# Restricted Fault Diameter of Hypercube Networks 

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#### Abstract

This paper studies restricted fault diameter of the $n$-dimensional hypercube networks $Q_{n}(n \geq 2)$. It is shown that for arbitrary two vertices $x$ and $y$ with the distance $d$ in $Q_{n}$ and any set $F$ with at most $2 n-3$ vertices in $Q_{n}-\{x, y\}$, if $F$ contains neither of neighbor-sets of $x$ and $y$ in $Q_{n}$, then the distance between $x$ and $y$ in $Q_{n}-F$ is given by $$
D\left(Q_{n}-F ; x, y\right) \begin{cases}=1, & \text { for } d=1 ; \\ \leq d+4, & \text { for } 2 \leq d \leq n-2, n \geq 4 \\ \leq n+1, & \text { for } d=n-1, n \geq 3 \\ =n, & \text { for } d=n\end{cases}
$$

Furthermore, the upper bounds are tight. As an immediately consequence, $Q_{n}$ can tolerate up to $2 n-3$ vertices failures and remain diameter 4 if $n=3$ and $n+2$ if $n \geq 4$ provided that for each vertex $x$ in $Q_{n}$, all the neighbors of $x$ do not fail at the same time. This improves Esfahanian's result.


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## 1 Introduction

In this paper, a graph $G=(V, E)$ always means a simple connected graph (undirected graph without loops and multiple edges) with the vertex-set $V$ and the edge-set $E$. We follow [2] for graph-theoretical terminologies and notations not defined here.

When the underlying topology of an interconnection network is modelled by a graph $G=$ $(V, E)$ in which $V$ represents the set of the processors and $E$ represents the bidirectional communication links connecting pairs of processors, some graph parameters such as connectivity, fault diameter ${ }^{[8]}$ and wide-diameter ${ }^{[7]}$ can be used to analyze the fault tolerance and efficiency of the network with faults (see [1], [7-11]). The study of these parameters is based on the concept of connectivity.

The concept of connectivity, however, has an obvious deficiency. That is that in investigating this concept it has tacitly been assumed that some subsets such as all neighbors of (or all incident edges with) any vertex in the graph can be removed at the same time. In fact, in many practical applications it can be safely assumed that any set of faults in some networks cannot contain all processors which are directly connected to some processor. Consequently, these parameters are inaccurate to measure the reliability and efficiency for such networks (see [3]).

To compensate for this shortcoming, Esfahanian and Hakimi ${ }^{[4]}$ generalized the concept of connectivity by introducing the restricted connectivity based on the assumption that a forbidden faulty set such as all the neighbors of (or all incident edges with) any given vertex cannot be

[^0]removed at the same time. A set $F$ of vertices in $G$ is said to be restricted if $F$ does not contain the neighbor-set of any vertex in $G$. The restricted connectivity $\kappa^{\prime}(G)$ of $G$ is the minimum cardinality $|F|$ of a restricted set $F$ such that $G-F$ is disconnected. It is not difficult to find some connected graphs that have no restricted connectivity. However, in any graph $G$ with at least three vertices, there exists a restricted set. Thus we can define the diameter $D(G-F)$ for any restricted set $F$ of $G$. As the restricted set $F$ is not known in advance, an interesting parameter is
$$
D_{f}(G)=\max \{D(G-F): F \text { is a restricted set of } G \text { and }|F| \leq f\},
$$
which is called the restricted fault diameter of $G$. It is clear that $D_{f}(G)$ is the fault diameter of $G$ if $f=\kappa(G)-1$, where $\kappa(G)$ is the connectivity of $G$. Thus the restricted fault diameter is a generalization of the fault diameter.

The restricted connectivity and the restricted fault diameter in conjunction with the abovementioned parameters and other well-known parameters can provide a more accurate fault tolerance analysis for reliability and efficiency of networks and received much attention (see, for example, $[1,3,4,8,10,11,13]$ ).

In particular, for the $n(\geq 3)$-dimensional hypercube network $Q_{n}$, Esfahanian ${ }^{[3]}$ obtained that $\kappa^{\prime}\left(Q_{n}\right)=2 n-2$ by proving $D_{2 n-3}\left(Q_{n}\right) \leq n+6$. In this paper, we will show that $D_{3}\left(Q_{3}\right)=4$ and $D_{2 n-3}\left(Q_{n}\right)=n+2$ for $n \geq 4$. Latifi ${ }^{[9]}$ also observed this result. His proof, however, is somewhat cumbersome. Using a method completely different than that used of Latifi, we will first prove the following theorem, from which Latif's result follows as a straightforward corollary.

