Discrete Mathematics 243 (2002) 291–298

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

On restricted edge-connectivity of graphs ☆

Jun-Ming Xu, Ke-Li Xu

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China

Received 7 July 1998; revised 19 July 1999; accepted 26 February 2001

Abstract

This paper considers the concept of restricted edge-connectivity, and relates that to the edge-degree of a connected graph. The author gives some necessary conditions for a graph whose restricted edge-connectivity is smaller than its minimum edge-degree, then uses these conditions to show some large classes of graphs, such as all connected edge-transitive graphs except a star, and all connected vertex-transitive graphs with odd order or without triangles, have equality of the restricted edge-connectivity and the minimun edge-degree. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 05C40

Keywords: Connectivity; Restricted edge-connectivity; Transitive graphs

1. Introduction

We follow [2] for graph-theoretical terminology and notation not defined here. A graph $G=(V,E)$ always means a simple graph (without loops and multiple edges), where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set. In the present paper, we consider the restricted edge-connectivity, which is a new graph-theoretical concept and introduced by Esfahanian and Hakimi [4].

In this paper, we call a disconnected graph, a triangle, or a star trivial and all other graphs non-trivial. Let G be a non-trivial graph and $S \subseteq E(G)$. If $G-S$ is disconnected and contains no isolated vertices, then S is called a restricted edge-cut of G. The restricted edge-connectivity of G, denoted by $\lambda'(G)$, is defined as the minimum cardinality over all restricted edge-cuts of G. The restricted edge-connectivity provides a more accurate measure of fault-tolerance of networks than the classical edge-connectivity (see [3]). Thus, it has received much attention recently (see, for example, [3,4,6–9,11,13–15]).

☆ The work was supported partially by NNSF of China (No.19971086).

E-mail address: xujm@ustc.edu.cn (J.-M. XU).

PII: S0012-365X(01)00232-1

Let G be a graph. For $e=xy\in E(G)$, let $\xi_G(e)=d_G(x)+d_G(y)-2$ and $\xi(G)=\min\{\xi_G(e)\colon e\in E(G)\}$. The parameter $\xi(G)$ is called the minimum edge-degree of G. It has been shown in [4] that for any non-trivial graph G, $\lambda'(G)$ certainly exists and satisfies the following inequality:

$$\lambda'(G)\leqslant\xi(G). \qquad (1)$$

Let G be a non-trivial graph. If $\lambda'(G)=\xi(G)$, then G is called optimal; otherwise G is non-optimal. We are interested in finding some classes of optimal graphs. Some of them have been found in [3,6–9,11,14,15]. In this paper, we will give some necessary conditions for a non-optimal graph. From these we will obtain some large classes of optimal graphs, such as all non-trivial edge-transitive graphs, and all connected vertex-transitive graphs with odd order or without triangles. Some classes of optimal graphs given in [3,7,9,14] can easily be deduced from our results.

2. Notation and preliminary results

Let $G=(V,E)$ be a graph. For two disjoint non-empty subsets X and Y of V, let $(X,Y)=\{e=xy\in E\colon x\in X$ and $y\in Y\}$. For the sake of convenience, we write x for the single vertex set $\{x\}$. If $Y=\bar{X}=V\setminus X$, then we write $\partial(X)$ for $(X,\bar{X})$ and $d(X)$ for $|\partial(X)|$. The following inequality is well known (see [10], Problem 6.48):

$$d(X\cap Y)+d(X\cup Y)\leqslant d(X)+d(Y). \qquad (2)$$

A restricted edge-cut S of G is called a λ'-cut if $|S|=\lambda'(G)$. It is clear for any λ'-cut S that $G-S$ has just two connected components. Let X be a proper subset of V. If $\partial(X)$ is a λ'-cut of G, then X is called a fragment of G. It is clear that if X is a fragment of G, then so is $\bar{X}$. Let

$$r(G)=\min\{|X|\colon X \text{ is a fragment of } G\}.$$

Obviously, $2\leqslant r(G)\leqslant\frac{1}{2}|V|$. A fragment X is called an atom of G if $|X|=r(G)$.

