An infinite family of 4-tight optimal double loop networks

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Abstract An infinite family of 4-tight optimal double loop networks is given in this paper.

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Because of their symmetry, simplicity and extensionality, the double loop networks (DLNs) have been widely used in the topological design of local networks, multi-module memory organizations, data alignments in parallel memory systems, and supercomputer architecture[1]. The graphical model of a DLN is a digraph (also called circulant digraph) $G(n;s)$ with the vertex set $\{0,1,2,\cdots,n-1\}$ and the edge set $\{i \to i+1 \pmod n,\ i \to i+s \pmod n:\ i=0,1,2,\cdots,n-1\}$, where s is a given integer with $1<s<n$. From the definition, it is clear that $G(n;s)$ is strongly connected and the diameter is only determined by n and s. Denote by $d(n;s)$ the diameter of $G(n,s)$ and let $d(n)=\min\{d(n;s): 1<s<n\}$. A network $G(n;s)$ is optimal if $d(n;s)=d(n)$. Wong and Coppersmith[2] have shown that $d(n)\geqslant lb(n)=\lceil\sqrt{3n}\,\rceil-2$, where the symbol $\lceil m\rceil$ denotes the smallest integer not less than the real number m.

An important problem is to determine the value of $d(n)$ and find s such that $G(n;s)$ is optimal for any given $n\geqslant 4$. The problem has attracted many authors' interest[2−10], although it seems impossible to express the function $d(n)$ in a closed form.

Let Z be the infinite set of all nonnegative integers. For $k\in Z$, an optimal $G(n;s)$ is said to be k-tight if $d(n;s)=lb(n)+k$. Generally, 0- or 1-tight optimal DLNs are called tight optimal and near-tight optimal, respectively[7,8]. We say that $\{G(n(t);s(t)):\ t\in Z,\ t\geqslant t_0\}$ is an infinite family of k-tight optimal DLNs if $G(n(t);s(t))$ is k-tight optimal for any $t\in Z$ and $t\geqslant t_0$, where $G(n(t_0);s(t_0))$ is the initial element. We say that $\{n(t):\ t\in Z,\ t\geqslant t_0\}$ contains no k-tight optimal DLN if $d(n(t);s(t))>lb(n(t))+k$ for any $s(t)$ and $t\geqslant t_0$. The functions $n(t)$ and $s(t)$ are polynomials in $t\in Z$ with integral coefficients.

Li, Xu et al.[7] have presented a systematic method to construct optimal DNLs and listed 102 infinite families of optimal DLNs, of which 69 are tight and 33 near-tight, to show that $d(n)\leqslant lb(n)+1$ for $n\leqslant 300$. Xu[9] has found 3 infinite families of 2-tight optimal DLNs. Erdös and Hsu[4] reported an exhaustive computer search. Chen showed that $d(n)\leqslant lb(n)+4$ for $n\leqslant 75000$, there exist only 3 n's, 53749, 64729 and 69283, for which the equality holds; the corresponding 4-tight optimal DLN's are $G(53749;985)$, $G(64729;394)$ and $G(69283;1764)$, and their diameters are 404, 443 and 458, respectively. With computer, we found that the fourth

4-tight optimal DLN is $G(94921;515)$, and its diameter is 536. However, as far as we know, no infinite family of 4-tight optimal DLNs has been found so far. In this paper, combining geometric method with number theoretic technics, we will construct an infinite family of 4-tight optimal DLNs with the initial element $G(69283;1764)$.

In the following, we consider $n(t)=3t^2+6t-26$, $t\in Z$. It is easy to verify $lb(n(t))=3t+1$ for $t\geqslant 14$. Please refer to ref. [1] or ref. [7] for terminology and notation not defined and explained in this paper.

1 Some lemmas

Lemma 1[7]. Let $L=L(n;l,h,x,y)$ be an L-tile. If $|y-x|\geqslant z_0\geqslant 1$, then $d(n)\geqslant \sqrt{3n-\frac{3}{4}z_0^2}+\frac{1}{2}z_0-2$.

Lemma 2[8]. Let $n(t)=3t^2+3t-26$. Then $\{n(t):\ t\in Z, t\geqslant 29\}$ contains no tight optimal DLN. Moreover, if $L(n,l,h,x,y)$ is near-tight optimal, then $|x-y|\leqslant 1$.