Theorem. Let $x$ and $y$ be arbitrary two vertices with distance $d$ in $Q_{n}(n \geq 2), F$ any set with at most $2 n-3$ vertices in $Q_{n}-\{x, y\}$. If $F$ contains neither of neighbor-sets of $x$ and $y$ in $Q_{n}$, then the distance between $x$ and $y$ in $Q_{n}-F$ is given by

$$
D\left(Q_{n}-F ; x, y\right) \begin{cases}=1, & \text { for } d=1 ; \\ \leq d+4, & \text { for } 2 \leq d \leq n-2, n \geq 4 \\ \leq n+1, & \text { for } d=n-1, n \geq 3 \\ =n, & \text { for } d=n\end{cases}
$$

Furthermore, the upper bounds are tight in the sense that there is a restricted set $F$ with $2 n-3$ vertices in $Q_{n}-\{x, y\}$ such that $D\left(Q_{n}-F ; x, y\right)$ can reach the upper bounds.

## 2 Some Properties of the $n$-dimensional Hypercube

For a given graph $G$ and two vertices $x$ and $y$ in $G$, the length of an $(x, y)$-path $P=\left(x_{0}(=\right.$ $x), x_{1}, \cdots, x_{p-1}, x_{p}(=y)$ ) is the number $p$ of edges in $P$ and will be denoted by $\varepsilon(P)$, where $x_{1}, x_{2}, \cdots, x_{p-1}$ are called internal vertices. For any $0 \leq i<j \leq p$, denote by $P\left(x_{i}, x_{j}\right)$ the subpath $\left(x_{i}, x_{i+1}, \cdots, x_{j-1}, x_{j}\right)$ of $P$. The distance between $x$ and $y$ in $G$, denoted by $D(G ; x, y)$, is the length of a shortest $(x, y)$-path in $G$. The diameter of $G$, denoted by $D(G)$, is the maximum distance between any pair of vertices of $G$. Let $F$ be a proper subset of $V(G)$. A subgraph $H$ of $G$ avoids $F$ if $H$ does not contain any vertex in $F$. For a given vertex $x$ in $G$, we use $N(G ; x)$ to denote the neighbor-set of $x$.

The $n$-dimensional hypercube, termed $n$-cube for short and denoted by $Q_{n}$, can be defined and characterized in a number of ways (cf. [5]). A convenient definition for our purpose is to
express $Q_{n}$ as the cartesian products of $n$ identical $K_{2}$, that is

$$
Q_{1}=K_{2}, \quad Q_{n}=\underbrace{K_{2} \times K_{2} \times \cdots \times K_{2}}_{n(\geq 2)} .
$$

Using this definition, we can express $Q_{n}$ as $K_{2} \times Q_{n-1}$ for $n \geq 2$. This implies that $Q_{n}$ can be obtained from two identical $Q_{n-1}$ by adding edges joining two vertices with the same label. It is often convenient to write $Q_{n}=Q_{n-1} \odot Q_{n-1}$ for $n \geq 2$. The edges between two $Q_{n-1}$ are called cross edges. Note that as an operation on graphs, the Cartesian products satisfy associative and communicative laws. This implies that for any an edge $e$ of $Q_{n}$ there exist two disjoint subgraph $L$ and $R$ of $Q_{n}$ that are isomorphic to $Q_{n-1}$ such that $Q_{n}=L \odot R$ and $e$ is a cross edge in $L \odot R$.

The $n$-cube $Q_{n}$ is widely used in network theory. Thus, it is investigated in depth from many different perspectives (see, for example, $[3,5,6,8,12-14]$ ). These studies have led to the discovery of many properties of $Q_{n}$, some of them will be mentioned below.

Property 1. $\quad Q_{n}$ is a vertex-transitive bipartite graph, and has diameter and connectivity $n$. Furthermore, for any vertex $x$ in $Q_{n}$, there is a unique vertex $y$ in $Q_{n}$ such that distance $D\left(Q_{n} ; x, y\right)=n$.

Property 2. For any pair of vertices $x$ and $y$ with $D\left(Q_{n} ; x, y\right)=d$, there are $n$ internally vertex-disjoint $(x, y)$-paths $P_{1}, P_{2}, \cdots, P_{n}$ such that $d$ of them have length $d$ and the rest have length $d+2$ if $d \leq n-1$; and all have length $n$ if $d=n$.

