. if G is k-connected, and x,y are two distinct vertices of G, then by Menger's theorem there are k internally vertex-disjoint (x,y)-paths P_1, $P_2,\ldots,P_k$. The set

$$\mathscr{P}_k(G;x,y)=\{P_1,P_2,\ldots,P_k\},\text{ where } |E(P_1)|\leqslant|E(P_2)|\leqslant\ldots\leqslant|E(P_k)|$$

is called a Menger (x,y)-path system. $\mathscr{P}_k(G;x,y)$ is called a minimum Menger (x,y)-path system if it is of the minimum sum of path-lengths over all Menger (x,y)-path systems. It is clear that if $d_k(G)=d$, then there are two vertices x and y in G and a minimum Menger (x,y)-path system $\mathscr{P}_k(G;x,y)$ such that $d_k(G;x,y)=d_k(G)=|E(P_k)|=d$. Let

$$R_i=V(P_i)\setminus\{x,y\},r_i=|R_i|,\quad i=1,2,\ldots,k,$$

$$B=R_1\cup R_2\cup\ldots\cup R_{k-1}\cup(x,y),\ A=V(G)\setminus(B\cup R_k),$$

then

$$n=2+|A|+\sum_{i=1}^{k}r_i. \tag{1}$$

Proposition 1. Suppose that $\mathscr{P}_k(G;x,y)=\{P_1,P_2,\ldots,P_k\}$ is a minimum Menger (x,y)-path system, then

(i) any two nonadjacent vertices of $P_i(i=1,2,\ldots,k)$ are not adjacent in G;

(ii) any two vertices of $P_i(i=1,2,\ldots,k)$ whose distance in P_i is at least 3 have no neighbor vertex in common in A;

(iii) for any independent set S of P_j and $r_i\neq 0(1\leqslant i\neq j\leqslant k)$, the number of edges between S and R_i in G is at most $r_i+|S|-1$.

Proof. The parts (i) and (ii) of Proposition 1 hold clearly by the minimum of $\mathscr{P}_k(G;x,y)$. To prove the rest, let $P_i=(x=u_0,u_1,u_2,\ldots,u_{r_i},u_{r_i+1}=y)$ and $P_j=(x=v_0,v_1,v_2,\ldots,v_{r_j},v_{r_j+1}=y)$, $i\neq j$. Denote by $E_G(S,R_i)$ the set of edges between S and R_i, and denote by H the subgraph of G induced by $E_G(S,R_i)$, we will show that H has no cycle.

Suppose, to the contrary, that H contains a cycle C. Since H is a bipartite graph with the bipartition $\{S,R_i\}$, the length of C is an even number at least four. And so there must be two vertices $u_a,u_b(a<b)$ in R_i and two vertices $v_h,v_t(h<t)$ in S such that $u_av_t,u_bv_h\in E(G)$. Since $b-a\geqslant 1$ and $t-h\geqslant 2$, replacing two paths P_i and P_j in $\mathscr{P}_k(G;x,y)$ by two new paths $P_i'=(x,u_1,u_2,\ldots,u_{a-1},u_a,v_t,v_{t+1},\ldots,v_{r_j},y)$ and $P_j'=(x,v_1,v_2,\ldots,v_{h-1},v_h,u_b,u_{b+1},\ldots,u_{r_i},y)$, we can obtain another Menger (x,y)- path system $\mathscr{P}_k'(G;x,y)$ whose sum of path-lengths is smaller than $\mathscr{P}_k(G;x,y)$'s. This contradicts the choice of $\mathscr{P}_k(G;x,y)$. Thus H has no cycle.

Since H has order $r_i+|S|$ and no cycle, we have $|E_G(R_i,S)|=|E(H)|\leqslant|V(H)|-1=r_i+|S|-1$. The proof is completed.

Before presenting the following propositions we describe a class of graphs called a generalized wheel. A generalized wheel denoted by $W(m,p)$ is a graph of order $(m+p)$ obtained from the union of a complete graph K_m of order m and a cycle C_p of order p by adding edges joining each of vertices of K_m to all vertices of C_p. For $k\geqslant 2$ and $p\geqslant 3$, the generalized wheel $W(k-2,p)$ has connectivity k and k-diameter $p-1$. Generalized wheels can provide some extremal examples related to the results obtained in this paper.