Lemma 3[7]. Let $n(t)=3t^2+3t-26$ and $L=L(n;l,h,x,y)$ be an L-tile, where $l=2t+a$, $h=2t+b$, $x=t+a+b-j$, $z=|x-y|$, a, b, x and y are polynomials in $t\in Z$ with integral coefficients. Then L is k-tight if and only if

$$(a+b-j)(a+b-j+z)-ab+(6+z-2j)-26=0 \tag{1}$$

holds for any $j=3+k$ $(k\in Z)$. Moreover, if there exist integers α and β such that $\alpha y+\beta(h-y)\equiv 1$, then there exists only one k-tight optimal $G(n(t);s(t)$, where $s\equiv \alpha l-\beta(x-l) \pmod n$.

2 Main results

Theorem 1. For $n\in Z$, a necessary condition1) that there exist $s,m\in Z$ such that $n=s^2+3m^2$ is that if n has a prime divisor p with $p=2$ or $p\equiv 5 \pmod 6$, then p has an even power in the prime decomposition of n.

Proof. We proceed by induction on $n\geqslant 3$. If $n=3$ there is nothing to prove, so we suppose that the theorem holds for any n less than an integer k with $k\geqslant 4$. Let $n=k$ and suppose that there exist $s,m\in Z$ such that $k=s^2+3m^2$ and k has a prime divisor p with $p=2$ or $p\equiv 5$ (mod 6).

If $p=2$, then s and m are of the same parity since 2 is a divisor of $k=s^2+3m^2$. If $s=2i$ and $m=2j$, then $k=2^2(i^2+3j^2)$, so the theorem holds by the induction hypothesis. If $s=2i+1$ and $m=2j+1$, then

$$k=s^2+3m^2=4i^2+4i+1+3(4j^2+4j+1)=2^2[i(i+1)+3j(j+1)+1].$$

The theorem holds by the induction hypothesis since the number in the square bracket is odd.

We now suppose $p\equiv 5 \pmod 6$. If p is not a divisor of sm, since $k=s^2+3m^2\equiv 0 \pmod p$, then $p\equiv 1 \pmod 6$, a contradiction. Thus, p is a divisor of s or m. Since p is a divisor of $k=s^2+3m^2$, it follows that p is a common divisor of s and m. Let $s=pi$ and $m=pj$. Then $k=p^2(i^2+3j^2)$, so the theorem holds by the induction hypothesis.

Corollary. Let $n=s^2+3m^2$ $(s,m\in Z)$ and suppose that 3 is not a divisor of n. Then $n\equiv 4 \pmod 6$ if 2 is a divisor of n, and $n\equiv 1 \pmod 6$ otherwise.

1) It is easy to prove that this condition is also sufficient.

Theorem 2. A necessary condition1) for eq. (1) in a and b to have an integral solution is that for any $z, j, t \in Z$, if

$$H_{z,j} = (2j-z)^2 - 3[j(j-z) + (6+z-2j)t - 26] \tag{2}$$

has a prime divisor p with $p = 2$ or $p \equiv 5 (\mathrm{mod}\ 6)$, then p has an even power in the prime decomposition of n.

Proof. Suppose that eq. (1) in a and b has an integral solution and rewrite it as

$$a^2 + [b - (2j-z)]a + b^2 - (2j-z)b + C = 0, \tag{3}$$

where $C = j(j-z) + (6+z-2j)t - 26$. Then eq. (3) in a has an integral solution by our assumption. Thus, there exists an $m \in Z$ such that

$$[b-(2j-z)]^2 - 4[b^2 - (2j-z)b + C] = m^2.$$

Express it as an equation in b:

$$3b^2 - 2(2j-z)b + [4C + m^2 - (2j-z)^2] = 0. \tag{4}$$

Then eq. (4) in b has an integral solution. Thus, there exists an $n \in Z$ such that

$$4(2j-z)^2 - 12[4C + m^2 - (2j-z)^2] = n^2.$$

This implies that n is even. Let $n = 2s$ in the above expression. We have $4(2j-z)^2 - 12C = s^2 + 3m^2$, that is, $4H_{z,j} = s^2 + 3m^2$. The theorem follows by Theorem 1.

Theorem 3. Let $n(t) = 3t^2 + 6t - 26$, $t(f) = 28f^2 + 132f + 151$, $f = 22 \cdot 85^2 e$. Then $\{n(t(f(e))) : e \in Z\}$ contains no k-tight optimal DLN for each $k = 0, 1, 2, 3$.