We will call a set of $n$ internally vertex-disjoint $(x, y)$-paths $P_{1}, P_{2}, \cdots, P_{n}$ in $Q_{n}$ determined by Property 2 to be an $(x, y)$-container with width $n$, denoted by $C_{n}\left(Q_{n} ; x, y\right)=$ $\left\{P_{1}, P_{2}, \cdots, P_{n}\right\}$. We call a $d$-cube $Q_{d}(1 \leq d \leq n)$ to be determined by $x$ and $y$ if it is a subgraph of $Q_{n}$ and $D\left(Q_{n} ; x, y\right)=d$. It is clear from Property 2 that if $D\left(Q_{n} ; x, y\right)=d$, then $d$ paths of length $d$ in $C_{n}\left(Q_{n} ; x, y\right)=\left\{P_{1}, P_{2}, \cdots, P_{n}\right\}$ all are located in the $Q_{d}$ determined by $x$ and $y$.

## 3 The Proof of Theorem

In this section we will give a proof of the Theorem stated in Introduction. Let $x$ and $y$ be arbitrary two vertices in $Q_{n}, F$ any set with at most $2 n-3$ vertices in $Q_{n}-\{x, y\}$ such that $F$ contains neither $N\left(Q_{n} ; x\right)$ nor $N\left(Q_{n} ; y\right)$. We will complete the proof of the Theorem by proving the following two Lemmas.

Lemma 1. If $D\left(Q_{n} ; x, y\right)=n \geq 2$, then $D\left(Q_{n}-F ; x, y\right)=n$.
Lemma 2. If $D\left(Q_{n} ; x, y\right)=d$, then

$$
D\left(Q_{n}-F ; x, y\right) \begin{cases}=1, & \text { for } d=1 \\ \leq d+4, & \text { for } 2 \leq d \leq n-2, n \geq 4 \\ \leq n+1, & \text { for } d=n-1, n \geq 3\end{cases}
$$

Furthermore, these upper bounds are tight in the sense that there is a restricted set $F$ with $2 n-3$ vertices in $Q_{n}-\{x, y\}$ such that $D\left(Q_{n}-F ; x, y\right)$ is equal to the upper bounds provided $D\left(Q_{n} ; x, y\right) \leq n-1$.

Proof of Lemma 1. It is easy to verify that Lemma 1 holds for $n=2$ and 3. We prove Lemma 1 for $n \geq 4$. Noting that $n=D\left(Q_{n} ; x, y\right) \leq D\left(Q_{n}-F ; x, y\right)$, we need only show that there is an $(x, y)$-path of length $n$ in $Q_{n}-F$.

Since $F$ does not contain $N\left(Q_{n} ; x\right)$, there is a vertex $z$ in $N\left(Q_{n} ; x\right)$ but not in $F$. Thus $Q_{n}$ can be represented as $Q_{n}=L \odot R$, each of both $L$ and $R$ is isomorphic to $Q_{n-1}$, such that the edge $x z$ is a cross edge in $L \odot R$. Without loss of generality, suppose that $x$ is in $L$ and $z$ is in $R$. Then $y$ must be located in $R$ since $D\left(Q_{n} ; x, y\right)=n$ and $D(L)=n-1$. Let $u$ be a vertex in $L$ such that the edge $u y$ is a cross edge in $L \odot R$. Then $D(L ; x, u)=D(R ; z, y)=D\left(Q_{n-1}\right)=n-1$, and so there are an $(x, u)$-container $C_{n-1}(L ; x, u)=\left\{L_{1}, L_{2}, \cdots, L_{n-1}\right\}$ and a $(z, y)$-container $C_{n-1}(R ; z, y)=\left\{R_{1}, R_{2}, \cdots, R_{n-1}\right\}$, in which each path is of length $n-1$.

Let $F_{L}=F \cap L$ and $F_{R}=F \cap R$. Since $|F| \leq 2 n-3$, at least one of $\left|F_{L}\right|$ and $\left|F_{R}\right|$ is at most $n-2$. If $\left|F_{R}\right| \leq n-2$, then there is a path in $C_{n-1}(R ; z, y)$, let us say $R_{i}$, such that $R_{i}$ avoids $F_{R}$. So the path $P=x z+R_{i}$ is an $(x, y)$-path in $Q_{n}-F$ and is of length $\varepsilon(P)=1+\varepsilon\left(R_{i}\right)=1+(n-1)=n$.