Theorem 1. *A non-trivial graph G is optimal if and only if $r(G)=2$.*

Proof. Let $r(G)=2$. Then there exists an atom $X=\{x,y\}$ such that $d(X)=\lambda'(G)=\xi_G(e)$ with $e=xy\in E(G)$. It follows from (1) and the definition of $\xi(G)$ that $\xi(G)\leqslant\xi_G(e)=d(X)=\lambda'(G)\leqslant\xi(G)$, and hence G is optimal.

Conversely, if G is optimal there exists an edge $e=xy$ of G such that

$$\lambda'(G)=\xi(G)=\xi_G(e)=d_G(x)+d_G(y)-2.$$

Now, let $X=\{x,y\}$. Then $r(G)=2$ if $G-\partial(X)$ has no isolated vertices. Suppose on the contrary that $G-\partial(X)$ contains an isolated vertex u. Obviously, $1\leqslant d_G(u)\leqslant 2$.

If $d_G(u)=1$, then we assume, without loss of generality, that u is adjacent to y. Thus

$$d_G(x)+d_G(y)-2=\xi(G)\leqslant d_G(y)+d_G(u)-2=d_G(y)-1.$$

This implies that $d_G(x)=1$. It follows that

$$\lambda'(G) \leqslant |\{yz\colon d_G(z) \geqslant 2\}| \leqslant d_G(y)-2=(d_G(x)+d_G(y)-2)-1=\xi(G)-1,$$

a contradiction.

In the case $d_G(u)=2$, the vertex u is adjacent to x and y. Then,

$$d_G(x)+d_G(y)-2=\xi(G) \leqslant d_G(y)+d_G(u)-2=d_G(y).$$

This yields that $d_G(x)=2$, and analogously, we obtain $d_G(y)=2$. Therefore, G is a triangle. This contradiction completes the proof. □

3. Properties of atoms of non-optimal graphs

Lemma 2. *Let G be a non-optimal graph, F a fragment of G, U a proper subset of F, and I the set of all isolated vertices in $G-\partial(U)$. If $I \subseteq U$ and $|(I,\bar{F})| \geqslant |(I,F\setminus U)|$, then $F\setminus I$ is a fragment of G.*

Proof. If $I=\emptyset$, then there is nothing to prove. Suppose $I\neq\emptyset$ below. Let $Y=F\setminus I$ and $F'=F\setminus U$. Then $Y\neq\emptyset$ and $F'\neq\emptyset$, since $I\subseteq U$ and U is a proper subset of F. Let Z be the set of all isolated vertices in $G-\partial(Y)$. If $Z=\emptyset$, then Y is a restricted edge-cut of G. By the assumption $|(I,\bar{F})| \geqslant |(I,F')|$, we have

$$\lambda'(G) \leqslant d(Y)=d(F)-|(I,\bar{F})|+|(I,F')| \leqslant d(F)=\lambda'(G).$$

This implies that Y is a fragment of G, and so the conclusion holds if $Z=\emptyset$.

The rest is to show $Z=\emptyset$. Suppose on the contrary that $Z\neq\emptyset$. Our aim is to deduce a contradiction.

First, we show that $(x,\bar{F})\neq\emptyset$ for any $x\in I$. At the end, we let $I'=\{x\in I\colon (x,\bar{F})=\emptyset\}$. If $I'\neq\emptyset$, then $N_G(I')\subseteq F'$, since $(I,U\setminus I)=\emptyset$ by the assumption. Let $Z'=(Z\cap F')\setminus N_G(I')$, and let $W=(Y\cup I')\setminus Z'$. Then it is easy to see that $G-\partial(W)$ has no isolated vertices. Thus, $\partial(W)$ is a restricted edge-cut of G. Noticing $|(I\setminus I',\bar{F})|=|(I,\bar{F})| \geqslant |(I,F')| \geqslant |(I',F')| \geqslant |I'|>0$, we have $I\setminus I'\neq\emptyset$, and

$$|(I\setminus I',\bar{F})| \geqslant |(I,F')| \geqslant |(I',F'\setminus Z')|+|(I\setminus I',F'\setminus Z')| > |(I\setminus I',F'\setminus Z')|.$$