Proof. (a) By Lemma 2, $\{n(t) : t \in Z, t \geq 29\}$ contains no tight optimal DLN.

(b) If $\{n(t) : t \in Z,\ t \geqslant 29\}$ contains a near-tight optimal DLN, then there exists a near-tight tile $L = L(n; l, h, x, y)$. Let $z = |x - y|$. Then $z \leqslant 1$ by Lemma 2. Counting $H_{0,4}$ and $H_{1,4}$ in expression (2), we have

$$\begin{aligned} H_{0,4} =& 6t + 94 = 6 \cdot 28 \cdot 22^2 \cdot 85^4 e^2 + 6 \cdot 132 \cdot 22 \cdot 85^2 e + 1000 \\ =& 2^3(3 \cdot 7 \cdot 22^2 \cdot 85^4 e^2 + 3 \cdot 33 \cdot 22 \cdot 85^2 e + 125); \\ H_{1,4} =& 3t + 91 = 3 \cdot 28 \cdot 22^2 \cdot 85^4 e^2 + 3 \cdot 132 \cdot 22 \cdot 85^2 e + 544 \\ =& 17(3 \cdot 28 \cdot 22^2 \cdot 85^3 \cdot 5e^2 + 3 \cdot 132 \cdot 22 \cdot 85 \cdot 5e + 32). \end{aligned}$$

Note that the number in the brackets of the expression of $H_{0,4}$ is odd, and, that of the expression of $H_{1,4}$ is not a multiple of 17. It follows that neither $H_{0,4}$ nor $H_{1,4}$ satisfies the condition in Theorem 2; that is, eq. (1) in a and b has no integral solution. This implies that $\{n(t(f(e))) : e \in Z\}$ contains no near-tight optimal DLN.

(c) If $\{n(t(f(e))) : e \in Z\}$ contains a 2-tight optimal DLN, then there exists a 2-tight tile $L = L(n; l, h, x, y)$. Let $z = |x - y|$. Then for $z \geqslant 5$ and $t \geqslant 151$, we have

$$3n(t) - \frac{3}{4}5^2 = \left(3t + \frac{5}{2}\right)^2 + 3t - 97 > \left(3t + \frac{5}{2}\right)^2.$$

By Lemma 1, a contradiction can be deduced as follows:

$$3t + 3 = d(n(t)) > 3t + \frac{5}{2} + \frac{5}{2} - 2 = 3t + 3.$$

1) It is easy to prove that this condition is also sufficient.

Therefore $z \leqslant 4$. For $0 \leqslant z \leqslant 4$ and $j = 5$, the values of $H_{z,5}$ in expression (2) can be counted as follows:

$H_{4,5} = 99 = 3^2 \cdot 11;$

$H_{3,5} = 3t + 97 = 2(3 \cdot 14f^2 + 3 \cdot 66f + 275);$

$H_{2,5} = 6t + 97 = 17(6 \cdot 28 \cdot 22^2 \cdot 85^2 \cdot 5e^2 + 6 \cdot 132 \cdot 22 \cdot 85 \cdot 5e + 59);$

$H_{1,5} = 9t + 99 = 2(9 \cdot 14f^2 + 9 \cdot 66f + 729);$

$H_{0,5} = 12t + 103 = 5(12 \cdot 28 \cdot 22^2 \cdot 17 \cdot 85^3 e^2 + 12 \cdot 132 \cdot 22 \cdot 17 \cdot 85e + 383).$

It is easy to verify that no value of $H_{z,5}$ satisfies the condition in Theorem 2. Therefore, eq. (1) in a and b has no integral solution, implying that $\{n(t(f(e))) : e \in Z\}$ contains no 2-tight optimal DLN.