Proposition 2. Let G be k-connected and $k\geqslant 2$, then $2\leqslant d_k(G)\leqslant n-k+1$. Furthermore, both the upper and the lower bounds are best possible.

Proof. Let x,y be two vertices of G, and $\mathscr{P}_k(G;x,y)=\{P_1,P_2,\ldots,P_k\}$ be a minimum Menger (x,y)-path system such that $d_k(G;x,y)=|E(P_k)|=d_k(G)$. Then by $k\geqslant 2$, the simpleness of G and the equality (1), we have

$$1\leqslant r_k=n-2-(r_1+r_2+\ldots+r_{k-1})-|A|\leqslant n-2-(k-2)=n-k. \quad (2)$$

It follows from Inequality (2) that $2\leqslant d_k(G)=|E(P_k)|=r_k+1\leqslant n-k+1$.

The upper and the lower bounds above for $d_k(G)$ can not be improved in general case. Namely, there are two k-connected graphs G and G', for example, $G=W(k-2,n-k+2)$ and $G'=K_{k+1}$, such that $d_k(G)=n-k+1$ and $d_k(G')=2$. The proof is completed.

Proposition 3. Let G be k-connected, $k\geqslant 2,d\geqslant 2$ and $\alpha(G)$ be the independence number of G, then $d_k(G)\leqslant 2\alpha(G)$. Whence $d_k(G)\leqslant d$ if $\alpha(G)\leqslant\lfloor d/2\rfloor$. Furthermore, these bounds are best possible.

Proof. Let x and y be two vertices of G such that $d_k(G;x,y)=d_k(G)$, and $\mathscr{P}_k(G;x,y)=\{P_1,\ldots,P_k\}$ be a minimum Menger (x,y)-path system, then $d_k(G;x,y)=|E(P_k)|=r_k+1=d_k(G)$. Let $P_k=(x=x_0,x_1,x_2,\ldots,x_{r_k-1},x_{r_k},y)$, and $I=\{x_{2j}:0\leqslant j\leqslant\lceil(r_k+1)/2\rceil-1\}$, then $y\notin I$ since $2(\lceil(r_k+1)/2\rceil-1)\leqslant r_k$. By the minimum of $\mathscr{P}_k(G;x,y)$ and $y\notin I$, any two vertices of I are not adjacent in G. Thus I is an independent set of G, and so

$$\alpha(G) \geqslant |I| = \lceil (r_k + 1)/2 \rceil \geqslant (r_k + 1)/2.$$

It follows from the above inequality that $d_k(G) = r_k + 1 \leqslant 2\alpha(G)$.

The generalized wheels $W(k-2, 2n+1)$ and $W(k-2, d+2)$ are examples showing that these bounds are the best ones. The proof is completed.

§ 3 Main Results

In this section, d is assumed to be a fixed positive integer. We will present our main results which give a certain sum of degree conditions to insure that $d_k(G) \leqslant d$ for a k-connected graph G. We assume by Proposition 2 below that $2 \leqslant d \leqslant n-k+1$. Let us start with a special case of $d=2$.

Theorem 1. If $k \geqslant 2$ and $d_G(x) + d_G(y) \geqslant n+k-1$ for any two vertices x and y of G, then $d_k(G) = 2$. Furthermore, the degree-condition is sharp.