Suppose now that $\left|F_{L}\right| \leq n-2$. If $u$ is not in $F_{L}$, then there is a path in $C_{n-1}(L ; x, u)$, let us say $L_{j}$, such that $L_{j}$ avoids $F_{L}$ and so the path $P=L_{j}+u y$ is an $(x, y)$-path in $Q_{n}-F$ and is of length $\varepsilon(P)=\varepsilon\left(L_{j}\right)+1=(n-1)+1=n$.

Suppose that $u$ is in $F_{L}$ below. Then there is a vertex $v$ in $N(R ; y)$ such that $v$ is not in $F$ since $N\left(Q_{n} ; y\right)$ is not contained in $F$. Furthermore, $v$ is not in $N(R ; z) \cup\{z\}$ since $D(R ; z, y)=n-1 \geq 3$. Consider the set $N=N\left(Q_{n} ; y\right) \cup N\left(Q_{n} ; v\right),|N|=2 n$ since $Q_{n}$ is a bipartite graph. There must be two adjacent vertices $w$ in $N$ and $w_{L}$ in $L$ such that the edge $w w_{L}$ is a cross edge and avoids $F$ because $|N|=2 n$ and $|F| \leq 2 n-3$. It is clear that $w_{L} \neq u$ since $u \in F_{L}$ and $D\left(L ; x, w_{L}\right) \leq n-2$ since $u$ is the unique vertex whose distance from $x$ is equal to $n-1$ in $L$ by Property 1. On the other hand, $D\left(L ; x, w_{L}\right) \geq n-3 \geq 1$ and $w_{L} \neq x$ since $n-1=D(R ; z, y) \leq D(R ; z, w)+2=D\left(L ; x, w_{L}\right)+2$. Select such a vertex $w_{L}$ in $L \backslash F$ that $D\left(L ; x, w_{L}\right)$ is as large as possible. Let $D\left(L ; x, w_{L}\right)=h$. Then $n-3 \leq h \leq n-2$.

Let $B$ be an $h$-cube determined by $x$ and $w_{L}$. Let $C_{h}\left(B ; x, w_{L}\right)=\left\{W_{1}, W_{2}, \cdots, W_{h}\right\}$ be an $\left(x, w_{L}\right)$-container. Then each of paths in $C_{h}\left(B ; x, x_{L}\right)$ is of length $h$.

We claim that $u$ is not in $B$. It is because $D(L ; x, u)=n-1$ and $D(L ; x, b) \leq D\left(L ; x, w_{L}\right)=$ $h \leq n-2$ for any vertex $b$ in $B$. Therefore $B$ contains at most $n-3$ vertices in $F_{L}$ since $\left|F_{L}\right| \leq n-2$.

If $h=n-2$, then $w_{L} \in N(L ; u)$. Otherwise there is another vertex $u^{\prime}$ different from $u$ such that $D\left(L ; x, u^{\prime}\right)=n-1$, which contradicts the uniqueness of such a vertex whose distance from $x$ is $n-1$ by Property 1. This implies that $w$ is in $N(R ; y)$. Since $B$ contains at most $n-3$ vertices in $F_{L}$, there is an $\left(x, w_{L}\right)$-path of length $n-2$ in $C_{n-2}\left(L ; x, w_{L}\right)$, let us say $W_{k}$, such that $W_{k}$ avoids $F_{L}$. Let $P=W_{k}+w_{L} w+w y$. Then $P$ is an $(x, y)$-path in $Q_{n}-F$ and is of length $\varepsilon(P)=\varepsilon\left(W_{k}\right)+2=(n-2)+2=n$.

If $h=n-3$, then $w$ is in $N(R ; v)-\{y\}$. Note that $\left|F_{L}\right| \leq n-2$ and $u$ is in $F_{L}$ but not in $B$, therefore, if $B$ contains at least $n-3$ vertices in $F_{L}$, then $u$ is the only vertex of $F_{L}$ outside $B$. Let $v_{L} \in N(L ; u)$ such that the $v_{L} v$ is a cross edge in $L \odot R$. Then $v_{L} v$ avoids $F$, but $D\left(L ; x, v_{L}\right)=n-2$, which contradicts our choice of $w_{L}$. Therefore, $B$ contains at most $n-4$ vertices in $F_{L}$. Hence there is an $\left(x, w_{L}\right)$-path of length $n-3$, let us say $W_{l}$, in $C_{n-3}\left(B ; x, w_{L}\right)$ such that $W_{l}$ avoids $F_{L}$. Then the path $P=W_{l}+w_{L} w+w v+v y$ is an $(x, y)$-path in $Q_{n}-F$ of length $\varepsilon(P)=\varepsilon\left(W_{l}\right)+3=(n-3)+3=n$.

