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On Bounded Paths in Some Networks

YINGFENG TAO JUNMING XU

(Department of Math. University of Science and Technology of China, Hefei 230026, China)

Abstract

The problem of paths with bounded length, being a variation and a generalization of Menger's theorem, is of very important significance in the design and analysis of real-time or fault-tolerant interconnection networks. For a given positive integer d, the symbol $A_d(D)$ denotes the maximum number of internally disjoint paths of length at most d between any two vertices with distance at least two in the network D; the symbol $B_d(D)$ denotes the minimum number of vertices whose deletion results in diameter larger than d. Obviously, $A_d(D) \le B_d(D)$ and it has been shown to determine the value of $A_d(D)$ is NP-complete. In the present paper, three well-known networks, hypercubes, de Bruijn and Kautz, are considered and the values of $A_d(D)$ and $B_d(D)$ are determined, which show $A_d(D) = B_d(D)$.

Key words: paths, Menger's theorem, hypercubes, de Bruijn digraphs, Kautz digraphs.

几个著名网络的限长路径

陶颖峰 徐俊明

(中国科学技术大学数学系，合肥， 230026)

摘 要：设给出了 $(h,\varphi)-\eta$ 限长路径问题是图论中的 Menger 定理的变形和推广，在实时容错网络设计和分析中有重要意义. 对于给定的正整数 d, $A_d(D)$ 表示网络 D 中任何距离至少为 2 的两顶点之间内点不交且长度都不超过 d 的路的最大条数； $B_d(D)$ 表示 D 的顶点子集 B 中的最小顶点数使得 $D-B$ 的直径大于 d. 已证明确定 $A_d(D)$ 的问题是 NPC 问题，而且显然有不等式 $A_d(D) \le B_d(D)$. 本文考虑 D 为超立方体网络、De Bruijn 网络和 Kautz 网络，对 d 的不同值确定了 $A_d(D)$ 及 $B_d(D)$, 而且均有 $A_d(D) = B_d(D)$.

关键词：路， Menger 定理，超立方体网络， De Bruijn 网络， Kautz 网络.

1. Introduction

We follow [1] for terminology and notation not defined and explained in this paper.

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In a real-time system, such as parallel computing or processing system, in order to be effective, one uses k internally disjoint paths to transmit messages simultaneously from one vertex to another. The message delay must be limited within a given period d since any message obtained beyond the bound may be worthless. On the other hand, if a real-time system is fault tolerant, then a question is whether the system can keep communicating when faults occur. Thus, for given network system and period d, determining the maximum value of k is useful.

When a network system is modelled by a graph D, the problem above can, in the language of graph theory, be explained as for two given vertices x, y in D and a positive integer d, determining the largest k such that there are k internally disjoint (x, y)-paths of length at most d in D. It is called the problem of paths with bounded length in the literature. As it has strong applications in networks, this problem has attracted much attention in the past years (see, for example, [2-5]). However, in general, it has been shown to be an NPC problem [4]. We are, in this paper, interested in the following problem.

Let x and y be any two distinct vertices in D with $(x, y) \notin E(D)$. The symbol $A_d(D; x, y)$ denotes the maximum number of internally disjoint (x, y)-paths of length at most d in D, the symbol $B_d(D; x, y)$ denotes the minimum number of vertices in D whose deletion destroys all (x, y)-paths of length at most d. In order to destroy all (x, y)-paths of length at most d, we need delete at least one vertex from each path of length at most d, which implies $A_d(D; x, y) \leq B_d(D; x, y)$ for any positive integer d. However, the equality does not hold in general. A natural question is which condition is satisfied for either d or D to ensure $A_d(D; x, y) = B_d(D; x, y)$.

To avoid the relatively non-significant case in which $d < d_D(x, y)$ or $d = 1$, we suppose $d \geq d_D(x, y) \geq 2$. Since length of any path in D with order n does not exceed $n-1$, we suppose $d \leq n-1$. For $d = n-1$ no restriction is imposed on the length of any path in D, thus, we have $A_{n-1}(D; x, y) = B_{n-1}(D; x, y)$ by Menger's theorem. Besides, it has been shown in [2,3] that $A_d(D; x, y) = B_d(D; x, y)$ for $d = 2, 3, 4, d_D(x, y)$, where $d_D(x, y)$ is distance from x to y in D. Let

$$A_d(D) = \min\{A_d(D; x, y); \forall x, y \in V(D), (x, y) \notin E(D)\};$$

$$B_d(D) = \min\{B_d(D; x, y); \forall x, y \in V(D), (x, y) \notin E(D)\}.$$

For a parallel network system D, it is great useful to determine $A_d(D)$ and $B_d(D)$. Apparently, if $d < d(D)$, the diameter of D, then $A_d(D) = B_d(D) = 0$. Thus we can always suppose $2 \leq d(D) \leq d \leq n-1$. And we have $A_d(D) \leq B_d(D) \leq \kappa(D)$, where $\kappa(D)$ is the connectivity of D.

