Note on Bounded Length Paths of De Bruijn Digraphs

XU Jun-ming, TAO Ying-feng, XU Ke-li

(Department of Mathematics, USTC, Hefei, Anhui, 230026, China)

Abstract: Imase et al showed that for any two distinct vertices \( x \) and \( y \) of the de Bruijn digraph \( B(d,k) \), there are \( d-1 \) internally disjoint \((x,y)\)-paths of length at most \( k+1 \). A very short proof is given in this note.

Key words: bounded length paths; Menger’s theorem; de Bruijn digraphs

CLC number: O157.5

Document code: A

AMS Subject Classifications (1991): Primary 05C12; 05C40

Following Fiol et al [1], the de Bruijn digraph, denoted by \( B(d,k) \), can be defined as the \((k-1)\)th iterated line digraph of \( K_k^d \), where \( K_k^d \) denotes a digraph obtained from the complete symmetric digraph with \( d(\geq 2) \) vertices by attaching a loop at each vertex. In other words, \( B(d,k) \) is recursively defined as follows.

\[
B(d,1) = K_k^d; \quad B(d,k) = L^{k-1}(K_k^d), \quad k \geq 2.
\]

The de Bruijn digraph has many desirable structural properties, the most of which are contained in an excellent survey by Bermond and Peyrat [2]. The de Bruijn digraph is a suitable model for interconnection networks in parallel and distributed processing systems, and is regarded to be a good competitor for the hypercube and might constitute the next generation of parallel architectures.

Let the vertex set of \( K_k^d \) be \( \{0,1,\ldots,d-1\} \). By the definition, any vertex \( x \) of \( B(d,k) \) is a directed walk \((x_1,x_2,\ldots,x_k)\) of length \( k - 1 \) in \( K_k^d \), where \( x_i \in \{0,1,\ldots,d-1\}, 1 \leq i \leq k \). We may write \( x = x_1x_2\ldots x_k \). The vertex \( x \) is adjacent to vertices of the form \( y = x_2x_3\ldots x_{i+1} \) with \((x_i,x_{i+1})\) being an edge of \( K_k^d \). It follows that a directed walk of length \( n \) with the origin \( x \) in \( B(d,k) \) can be expressed as a sequence \((x_1,x_2,\ldots,x_k,x_{k+1},\ldots,x_{k+n})\) of the vertices in \( K_k^d \), where \((x_i,x_{i+1})\) is an edge of \( K_k^d \) for each \( i = 1,2,\ldots,k+n-1 \).
It is clear that, then it can be shown that the theorem holds for any two vertices of $B(d, k)$.

**Theorem 1** For any two distinct vertices $x$ and $y$ of $B(d, k)$, there are $d - 1$ internally disjoint $(x, y)$-paths of length at most $k + 1$.

**Proof** We proceed by induction on $k \geq 1$. Since $B(d, k) = K_d$, the theorem is true for $k = 1$ clearly. Suppose $k \geq 2$ and the theorem holds for any two vertices of $B(d, k - 1)$. Assume that $x$ and $y$ are two distinct vertices of $B(d, k)$. Then $x$ and $y$ correspond to two edges of $B(d, k - 1)$ since $B(d, k) = L(B(d, k - 1))$. Let such two edges be $x = (w, w')$ and $y = (v, v')$.

If $w' \neq v$, then by the induction hypothesis, there are $d - 1$ internally disjoint $(w', v)$-paths of length at most $k$ in $B(d, k - 1)$, from which we can easily induce $d - 1$ internally disjoint $(x, y)$-paths of length at most $k + 1$ in $B(d, k)$.

If $w' = v$, then $(x, y)$ is an edge of $B(d, k)$, and $x$ and $y$ can be written as

$$x = x_1x_2 \ldots x_k, 1 \neq x_1x_2 \ldots x_kx_{k+1},$$

where $x_1, x_2, \ldots, x_k, x_{k+1} \in \{0, 1, \ldots, d - 1\}$, and, hence $(x_1, x_2, \ldots, x_k, x_{k+1})$ is a walk of length $k$ in $K_d$. We construct $d - 1$ internally disjoint $(x, y)$-walks $W_1, W_2, \ldots, W_{d-1}$ of length at most $k + 1$ in $B(d, k)$ as follows.

$$W_1 = (x_1, x_2, \ldots, x_k, 1, x_k, x_{k+1}),$$

$$W_j = (x_1, x_2, \ldots, x_k, u_j, x_2, x_3, \ldots, x_k, x_{k+1}), \quad j = 2, 3, \ldots, d - 1,$$

where $u_2, \ldots, u_{d-1}$ are $d - 2$ distinct elements in $\{0, 1, \ldots, d - 1\} \setminus \{x_1, x_{k+1}\}$. It is clear that $W_1$ is of length one and $W_j$ is of length $k + 1$ for each $j = 2, 3, \ldots, d - 1$. In order to prove these $(x, y)$-walks are internally disjoint in $B(d, k)$, it is sufficient to prove $W_2, \ldots, W_{d-1}$ are internally disjoint in $B(d, k)$.

Suppose to the contrary that there are some $i$ and $j$ ($2 \leq i \neq j \leq d - 1$) such that $W_i$ and $W_j$ have common vertices rather than $x$ and $y$. Let $u$ be the first internally common vertex of $W_i$ and $W_j$ from $x$ to $y$. Assume the section $W_i(x, u)$ is of length $a$ and the section $W_j(x, u)$ is of length $b$. Then $2 \leq a, b \leq k - 1$. Let $u'$ and $u''$ be in-neighbors of $u$ on $W_i$ and $W_j$, respectively. Then $u'' \neq u'$. Since $u$ can be reached in $a$ steps from $x$ along $W_i$, and in $b$ steps from $x$ along $W_j$, then it can be written as

$$u = x_{a+1}x_{a+2} \ldots x_ku_ix_2 \ldots x_ax_{a+1}$$

$$= x_{b+1}x_{b+2} \ldots x_ku_jx_2 \ldots x_bx_{b+1}.$$
References


关于 de Bruijn 图中限长路的注记
徐俊明, 陶颖峰, 徐克力
(中国科学技术大学数学系, 合肥 230026)

摘要: Imase 等人证明了对于 de Bruijn 图 \(B(d, k)\) 中任何两个不同的顶点 \(x, y\), 存在 \(d - 1\) 条内点不交且长度都不超过 \(k + 1\) 的 \((x, y)\) 路。本文给出它的简单证明。

关键词: 限长路; Menger 定理; de Bruijn 图