The proof of Lemma 1 is completed.
Proof of Lemma 2. Suppose $D\left(Q_{n} ; x, y\right)=d$. If $d=1$, then clearly Lemma 2 holds. We prove Lemma 2 for $d \geq 2$ by using an induction on $n(\geq 3)$.

Clearly, Lemma 2 holds for $n=3$. Assume that Lemma 2 is true for $n-1 \geq 3$, and consider $Q_{n}(n \geq 4)$.

Let $x$ and $y$ be two vertices in $Q_{n}$ with $D\left(Q_{n} ; x, y\right)=d \geq 2, F$ be any set of at most $2 n-3$ vertices in $Q_{n}-\{x, y\}$ that contains neither of $N\left(Q_{n} ; x\right)$ and $N\left(Q_{n} ; y\right)$. Then $Q_{n}$ can be represented as $Q_{n}=L \odot R$, where $L$ and $R$ are isomorphic to $Q_{n-1}$, such that both $x$ and $y$ are located in $L$ or $R$ since $d \leq n-1$. We can, without loss of generality, suppose that both $x$ and $y$ are in $L$. Let $x x_{R}$ and $y y_{R}$ be two cross edges in $Q_{n}=L \odot R, x_{R}, y_{R} \in R$. Let $C_{n-1}(L ; x, y)=$ $\left\{L_{1}, L_{2}, \cdots, L_{n-1}\right\}$ be an $(x, y)$-container, and $C_{n-1}\left(R ; x_{R}, y_{R}\right)=\left\{R_{1}, R_{2}, \cdots, R_{n-1}\right\}$ be an $(x, y)$-container. Let $F_{L}=V(L) \cap F$ and $F_{R}=V(R) \cap F$. We will distinguish two cases.
Case 1. $\quad\left|F_{L}\right| \leq 2 n-5$.
Suppose that $F_{L}$ contains neither of $N(L ; x)$ and $N(L ; y)$. Note that $L$ is isomorphic to $Q_{n-1}$ and $\left|F_{L}\right| \leq 2 n-5=2(n-1)-3$. If $D(L ; x, y)=d=n-1$, then $D\left(Q_{n}-F ; x, y\right)=$ $D(L-F ; x, y)=n-1$ by Lemma 1 . If $D(L ; x, y)=d \leq n-2$, by our induction hypothesis there is an $(x, y)$-path $P$ in $L-F_{L}$ such that $P$ is of length

$$
\varepsilon(P) \leq \begin{cases}d+4, & \text { for } \quad 2 \leq d \leq n-3, n \geq 5 \\ n, & \text { for } \quad d=n-2, n \geq 4\end{cases}
$$

This implies that $\varepsilon(P) \leq d+4$ for $2 \leq d \leq n-2$, and thus lemma 2 holds.
Now suppose that $F_{L}$ contains either $N(L ; x)$ or $N(L ; y)$. Then $|N(L ; x) \cup N(L ; y)| \geq 2 n-4$ since $|N(L ; x) \cap N(L ; y)| \leq 2$. Thus only one of $N(L ; x) \subset F_{L}$ and $N(L ; y) \subset F_{L}$ is true. We can, without loss of generality by transitivity of $Q_{n}$, suppose that $N(L ; x) \subset F_{L}$. Then $\left|F_{L}\right| \geq n-1$, and thus $\left|F_{R}\right| \leq n-2$.

Since $N\left(Q_{n} ; x\right)$ is not included in $F, x_{R}$ is not in $F$. If $d=n-1$, then $D\left(Q_{n} ; x_{R}, y\right)=n$. By Lemma 1 there is an $\left(x_{R}, y\right)$-path $P^{\prime}$ of length $n$ in $Q_{n}-F$. Let $P=x x_{R}+P^{\prime}$, then $P$ is an $(x, y)$-path of length $n+1$ in $Q_{n}-F$. This proves the case 3 in Lemma 2. So we suppose that $2 \leq d \leq n-2$ below, and need only prove that there is an $(x, y)$-path $P$ of length at most $d+4$ in $Q_{n}-F$.