Note that determining $A_d(D)$ is NP-hard since determining $A_d(D; x, y)$ is an NPC problem. Thus, it is significant to determine $A_d(D)$ for some well-known networks D. In the present paper, three well-known networks D, the hypercubes, de Bruijn and Kautz, are considered and the values of $A_d(D)$ and $B_d(D)$ are determined, which show $A_d(D) = B_d(D)$.

2. Hypercubes

The n-dimensional hypercube Q_n is an undirected graph with the vertex-set $V = \{x_1x_2\cdots x_n : x_i \in \{0,1\}, i = 1,2,\cdots,n\}$, and two vertices $x_1x_2\cdots x_n$ and $y_1y_2\cdots y_n$ are adjacent if and only if they differ exactly in one coordinate. It has been known that Q_n has 2^n vertices, n regularity, diameter $d(Q_n) = n$ and connectivity $\kappa(Q_n) = \lambda(Q_n)$.

Lemma 2.1. [6] *Suppose that x and y are arbitrary two vertices in Q_n with distance d, then there exist n internally disjoint xy-paths, d of length d, others $d+2$.*

Clearly, $A_d(Q_2) = B_d(Q_2) = 2$ for $2 \le d \le 3$. So suppose $n \ge 3$ below.

Theorem 2.2. *Suppose* $3 \le n \le d \le 2^n - 1$. *Then*

$$A_d(Q_n) = B_d(Q_n) = \begin{cases} n-1, & \text{if } d = n; \\ n, & \text{if } d > n. \end{cases}$$

Proof: For any $x, y \in V(Q_n)$, by Lemma 2.1, Q_n has n internally disjoint xy-paths of length at most $n+1$. Thus, if $d \ge n+1$, then

$$n = \kappa(Q_n) \ge B_d(Q_n) \ge A_d(Q_n) \ge n,$$

that is, $A_d(Q_n) = B_d(Q_n) = n$.

Let s and t be two vertices in Q_n with distance $n-1$. Then s and t are not adjacent since $n \ge 3$. In any n internally disjoint st-paths, exactly one is of length at least $n+1$ by Lemma 2.1. It follows that

$$n - 1 \le A_n(Q_n) \le B_n(Q_n) \le B_n(Q_n; s, t) = n - 1,$$

that is, $A_n(Q_n) = B_n(Q_n) = n - 1$. ∎

3. De Bruijn Networks

The de Bruijn digraph, denoted by $B(n,k)$ $(n \ge 2, k \ge 1)$, is a digraph with the vertex-set $V(B(n,k)) = \{x_1x_2\cdots x_k : x_i \in \{0,1,\cdots,n-1\}\}$, and the edge-set consisting of all edges from one vertex $x_1x_2\cdots x_k$ to n others $x_2x_3\cdots x_k\alpha$, where $\alpha \in \{0,1,\cdots,n-1\}$. It has been known that $B(n,k)$ has n^k vertices, n regularity, diameter $d(B(n,k)) = k$ and connectivity $\kappa(B(n,k) = n - 1$.

Lemma 3.1.[7] *Suppose that (x, y) is an arbitrary ordered pair of vertices in $B(n,k)$, then there is unique shortest directed (x,y)-path in $B(n,k)$.*

Lemma 3.2.[8] *For any two distinct vertices x and y of $B(n,k)$, there are $n-1$ internally disjoint (x,y)-paths of length at most $k+1\sum_{i}^{b}$.*

Theorem 3.3. *Suppose $2 \le k \le d \le n^k - 1$. Then*

$$A_d(B(n,k)) = B_d(B(n,k)) = \begin{cases} 1, & \text{if } d = k; \\ n-1, & \text{if } d > k. \end{cases}$$

Proof: For any $x, y \in V(B(n,k))$, by Lemma 3.2, $B(n,k)$ has $n-1$ internally disjoint (x,y)-paths of length at most $k+1$. Thus, if $d \ge k+1$, then

$$n - 1 = \kappa(B(n,k)) \ge B_d(B(n,k)) \ge A_d(B(n,k)) \ge n - 1,$$

that is, $A_d(B(n,k)) = B_d(B(n,k)) = n-1$.

Let s and t be two vertices $B(n,k)$ with distance k. By Lemma 3.1, there is the unique directed (s,t)-path of length k in $B(n,k)$. So

$$1 \le A_k(B(n,k)) \le B_k(B(n,k)) \le B_k(B(n,k); s,t) = 1,$$

that is, $A_k(B(n,k)) = B_k(B(n,k)) = 1$. ∎

4. Kautz Networks

The Kautz digraph,denoted by $K(n,k)\,(n \ge 2, k \ge 1)$, is a digraph with the vertex-set $V(K(n,k)) = \{x_1x_2\cdots x_k : x_i \in \{0,1,\cdots,n\}, x_i \ne x_{i+1}, i = 1,2,\cdots,k-1\}$, and the edge-set consisting of all edges from one vertex $x_1x_2\cdots x_k$ to n others $x_2x_3\cdots x_k\alpha$, where $\alpha \in \{0,1,\cdots,n\}, \alpha \ne x_k$. It has been known that $K(n,k)$ has n^k+n^{k-1} vertices, n regularity, diameter $d(K(n,k)) = k$ and connectivity $\kappa(B(n,k) = n$.