Proof. It is not difficult to see that G is k-connected under the assumed condition and $d_k(G) \geqslant 2$ since $k \geqslant 2$. We show $d_k(G) \leqslant 2$ below. Suppose, to the contrary, that $d_k(G) \geqslant 3$, then there are two vertices x and y in G and a minimum Menger (x, y)-path system $\mathscr{P}_k(G; x, y) = \{P_1, P_2, \ldots, P_k\}$ such that $d_k(G; x, y) = |E(P_k)| = d_k(G) \geqslant 3$. We estimate the degree-sum of two vertices x and y in G. It is clear that

$$d_B(x) + d_B(y) = 2(k-1). \tag{3}$$

Since $|E(P_k)| = d_k(G; x, y) \geqslant 3$ and the minimum of $\mathscr{P}_k(G; x, y)$, x and y have no neighbor vertex in common in $T = A \cup R_k$. By $k \geqslant 2$ and the simpleness of G, $r_1 \geqslant 0$ and $r_i \geqslant 1$ $(i = 2, 3, \ldots, k)$. Thus by the equality (1) we have

$$d_T(x) + d_T(y) \leqslant |A| + |R_k| = n - 2 - \sum_{i=1}^{k-1} r_i \leqslant n - k. \tag{4}$$

It follows from Equality (3) and Inequality (4) that $d_G(x) + d_G(y) \leqslant 2(k-1) + (n-k) = n + k - 2$, which contradicts our assumption. Therefore, $d_k(G) \leqslant 2$.

For $k \geqslant 2$, the generalized wheel $W(k-2, 4)$ has order $k+2$, connectivity k, k-diameter is 3. However, the degree-sum of any two vertices is equal to $2k = n+k-2$, which does not satisfy our condition. This shows the sharpness of the sum of degree condition.

Theorem 2. Let G be k-connected, $d \geqslant 3$, and $l = \lfloor d/3 \rfloor$. If there is an integer $s (2 \leqslant s \leqslant l+1)$ such that any independent set S with s vertices in G satisfies the condition

$$\sum_{x \in S} d_G(x) \geqslant \begin{cases} n + k - d + 1, & \text{if } s = 2, \\ n + (k+1)(s-1) - d - 1, & \text{if } s \geqslant 3, \end{cases} \tag{5}$$

then $d_k(G) \leqslant d$. Furthermore, Condition (5) is sharp and the upper bound d of k-diameter is best possible.

Proof. Suppose, to the contrary, that $d_k(G) > d$, then there are two vertices x, y in G and a minimum Menger (x, y)-path system $\mathscr{P}_k(G; x, y) = \{P_1, P_2, \ldots, P_k\}$ such that $d_k(G; x,$

$y)=|E(P_k)|=d_k(G)\geqslant d+1\geqslant 4$. Let $P_k=(x=x_0,x_1,x_2,\ldots,x_{r_k},y)$. Consider the subset of the vertices of P_k

$$S=\begin{cases}\{x_{3j}:0\leqslant j\leqslant s-2\}\cup\{y\}, & \text{if } xy\notin E(G),\\ \{x_{3j}:0\leqslant j\leqslant s-1\}, & \text{if } xy\in E(G),\end{cases}$$

then $2\leqslant|S|=s\leqslant l+1$. Noting that $r_k\geqslant d\geqslant 3$, we have $3l\leqslant d\leqslant r_k$ and so $y\notin S$. Since any two vertices of S are not adjacent in G, S is an independent set of G with s vertices, and

$$\sum_{u\in S}d_{P_k}(u)=\begin{cases}2+2(s-2), & \text{if } xy\notin E(G),\\ 1+2(s-1), & \text{if } xy\in E(G).\end{cases}\tag{6}$$

For any two vertices u and v of S, since the diameter between them is at least three on P_k, by Proposition 1 (ii), they have no neighbor vertex in common in A. It follows from Equality (1) that

$$\sum_{u\in S}d_A(u)\leqslant|A|=n-2-\sum_{i=1}^{k}r_i.\tag{7}$$

If $xy\notin E(G)$, then by Proposition 1(iii) we have

$$\sum_{u\in S}d_{R_i}(u)\leqslant r_i+s-1,\ \text{for any } i=1,2,\ldots,k-1.$$

And so, we have

$$\sum_{u\in S}d_B(u)\leqslant\sum_{i=1}^{k-1}(r_i+s-1)=\sum_{i=1}^{k-1}r_i+(k-1)(s-1).\tag{8}$$

It follows from Equality (6), Inequalities (7) and (8) that

$$\begin{aligned}\sum_{u\in S}d_G(u)&=\sum_{u\in S}d_{R_k}(u)+\sum_{u\in S}d_A(u)+\sum_{u\in S}d_B(u)\leqslant\\ &2+2(s-2)+n-2-\sum_{i=1}^{k}r_i+\sum_{i=1}^{k-1}r_i+(k-1)(s-1)=\\ &n+(k+1)(s-1)-2-r_k\leqslant n+(k+1)(s-1)-2-d.\end{aligned}$$

This is contrary to Condition (5).

