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Note on Bounded Length Paths of De Bruijn Digraphs*

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Abstract: Imase *et al* showed that for any two distinct vertices x and y of the de Bruijn digraph $B(d,k)$, there are $d-1$ internally disjoint (x,y)-paths of length at most $k+1$. A very short proof is given in this note.

Key words: bounded length paths; Menger's theorem; de Bruijn digraphs

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Following Fiol *et al* [1], the de Bruijn digraph, denoted by $B(d,k)$, can be defined as the $(k-1)$th iterated line digraph of K_d^+, where K_d^+ denotes a digraph obtained from the complete symmetric digraph with $d(\geq 2)$ vertices by attaching a loop at each vertex. In other words, $B(d,k)$ is recursively defined as follows.

$$B(d,1) = K_d^+; \quad B(d,k) = L^{k-1}(K_d^+),\ k \geq 2.$$

The de Bruijn digraph has many desirable structural properties, the most of which are contained in an excellent survey by Bermond and Peyrat[2]. The de Bruijn digraph is a suitable model for interconnection networks in parallel and distributed processing systems, and is regarded to be a good competitor for the hypercube and might constitute the next generation of parallel architectures.

Let the vertex set of K_d^+ be $\{0,1,\cdots,d-1\}$. By the definition, any vertex x of $B(d,k)$ is a directed walk $(x_1,x_2,\cdots,x_k)$ of length $k-1$ in K_d^+, where $x_i \in \{0,1,\cdots,d-1\}$, $1 \leq i \leq k$. We may write $x = x_1x_2\cdots x_k$. The vertex x is adjacent to vertices of the form $y = x_2x_3\cdots x_kx_{k+1}$ with (x_k,x_{k+1}) being an edge of K_d^+. It follows that a directed walk of length n with the origin x in $B(d,k)$ can be expressed as a sequence $(x_1,x_2,\cdots,x_k,x_{k+1},\cdots,x_{k+n})$ of the vertices in K_d^+, where (x_i,x_{i+1}) is an edge of K_d^+ for each $i = 1,2,\cdots,k+n-1$.

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Imase *et al*[3] showed the following theorem, which is a classic and basic result and frequently occurs in applications and literature. But the original proof is very long. We give a very short proof here.

Theorem 1 For any two distinct vertices x and y of $B(d,k)$, there are $d-1$ internally disjoint (x,y)-paths of length at most $k+1$.

Proof We proceed by induction on $k \geq 1$. Since $B(d,k) = K_d^+$, the theorem is true for $k=1$ clearly. Suppose $k \geq 2$ and the theorem holds for any two vertices of $B(d,k-1)$. Assume that x and y are two distinct vertices of $B(d,k)$. Then x and y correspond to two edges of $B(d,k-1)$ since $B(d,k) = L(B(d,k-1))$. Let such two edges be $x = (w,w')$ and $y = (v,v')$.

If $w' \neq v$, then by the induction hypothesis, there are $d-1$ internally disjoint (w',v)-paths of length at most k in $B(d,k-1)$, from which we can easily induce $d-1$ internally disjoint (x,y)-paths of length at most $k+1$ in $B(d,k)$.

If $w' = v$, then (x,y) is an edge of $B(d,k)$, and x and y can be written as

$$x = x_1x_2\cdots x_{k-1}x_k, \quad y = x_2x_3\cdots x_kx_{k+1},$$

where $x_1, x_2, \cdots, x_k, x_{k+1} \in \{0,1,\cdots,d-1\}$, and, hence $(x_1,x_2,\cdots,x_k,x_{k+1})$ is a walk of length k in K_d^+. We construct $d-1$ internally disjoint (x,y)-walks $W_1, W_2, \cdots, W_{d-1}$ of length at most $k+1$ in $B(d,k)$ as follows.

$$W_1 = (x_1,x_2,\cdots,x_{k-1},x_k,x_{k+1}),$$

$$W_j = (x_1,x_2,\cdots,x_k,u_j,x_2,x_3,\cdots,x_k,x_{k+1}),\ j = 2,3,\cdots,d-1,$$

where $u_2,\cdots,u_{d-1}$ are $d-2$ distinct elements in $\{0,1,\cdots,d-1\} \setminus \{x_1,x_{k+1}\}$. It is clear that W_1 is of length one and W_j is of length $k+1$ for each $j = 2,3,\cdots,d-1$. In order to prove these (x,y)-walks are internally disjoint in $B(d,k)$, it is sufficient to prove $W_2,\cdots,W_{d-1}$ are internally disjoint in $B(d,k)$.

Suppose to the contrary that there are some i and j $(2 \leq i \neq j \leq d-1)$ such that W_i and W_j have common vertices rather than x and y. Let u be the first internally common vertex of W_i and W_j from x to y. Assume the section $W_i(x,u)$ is of length a and the section $W_j(x,u)$ is of length b. Then $2 \leq a, b \leq k-1$. Let u' and u'' be in-neighbors of u on W_i and W_j, respectively. Then $u'' \neq u'$. Since u can be reached in a steps from x along W_i and in b steps from x along W_j, then it can be written as

$$\begin{aligned} u &= x_{a+1}x_{a+2}\cdots x_ku_ix_2\cdots x_ax_{a+1} \\ &= x_{b+1}x_{b+2}\cdots x_ku_jx_2\cdots x_bx_{b+1}. \end{aligned}$$

From this expression, we have $x_a = x_b$ since $2 \leq a, b \leq k-1$, namely

$$u' = x_ax_{a+1}\cdots x_ku_ix_2\cdots x_a = x_bx_{b+1}\cdots x_ku_jx_2\cdots x_b = u'',$$

a contradiction. Note $W_2,\cdots,W_{d-1}$ may be not paths, but each of them must contain a path as its subgraph, and, thus, the theorem follows.

References

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关于 de Bruijn 图中限长路的注记

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摘要: Imase 等人证明了: 对于 de Bruijn 有向图 $B(d, k)$ 中任何两个不同的顶点 x 和 y，存在 $d-1$ 条内点不交且长度都不超过 $k+1$ 的 (x, y) 路. 但证明很长而且包含许多令人厌烦的验证. 本文给出它的简单证明.

关键词: 限长路; Menger 定理; de Bruijn 有向图