Thus we have

$$\begin{aligned}\lambda'(G) &\leqslant d(W)=d(F)-|(Z',\bar{F})|-|(I\setminus I',\bar{F})|+|(I\setminus I',F'\setminus Z')|\\ &< d(F)-|(Z',\bar{F})| \leqslant d(F)=\lambda'(G).\end{aligned}$$

The contradiction implies $I'=\emptyset$, i.e., $(x,\bar{F})\neq\emptyset$ for any $x\in I$. Thus we have

$$|(y,I'')|=|N_G(y)\cap I''| \leqslant |(N_G(y)\cap I'',\bar{F})|, \quad \forall\ y\in Z,\ I''\subseteq I. \tag{3}$$

Second, we assert that $Z=N_G(I)\cap F'$. The fact $I'=\emptyset$ implies $Z\subseteq Y$. Since Z is the set of all isolated vertices in $G-\partial(F\setminus I)$ and $G-\partial(F)$ has no isolated vertices, $Z\subseteq N_G(I)$. But $N_G(I)\cap U=\emptyset$ since $I\subseteq U$ is the set of the isolated vertices in

$G - \partial(U)$ by the assumption. Thus we have $Z \subseteq N_G(I) \cap F'$. On the other hand, if $(N_G(I) \cap F') \backslash Z \neq \emptyset$, then $Y \backslash Z \neq \emptyset$ and $G - \partial(Y \backslash Z)$ contains no isolated vertices, because F is a fragment of G. It is clear that $(I,Z) \neq \emptyset$ since $I \neq \emptyset$ and $\emptyset \neq Z \subseteq N_G(I) \cap F'$. Combining these with the assumption $|(I,\bar{F})| \geqslant |(I,F')|$, we have

$$\lambda'(G) \leqslant d(Y \setminus Z) = d(F) - |(I,\bar{F})| - |(Z,\bar{F})| + |(I,F' \setminus Z)|$$
$$< d(F) - (|(I,\bar{F})| - |(I,F')|) \leqslant d(F) = \lambda'(G).$$

The contradiction implies $(N_G(I) \cap F') \backslash Z = \emptyset$. Thus $Z = N_G(I) \cap F'$.

Third, we have that $(z,\bar{F}) \neq \emptyset$ for any $z \in Z$. Otherwise z is an isolated vertex in $G - \partial(U)$, which implies $z \in I$, a contradiction. Thus we have

$$|(x,Z'')| = |N_G(x) \cap Z''| \leqslant |(N_G(x) \cap Z'', \bar{F})|, \quad \forall\, x \in I, \;\; Z'' \subseteq Z. \tag{4}$$

Lastly, let $y \in Z$ and let $x \in N_G(y) \cap I$. A contradiction can be deduced as follows:

$$\begin{aligned}
\xi(G) &\leqslant d_G(x) + d_G(y) - 2 = |(x,\bar{F})| + |(x, Z \setminus \{y\})| + |(y,\bar{F})| + |(y, I \setminus \{x\})| \\
&\leqslant |(x,\bar{F})| + |(N_G(x) \cap (Z \setminus \{y\}), \bar{F})| + |(y,\bar{F})| + |(N_G(y) \cap (I \setminus \{x\}), \bar{F})| \\
&= (|(x,\bar{F})| + |(N_G(y) \cap (I \setminus \{x\}), \bar{F})|) \\
&\quad + (|(y,\bar{F})| + |(N_G(x) \cap (Z \setminus \{y\}), \bar{F})|) \\
&= |(N_G(y) \cap I), \bar{F}| + |(N_G(x) \cap Z, \bar{F})| \\
&\leqslant |(I,\bar{F})| + |(Z,\bar{F})| \leqslant |(F,\bar{F})| = d(F) = \lambda'(G) < \xi(G),
\end{aligned}$$

where the first equality holds because of the fact $Z = N_G(I) \cap F'$ and $(Z, Y \backslash Z) = \emptyset$, and the second inequality holds from (3) and (4). The proof is complete. □

Theorem 3. *Let G be a non-optimal graph. Then any two distinct atoms of G are disjoint.*

Proof. Let X and X' be two distinct atoms of G. Then $d(X) = d(X') = \lambda'(G) < \xi(G)$ and $|X| = |X'| = r(G) \geqslant 3$ by Theorem 1. Let

$$A = X \cap X', \quad B = X \cap \overline{X'}, \quad C = \bar{X} \cap X', \text{ and } D = \bar{X} \cap \overline{X'}.$$

Then $|B| = |C| = r(G) - |A| \geqslant 1$ and $|D| \geqslant |A|$. Suppose on the contrary that $A \neq \emptyset$. We will derive contradictions by considering two cases, separately.