(d) If $\{n(t(f(e))) : e \in Z\}$ contains a 3-tight optimal DLN, then there exists a 3-tight tile $L = L(n; l, h, x, y)$. Let $z = |x - y|$. Then for $z \geqslant 7$ and $t \geqslant 151$,

$$3n(t) - \frac{3}{4}7^2 = \left(3t + \frac{5}{2}\right)^2 + 3t - 115 > \left(3t + \frac{5}{2}\right)^2.$$

By Lemma 1, a contradiction can be deduced as follows:

$$3t + 4 = d(n(t)) > 3t + \frac{5}{2} + \frac{7}{2} - 2 = 3t + 4.$$

Therefore $z \leqslant 6$. For $0 \leqslant z \leqslant 6$ and $j = 6$, the values of $H_{z,6}$ in expression (2) can be counted as follows:

$H_{6,6} = 114 = 2 \cdot 57;$

$H_{5,6} = 3t + 109 = 2(3 \cdot 14f^2 + 3 \cdot 66f + 281);$

$H_{4,6} = 6t + 106 = 11(6 \cdot 28 \cdot 22 \cdot 2 \cdot 85^4 e^2 + 6 \cdot 12 \cdot 22 \cdot 85^2 e + 92);$

$H_{3,6} = 9t + 105 = 3 \cdot 2^3(3 \cdot 7 \cdot 11 \cdot 22 \cdot 85^4 e^2 + 3 \cdot 33 \cdot 22 \cdot 85e + 61);$

$H_{2,6} = 12t + 106 = 2(6t + 53);$

$H_{1,6} = 15t + 109 = 2(15 \cdot 14f^2 + 15 \cdot 66f + 1187);$

$H_{0,6} = 18t + 114 = 3(6t + 38).$

It is easy to verify that for each $z = 1, 2, \cdots, 6$, $H_{z,6}$ does not satisfy the condition in Theorem 2; that is, eq. (1) has no integral solution. For $H_{0,6}$, if eq. (1) has an integral solution, then, by Theorem 2, $6t+38$ can be expressed as the form $s^2 + 3m^2$. Note that 3 is not a divisor of $(6t+38)$, but 2 is. By Corollary of Theorem 1, we should have $6t + 38 \equiv 4 \pmod 6$. But $6t + 38 \equiv 2 \pmod 6$, a contradiction.

To sum up, $\{n(t(f(e))) : e \in Z\}$ contains no k-tight optimal DLN for each $k = 0, 1, 2, 3$.

Theorem 4. Let $n(t) = 3t^2 + 6t - 26$, $t = t(g) = 14812g^2 + 3036g + 151$, $s(g) = 14308392g^4 + 6176604g^3 + 984630g^2 + 68625g + 1764$. Then $\{G(n(t(g)); s(g)) : g = g(e) = 22 \cdot 85^2 e, e \in Z\}$ is an infinite fimily of 4-tight optimal DLNs, diameter $3t + 5$, and initial element $G(69283; 1764)$.

Proof. Let $n(t) = 3t^2 + 6t - 26$, $t(f) = 28f^2 + 132f + 151$, $f = 23g$, $g = 22 \cdot 85^2 e$. Then $t = 14812g^2 + 3036g + 151$. By Theorem 6, $\{n(t(g(e))) : e \in Z\}$ contains no k-tight optimal DLN for each $k = 0, 1, 2, 3$. To complete the proof, it suffices to show that it contains a 4-tight optimal DLN. To this end, let $z = x - y = 1$ and $j = 7$. By (1) we have

$$(a + b - 6)(a + b - 7) - ab - 7t - 26 = 0. \tag{5}$$

We find that $(a, b) = (1, -322g - 27)$ is a solution of (5). It follows that

$$l(g) = 2t + a = 29624g^2 + 6072g + 303;$$
$$h(g) = 2t + b = 29624g^2 + 5750g + 275;$$
$$x(g) = t + a + b - 7 = 14812g^2 + 2714g + 119;$$
$$y(g) = x - 1 = 14812g^2 + 2714g + 118;$$
$$h'(g) = h - y = 14182g^2 + 3036g + 157;$$
$$l'(g) = l - x = 14182g^2 + 3358g + 184.$$

Choose $\alpha(g) = 322g^2 + 73g + 4$, $\beta(g) = -322g^2 - 66g - 3$. Then $\alpha(g)y(g) + \beta(g)h'(g) = 1$ and g.c.d.$(y(g), h'(g)) = 1$ for $g = 22 \cdot 85^2 e$, $e \in Z$. By Lemma 3, the L-tile $L(n; l, h, x, y)$ is $(1, s)$-realizable, where $s(g) = \alpha(g)l(g) - \beta(g)l'(g) = 14308392g^4 + 6176604g^3 + 984630g^2 + 68625g + 1764$.

Corollary. $G(69283; 1764)$ is a 4-tight optimal DLN; its diameter is 458.

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