If $y_{R}$ is not in $F$, then since $\left|F_{R}\right| \leq n-2$, there is a path, let us say $R_{i}$, in $C_{n-1}\left(R ; x_{R}, y_{R}\right)$, such that $R_{i}$ avoids $F_{R}$. The path $P=x x_{R}+R_{i}+y_{R} y$ is an $(x, y)$-path in $Q_{n}-F$ of length

$$
\varepsilon(P)=\varepsilon\left(R_{i}\right)+2 \leq D\left(R ; x_{R}, y_{R}\right)+2+2=d+4
$$

Suppose that $y_{R}$ is in $F$. Let $N\left(R ; y_{R}\right)=\left\{u_{1}, u_{2}, \cdots, u_{n-1}\right\}$ such that $u_{i} \in R_{i}(i=$ $1,2, \cdots, n-1)$ and let $N(L ; y)=\left\{z_{1}, z_{2}, \cdots, z_{n-1}\right\}$ such that the edges $z_{i} u_{i}(i=1,2, \cdots, n-1)$ are cross edges. Let

$$
H_{i}=R_{i}\left(x_{R}, u_{i}\right)+u_{i} z_{i}+z_{i} y, \quad i=1,2, \cdots, n-1
$$

Then

$$
\varepsilon\left(H_{i}\right)=\varepsilon\left(R_{i}\right)+1 \leq D\left(R ; x_{R}, y_{R}\right)+3=d+3,
$$

and $H_{1}, H_{2}, \cdots, H_{n-1}$ are internally vertex-disjoint $\left(x_{R}, y\right)$-paths in $Q_{n}$. Let $H=\left\{H_{1}, H_{2}, \cdots\right.$, $\left.H_{n-1}\right\}$.

If there is a path in $H$, let us say $H_{i}$, such that $H_{i}$ avoids $F$, then $P=x x_{R}+H_{i}$ is an $(x, y)$-path in $Q_{n}-F$. Thus we have

$$
\varepsilon(P)=1+\varepsilon\left(H_{i}\right) \leq d+4
$$

We now suppose that for any $i=1,2, \cdots, n-1, H_{i}$ does not avoid $F$. Note that since $\left|F_{L} \cap N(L ; x)\right|=n-1$ and $|F|-\left|F_{L} \cap N(L ; x)\right|-\left|\left\{y_{R}\right\}\right| \leq n-3$, for each $i=1,2, \cdots, n-1$, $H_{i}$ contains a unique vertex in $F$ under our assumption. Furthermore, at least two paths in $H$
cannot avoid $F_{L} \cap N(L ; x)$. Thus $d=D\left(Q_{n} ; x, y\right)=D(L ; x, y)=2$. We need only show there is an $(x, y)$-path $P$ of length 6 in $Q_{n}-F$.

Of all paths in $C_{n-1}(L ; x, y)=\left\{L_{1}, L_{2}, \cdots, L_{n-1}\right\}$, two are of length 2 and the rest are of length 4 by Property 2. We can, without loss of generality, suppose that $z_{1}, z_{2} \in N(L ; x) \cap$ $N(L ; y)$ and $z_{i} \in L_{i}(i=1,2, \cdots, n-1)$. Then both $z_{1}$ and $z_{2}$ are in $F_{L}$ and for each $i$, $3 \leq i \leq n-1, H_{i}$ contains a unique vertex in $F \backslash(N(L ; x) \cup\{y\})$.

Since $N(L ; y)$ is not included in $F_{L}$, there is a vertex in $N(L ; y)$, let us say $z_{i}(3 \leq i \leq$ $n-1$ ), such that $z_{i}$ is not in $F_{L}$. Let $L_{i}=x a_{i} b_{i} z_{i} y$, then $b_{i}$ and $z_{i}$ are not in $F$. Let $R_{i} \in C_{n-1}\left(R ; x_{R}, y_{R}\right)$ be a path corresponding to the path $L_{i}, c_{i} \in R_{i}$ and $b_{i} c_{i}$ be the cross edge. Thus at least one of two edges $b_{i} c_{i}$ and $z_{i} u_{i}$ avoids $F$.

If $z_{i} u_{i}$ does not avoid $F$, then $u_{i} \in F$ and $b_{i} c_{i}$ avoids $F$. Since $H_{i}$ contains a unique vertex in $F$, the subpath $H_{i}\left(x_{R}, c_{i}\right)$ avoids $F$. The path $P=x x_{R}+H_{i}\left(x_{R}, c_{i}\right)+c_{i} b_{i}+b_{i} z_{i}+z_{i} y$ is an $(x, y)$-path in $Q_{n}-F$ and is of length 6.