Case 1: If $G - \partial(A)$ contains some isolated vertices, then let I be the set of all isolated vertices in $G - \partial(A)$. Then obviously, $I \subseteq A$, $(I,B) \neq \emptyset$ and $(I,C) \neq \emptyset$, because $\partial(X)$ and $\partial(X')$ are λ'-cuts of G. We can assume, without loss of generality, that $|(I,C)| \geqslant |(I,B)|$. Let $F = X$, $U = A \subset F$. Then $X \setminus I (= F \setminus I)$ is a fragment of G by Lemma 2. However, $X \backslash I$ is a proper subset of X. This contradicts the assumption that X is an atom of G.

Case 2: If $G - \partial(A)$ contains no isolated vertices, then A is a restricted edge-cut of G. This implies that $|A| \geqslant 2$ and $d(A) > \lambda'(G)$. By (2) we have

$$d(D) = d(X \cup X') \leqslant d(X) + d(X') - d(X \cap X') < \lambda'(G). \tag{5}$$

This implies that $G - \partial(D)$ certainly contains some isolated vertices, so let I be the set of all isolated vertices in $G - \partial(D)$. Obviously, $I \subseteq D$. If $D' = D \backslash I \neq \emptyset$, then $\partial(D')$ is a restricted edge-cut of G since $G - \partial(D')$ has no isolated vertices, and so from (5) we have

$$\lambda'(G) \leqslant d(D') = d(D) - \sum_{u \in I} d_G(u) < d(D) < \lambda'(G).$$

This contradiction implies $I = D$. Without loss of generality, we assume that $|(D,B)| \geqslant |(D,C)|$. Let $F = \bar{X}$, $I = U = D \subset F$. Thus, $C(= \bar{X} \backslash D = F \backslash I)$ is a fragment of G that is properly contained in X' by Lemma 2. This contradicts the assumption that X' is an atom of G. The proof is complete. □

Remark. Any cycle of length greater than three shows that Theorem 3 is not valid if G is optimal.

Theorem 4. *Let G be a non-optimal graph. If G is k-regular, then $r(G) \geqslant k \geqslant 3$.*

Proof. By Theorem 1, $r(G) \geqslant 3$, and obviously $k \geqslant 3$. Let X be an atom of G. Then $r = r(G) = |X|$ and $d(X) = \lambda'(G) < \xi(G) = 2k - 2$. Considering the sum of degrees of all vertices in X, we have

$$kr = \sum_{x \in X} d_G(x) \leqslant r(r-1) + d(X) < r^2 - r + 2k - 2 = kr - (k - r - 1)(r - 2).$$

This implies $r(G) \geqslant k$ since $k \geqslant 3$. □

Theorem 5. *Let G be a connected vertex-transitive graph with degree k ($\geqslant 3$), and let X be an atom of G. If G is non-optimal, then,*

(i) *$G[X]$ is a vertex-transitive subgraph of G with degree of $k - 1$ containing a triangle;*
(ii) *G has even order and there is a partition $\{X_1, X_2, \ldots, X_m\}$ of V such that $G[X_i] \cong G[X]$ for each $i = 1, 2, \ldots, m$, $m \geqslant 2$.*

Proof. (i) Since a vertex-transitive graph is regular, we have $|X| \geqslant k \geqslant 3$ by Theorem 4. Let x and y be two distinct vertices in X. Then there exists $\pi \in \Gamma(G)$, the automorphism group of G, such that $\pi(x) = y$ because G is vertex-transitive. Hence $\pi(X)$ is also an atom of G. Let $X' = \pi(X)$. Then $X \cap X' \neq \emptyset$ since $y \in X \cap X'$. This implies $X' = X$ by Theorem 3. Let

$$\Pi = \{\pi \in \Gamma(G)\colon \pi(X) = X\}, \quad \Psi = \{\pi \in \Pi\colon x \in X \Rightarrow \pi(x) = x\}.$$

Clearly Π is a subgroup of $\Gamma(G)$, and the constituent of Π on X acts transitively and Ψ is a normal subgroup of Π. Thus there is an injective homomorphism from the quotient group Π/Ψ to $\Gamma(G[X])$ whereby each coset of Ψ is associated with the restriction to X of any representative. This shows that $G[X]$ is vertex-transitive.