Lemma 4.1. *Suppose that (x,y) is an arbitrary ordered pair of vertices in $K(n,k)$, then there is unique shortest directed (x,y)-path in $K(n,k)$.*

Proof: Suppose that $x = x_1x_2\cdots x_k$ and $y = y_1y_2\cdots y_k$ are two vertices in $K(n,k)$ with distance $k - l = m$. Then $K(n,k)$ has a directed (x,y)-path of length m:

$$x = x_1x_2\cdots x_k \to x_2x_3\cdots x_ku_1 \to \cdots \to x_{m+1}\cdots x_ku_1u_2\cdots u_m = y_1y_2\cdots y_k = y$$

Hence there are $k - m = l$ overlapped components by the end part of x and the head part of y. Suppose $x = x_1x_2\cdots x_{k-l}z_1z_2\cdots z_l$, $y = z_1z_2\cdots z_ly_{l+1}y_{l+2}\cdots y_k$. Then

$$\begin{aligned} P(x,y) = x \;&\to x_2x_3\cdots x_{k-l}z_1z_2\cdots z_ly_{l+1} \to x_3\cdots z_ly_{l+1}y_{l+2} \\ &\to \cdots \to z_1z_2\cdots z_ly_{l+1}y_{l+2}\cdots y_k = y \end{aligned}$$

is shortest directed (x,y)-path in $K(n,k)$. We now prove the path $P(x,y)$ is unique. Suppose that

$$\begin{aligned} W: x = x_1x_2\cdots x_k &\to x_2x_3\cdots x_kw_1 \to \cdots \\ &\to x_{k-l+1}\cdots x_kw_1\cdots w_{k-l} = y_1y_2\cdots y_ly_{l+1}\cdots y_k = y \end{aligned}$$

is a directed (x,y)-path of length $k-l$. Then

$$x_{k-l+1} = y, \ i = 1, 2, \cdots, l; \ \ w_j = y_{l+j}, \ \ j = 1, 2, \cdots, k-l.$$

This implies $W = P(x,y)$. ∎

Lemma 4.2. [9] *For any two distinct vertices x and y of $K(n,k)$, there are n internally disjoint (x,y)-paths, two of length at most k and $k+2$, respectively, $n-2$ at most $k+1$.*

$K(n,1)$ is a complete digraph with $n+1$ vertices and it is trivial to determine A_d and B_d. So suppose $k \geq 2$ below.

Theorem 4.3. *Suppose $2 \leq k \leq d \leq n^k + n^{k-1} - 1$. Then*

$$A_d(K(n,k)) = B_d(K(n,k)) = \begin{cases} 1, & \text{if } d = k; \\ n-1, & \text{if } d = k+1; \\ n, & \text{if } d > k+1. \end{cases}$$

Proof: Let (x,y) be an ordered pair of vertices in $K(n,k)$ with distance k. By Lemma 4.1, $K(n,k)$ has a unique directed (x,y)-path of length k. Hence

$$1 \leq A_k(K(n,k)) \leq B_k(K(n,k)) \leq B_k(K(n,k);x,y) = 1,$$

that is, $A_k(K(n,k)) = B_k(K(n,k)) = 1$.

For any two vertices s and t in $K(n,k)$, by Lemma 4.2, there are n internally disjoint (s,t)-paths of length at most $k+2$. Hence for $d \geq k+2$, we have

$$n \leq A_d(K(n,k)) \leq B_d(K(n,k)) \leq \kappa(K(n,k)) = n,$$

that is, $A_d(B(n,k)) = B_d(B(n,k)) = n$.

By Lemma 4.2, $K(n,k)$ has at least $n-1$ internally disjoint (s,t)-paths of length at most $k+1$, that is, $A_{k+1}(K(n,k)) \geq n-1$. Consider two vertices $e = (\ldots 01010)$ and $f = (02020\ldots)$. Then one of any n (e,f)-paths must pass through the vertex $g = (\ldots 010101)$, which is adjacent to e, and any (g,f)-path which does not pass e is of length at least $k+1$. Thus, of any n internally disjoint (e,f)-paths, at least one is of length $k+2$. It follows that

$$n-1 \leq A_{k+1}(K(n,k)) \leq B_{k+1}(K(n,k)) \leq B_{k+1}(K(n,k);e,f) = n-1,$$

that is, $A_{k+1}(B(n,k)) = B_{k+1}(B(n,k)) = n-1$. ∎

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