Suppose that $z_{i} u_{i}$ avoids $F$. Since $D\left(R ; x_{R}, u_{i}\right)=3$, by Property 2 there are an $\left(x_{R}, u_{i}\right)$ container $C_{n-1}\left(R ; x_{R}, u_{i}\right)=\left\{T_{1}, T_{2}, \cdots, T_{n-1}\right\}$, in which three of all paths are of length 3 and the rest of length 5 , and a 3 -cube $B^{\prime}$ determined by $x_{R}$ and $u_{i}$ in $R$ such that all paths of length 3 in $C_{n-1}\left(R ; x_{R}, u_{i}\right)$ are located in $B^{\prime}$.

We first claim that $\left|F_{R} \cap V\left(B^{\prime}\right)\right| \leq 2$. It holds clearly if $n=4$ since $\left|F_{R}\right| \leq 2$. Suppose $n \geq 5$. Since $D\left(R ; x_{R}, u_{i}\right)=3, B^{\prime}$ is isomorphic to $Q_{3}$ and $y_{R} \in F_{R} \cap V\left(B^{\prime}\right)$. In $C_{n-1}\left(R ; x_{R}, y_{R}\right)$, two paths of length 2 and a path of length 4 must be located in $B^{\prime}$ and the other $n-4$ paths of length 4 must not pass through $B^{\prime}$. This implies that there are $n-4$ paths in $H$, each of which contains a unique vertex in $F$ and does not pass through $B^{\prime}$. In other words, $B^{\prime}$ contains at most two vertices in $F$. Hence there is a path of length 3 in $C_{n-1}\left(R ; x_{R}, u_{i}\right)$, let us say $T_{j}$, such that $T_{j}$ avoids $F$. So $P=x x_{R}+T_{j}+u_{i} z_{i}+z_{i} y$ is an $(x, y)$-path in $Q_{n}-F$ and is of length 6 .
Case 2. $\quad\left|F_{L}\right| \geq 2 n-4$.
In this case, $\left|F_{R}\right| \leq 1$. Arbitrarily select a shortest $\left(x_{R}, y_{R}\right)$-path $S$ in $R$ if $F_{R}$ is empty, then $P=x x_{R}+S+y_{R} y$ is an $(x, y)$-path in $Q_{n+1}-F$ and is of length $\varepsilon(P)=\varepsilon(S)+2=$ $D\left(R ; x_{R}, y_{R}\right)+2=d+2$. We suppose that $\left|F_{R}\right|=1$ below.

Subcase 2.1. $\quad$ Neither $x_{R}$ nor $y_{R}$ are in $F_{R}$.
Since $D\left(R ; x_{R}, y_{R}\right)=D\left(Q_{n} ; x, y\right)=d \geq 2$, in $C_{n-1}\left(R ; x_{R}, y_{R}\right)$ there are $d$ paths of length $d$. Also since $\left|F_{R}\right|=1$, one of these paths, let us say $R_{k}$, avoids $F_{R}$. Let $P=x x_{R}+R_{k}+y_{R} y$. Then $P$ is an $(x, y)$-path in $Q_{n}-F$ and is of length $\varepsilon(P)=\varepsilon\left(R_{k}\right)+2=d+2$.

Subcase 2.2. One of $x_{R}$ and $y_{R}$ is in $F_{R}$.
Without loss of generality, suppose that $x_{R} \in F_{R}$. Then $y_{R}$ is not in $F_{R}$. Since $N\left(Q_{n} ; x\right)$ is not included in $F$, there is a vertex $z \in N(L ; x)$ such that $z$ is not in $F$. It is clear that $z$ is not $y$ since $D(L ; x, y)=d \geq 2$. Let $z z_{R}$ is the cross edge in $Q_{n}=L \odot R, z_{R} \in R$. And let $C_{n-1}\left(R ; z_{R}, y_{R}\right)=\left\{Z_{1}, Z_{2}, \cdots, Z_{n-1}\right\}$ be a $\left(z_{R}, y_{R}\right)$-container. Since $D\left(R ; z_{R}, y_{R}\right)$ paths in $C_{n-1}\left(R ; z_{R}, y_{R}\right)$ are of length $D\left(R ; z_{R}, y_{R}\right)$, at least one of them, let us say $Z_{j}$, does not contain $x_{R}$. Let $P=x z+z z_{R}+Z_{j}+y_{R} y$ and

$$
\varepsilon(P)=\varepsilon\left(Z_{j}\right)+3=D\left(R ; z_{R}, y_{R}\right)+3 .
$$

Note that $z$ must be located in some path, let us say $L_{i}$, in $C_{n-1}(L ; x, y)$ since $z \in N(L ; x)$. Thus, $\varepsilon\left(L_{i}\right)$ is of $D(L ; z, y)$ or $D(L ; z, y)+2$.