Let $G[X]$ have degree t. Then $t \leqslant k-1$; from this and Theorem 4 we have

$$2(k-1) \geqslant d(X)+1=(k-t)r(G)+1 \geqslant (k-t)k+1.$$

This implies that

$$t \geqslant \left\lceil \frac{k^2-2k+3}{k} \right\rceil = k-1.$$

Namely $G[X]$ has degree $(k-1)$. Note that $G[X]$ certainly contains cycles since $G[X]$ is $(k-1)$-regular and $k \geqslant 3$, and has at least $2k-2$ vertices if it contains no triangles (see Exercise 1.7.4(a) in [2]). In this case, however, we see that $2k-2 \leqslant |X| = \lambda'(G) \leqslant 2k-3$, a contradiction. It follows that $G[X]$ contains a triangle.

(ii) Let y be any element in $\bar{X}$. Since G is vertex-transitive, there exists $\sigma \in \Gamma(G)$ such that $\sigma(x)=y$ for a fixed x in X. $\sigma(X)$ is an atom of G. Let $X_y=\sigma(X)$. Then $X \cap X_y = \emptyset$ by Theorem 3 since $y \notin X$, and $G[X] \cong G[X_y]$ since there exists an isomorphism σ between $G[X]$ and $G[X_y]$. Thus there are at least two atoms of G. It follows that for each vertex y in G there is an atom X_y that contains y such that $G[X_y] \cong G[X]$, and either $X_y = X_z$ or $X_y \cap X_z = \emptyset$ for any two distinct vertices y and z of G. These atoms $X_1, X_2, \ldots, X_m$ of G form a partition of $V(G)$, and $G[X_i] \cong G[X]$ for each $i=1,2,\ldots,m, m \geqslant 2$. So

$$|V| = m|X| = |V|k - m|X|(k-1) = 2|E(G)| - 2m|E(G[X])|.$$

This implies that G has even order. The proof is complete. □

4. Some classes of optimal graphs

Theorem 6 (Xu, [14]). *Let G be a connected vertex-transitive graph. If it either contains no triangles or has odd order, then G is optimal.*

This is a direct consequence of Theorem 5.

A well-known class of vertex-transitive graphs, very frequently employed in the construction of distributed-memory parallel computing systems, is the k-cube Q_k $(k \geqslant 2)$. It is k-regular bipartite, and so contains no triangles. Thus from Theorem 6 we have the following result immediately.

Corollary 7 (Esfahanian, [3]). *The k-cube Q_k is optimal.*

Let C_d be a cycle of length d. The k-dimensional toroidal mesh $C(d_1, d_2, \ldots, d_k)$, studied by Ishigami [5], can be represented as the cartesian product $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_k}$.

It is vertex-transitive, and contains no triangles if $d_i \geqslant 4$ (see, for example, [12]). Hence, from Theorem 6 we deduce the following result immediately.

Corollary 8. *The k-dimensional toroidal mesh* $C(d_1, d_2, \ldots, d_k)$ *is optimal if* $d_i \geqslant 4$ *for each* $i = 1, 2, \ldots, k$.

Another important class of vertex-transitive graphs used in the design of networks are the circulant graphs. A circulant graph, denoted by $G(n; a_1, a_2, \ldots, a_k)$, where $0 < a_1 < \cdots < a_k \leqslant n/2$, has vertices $0, 1, 2, \ldots, n-1$ and edge ij if and only if $|j-i| \equiv a_t \pmod n$ for some t, $1 \leqslant t \leqslant k$ (see, for example, [1]). If $a_k \neq n/2$, it is $2k$-regular. Otherwise, it is $(2k-1)$-regular.