If $xy\in E(G)$, then $r_1=0$. Similar to Inequality (8), we have

$$\sum_{u\in S}d_B(u)\leqslant 1+\sum_{i=2}^{k-1}(r_i+s-1)=\sum_{i=1}^{k-1}r_i+(k-2)(s-1)+1.\tag{9}$$

It follows from Equality (6), Inequalities (7) and (9) that

$$\begin{aligned}\sum_{u\in S}d_G(u)&=\sum_{u\in S}d_{R_k}(u)+\sum_{u\in S}d_A(u)+\sum_{u\in S}d_B(u)\leqslant\\ &1+2(s-1)+n-2-\sum_{i=1}^{k}r_i+\sum_{i=1}^{k-1}r_i+(k-2)(s-1)+1=\\ &n+(k+1)(s-1)-d-2-(s-3).\end{aligned}$$

This is contrary to Condition (5).

In order to show that Condition (5) is sharp, we consider the generalized wheel $W(k-2,d+2)$, $d\geqslant 3$, $k\geqslant 2$. It has order $n=k+d$, connectivity k, k-diameter $d+1$. However, the degree-sum of vertices in any independent set with $s(\geqslant 2)$ vertices is

$$sk\begin{cases} = n + k - d, & \text{if } s = 2, \\ < n + (k+1)(s-1) - d - 1, & \text{if } s \geqslant 3, \end{cases}$$

which do not satisfy Condition (5). This shows the sharpness of Condition (5).

For $d \geqslant 3$ and $k \geqslant 2$, the generalized wheel $W(k-2, d+1)$ is an example showing that the upper bound d of k-diameter can not be improved under Condition (5) for $s=2$. To show the upper bound d of k-diameter is best for $s \geqslant 3$, we construct a class of graphs $H(k,d)$ obtained from the union of a complete graph K_{k-1} of order $k-1$ and a path P_{d+1} of order $d+1$ by adding edges joining each of vertices of K_{k-1} to all the vertices of P_{d+1}. For $k \geqslant 2$ and $d \geqslant 3$, $H(k,d)$ has order $k+d$, connectivity k, independence number $\lfloor d/2 \rfloor + 1$ and k-diameter d. Any independent set S with $s(s \geqslant 2)$ vertices in $H(k,d)$ is located in P_{d-1}, and the degree-sum of vertices of S

$$\sum_{u \in S} d_H(u) \geqslant 2k + (k+1)(s-2) = n + (k+1)(s-1) - d - 1$$

satisfies Condition (5) for $s \geqslant 3$. However, if $s=2$, then there are two nonadjacent vertices in G, for example, the two end-vertices of P_{d+1}, such that their degree-sum is equal to $2k = n+k-d$, which does not satisfy Condition (5) for $s=2$. The proof thus is completed.

Remark. The condition $d \geqslant 3$ in Theorem 2 can not be modified as $d=2$. For example, let G be a graph obtained from the union of K_4 and K_2 with vertices x and y by adding four edges joining x to two vertices of K_4 and joining y to the remaining vertices of K_4. G has order $n=6$, the independence number 2, the connectivity 3. If let $k=2$, then the sum of degree condition in Theorem 2 is satisfied for $s=d=2$. However, $d_2(G)=3>2$. Note that x and y do not satisfy the sum of degree condition in Theorem 1. Thus Theorem 1 and Theorem 2 hold independently.

Corollary. Let G be k-connected, $d \geqslant 3$ and $k \geqslant 2$. If $d_G(x)+d_G(y) \geqslant n+k+1-d$ for any two nonadjacent vertices x and y of G, then $d_k(G) \leqslant d$.

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