If $\varepsilon\left(L_{i}\right)=D(L ; x, y)$, then $d=D\left(Q_{n} ; x, y\right)=D(L ; z, y)+1=D\left(R ; z_{R}, y_{R}\right)+1$. It follows that

$$
\varepsilon(P)=D\left(R ; z_{R}, y_{R}\right)+3=d+2
$$

If $\varepsilon\left(L_{i}\right)=D(L ; x, y)+2$, then $D(L ; x, y)=D\left(Q_{n} ; x, y\right) \leq n-2$ and $D\left(Q_{n} ; x, y\right)=$ $D(L ; z, y)-1=D\left(R ; z_{R}, y_{R}\right)-1$. It follows that

$$
\varepsilon(P)=D\left(R ; z_{R}, y_{R}\right)+3=D\left(Q_{n} ; x, y\right)+4=d+4 .
$$

To sum up, we complete the proof of the upper bounds of $D\left(Q_{n}-F ; x, y\right)$ given in Lemma 2. We now show that these two upper bounds can not be improved in general case by selecting a restricted set $F$ with $(2 n-3)$ vertices in $Q_{n}-\{x, y\}$ such that $D\left(Q_{n}-F ; x, y\right)$ is equal to the upper bounds.

Since $2 \leq D\left(Q_{n} ; x, y\right) \leq n-1, Q_{n}$ can be represented as $Q_{n}=L \odot R$, where each of $L$ and $R$ is isomorphic to $Q_{n-1}$ such that both $x$ and $y$ are located in $L$ or $R$. Let $x$ and $y$ be in $L$, and $x x_{R}, y y_{R}$ be the cross edges in $L \odot R, x_{R}, y_{R} \in R$. Let $C_{n-1}\left(R ; x_{R}, y_{R}\right)$ be an $\left(x_{R}, y_{R}\right)$-container, $R_{i}$ be a longest path in $R_{n-1}\left(R ; x_{R}, y_{R}\right)$ and $u_{i} \in N\left(R ; y_{R}\right) \cap V\left(R_{i}\right)$. Define

$$
F=N(L ; y) \cup N\left(R ; y_{R}\right)-\left\{y, y_{R}, u_{i}\right\} .
$$

Then $|F|=2 n-3$. Since $F$ does not contain all neighbors of any vertex in $Q_{n}, F$ is a restricted set of $Q_{n}$. Also $P=x x_{R}+R_{i}+y_{R} y$ is an $(x, y)$-path in $Q_{n}-F$. Therefore $D\left(Q_{n}-F ; x, y\right)=$ $\varepsilon(P)=\varepsilon\left(R_{i}\right)+2$. Note that $\varepsilon\left(R_{i}\right)=D\left(R ; x_{R}, y_{R}\right)+2$ if $2 \leq D\left(R ; x_{R}, y_{R}\right) \leq n-2$, and that $\varepsilon\left(R_{i}\right)=n-1$ if $D\left(R, x_{R}, y_{R}\right)=n-1$ and also $D\left(R ; x_{R}, y_{R}\right)=D\left(Q_{n} ; x, y\right)=d$. It follows that

$$
D\left(Q_{n}-F ; x, y\right)=\varepsilon\left(R_{i}\right)+2= \begin{cases}d+4, & \text { for } \quad 2 \leq d \leq n-2 \\ n+1, & \text { for } \quad d=n-1\end{cases}
$$

The proof of Lemma 2 is completed.
Corollary $1^{[9]} . \quad D_{3}\left(Q_{3}\right)=4$ and $D_{2 n-3}\left(Q_{n}\right)=n+2$ for $n \geq 4$.
Proof. It is easy to verify $D_{3}\left(Q_{3}\right)=4$ by enumeration. We need only prove that $D_{2 n-3}\left(Q_{n}\right)=$ $n+2$ for $n \geq 4$. But it is a direct consequence of the Theorem. By the theorem $D_{2 n-3}\left(Q_{n}\right) \leq$ $n+2$. On the other hand, $D_{2 n-3}\left(Q_{n}\right) \geq n+2$ since there are two vertices $x$ and $y$, and a restricted set $F$ with $2 n-3$ vertices in $Q_{n}$ such that $D\left(Q_{n}-F ; x, y\right)=n+2$ provided $D\left(Q_{n} ; x, y\right)=n-2$ and $n \geq 4$ by the Theorem.

Corollary $2^{[3]} . \quad \kappa^{\prime}\left(Q_{n}\right)=2 n-2$ for $n \geq 2$.

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