Corollary 9 (Li and Li, [7]). *Any connected circulant graph* $G(n; a_1, a_2, \ldots, a_k)$, $n \geqslant 4$, *is optimal if either it contains no triangles or* $a_k \neq n/2$.

Proof. Let $G = G(n; a_1, a_2, \ldots, a_k)$. By Theorem 6, we need to only prove that G is optimal if $a_k \neq n/2$. Suppose on the contrary that G is non-optimal. Then, by Theorem 5, there is an integer $m \geqslant 2$ such that for any atom X of G, $n = m|X|$ and $G[X]$ is $(2k-1)$-regular. Thus there is a subset $\{b_1, b_2, \ldots, b_t\}$ of the set $\{a_1, a_2, \ldots, a_k\}$ such that g.c.d.$(n, b_1, b_2, \ldots, b_t) = m$ and $G[X] \cong G(n/m; b_1/m, b_2/m, \ldots, b_t/m)$, where $b_1 < b_2 < \cdots < b_t$. Since $a_k \neq \frac{n}{2}$ we have $b_t/m \neq n/2m$. Thus $G[X]$ is even regular, which contradicts the fact that $G[X]$ is $(2k-1)$-regular. □

Theorem 10 (Li and Li, [10]). *All non-trivial edge-transitive graphs are optimal.*

Proof. Let G be a non-trivial edge-transitive graph. Suppose on the contrary that G is non-optimal. Let X be an atom of G. Then $|X| = r(G) \geqslant 3$ by Theorem 1. Let $e = xy$ be an edge in $G[X]$ and $e' = yz$ be an edge in $\partial(X)$, $z \in \bar{X}$. Since G is edge-transitive, there is $\sigma \in \Gamma(G)$ such that $\sigma(\{x, y\}) = \{y, z\}$. Hence $\sigma(X)$ is also an atom of G. Let $X' = \sigma(X)$. Then $X \neq X'$ since $z \in X'$ and $z \notin X$. But since $y \in X \cap X'$, we have $X = X'$ by Theorem 3. This is a contradiction and so G is optimal. This completes the proof. □

Acknowledgements

The author would like to thank the three anonymous referees for their kind help and valuable suggestions which led to an improvement in the presentation form.

References

[1] F. Boesch, R. Tindell, Circulants and their connectivities, J. Graph Theory 8 (1984) 487–499.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.

[3] A.H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Trans. Comput. 38 (11) (1989) 1586–1591.
[4] A.H. Esfahanian, S.L. Hakimi, On computing a conditional edge-connectivity of a graph, Inform. Process. Lett. 27 (1988) 195–199.
[5] Y. Ishigami, The wide-diameter of the n-dimensional toroidal mesh, Networks 27 (1996) 257–266.
[6] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large miltiprocessor systems, IEEE Trans. Comput. 43 (2) (1994) 218–221.
[7] Q.L. Li, Q. Li, Reliability analysis of circulants, Networks 31 (1998) 61–65.
[8] Q.L. Li, Q. Li, Refined connectivity properties of abelian Cayley graphs, Chinese Ann. Math. B 19 (4) (1998) 409–414.
[9] Q.L. Li, Q. Li, Super edge connectivity properties of connected edge symmetric graphs, Networks 33 (1999) 147–159.
[10] L. Lovasz, Combinatorial Problems and Exercises, North-Holland, Amsterdam, New York, Oxford, 1979.
[11] J. Wu, G. Guo, Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets, IEEE Trans. Comput. 47 (8) (1998) 888–893.
[12] J.M. Xu, Connectivity of cartesian product digraphs and fault-tolerant routings of generalized hypercubes, Appl. Math.-JCU 13B (2) (1998) 179–187.
[13] J.M. Xu, Some results on R_2-edge-connectivity of even regular graphs, Appl. Math. -JCU 14B (3) (1999) 366–370.
[14] J.M. Xu, Restricted edge-connectivity of vertex-transitive graphs, Chinese J. Contemp. Math. 21 (4) (2000) 369–374.
[15] J.M. Xu, On conditional edge-connectivity of graphs, Acta Math. Appl. Sinica 16B (4) (2000) 414–419.