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On Diameters of Altered Graphs

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Abstract For given positive integers t and $d\,(\geq 2)$, let $F(t,d)$ and $P(t,d)$ denote the minimum diameter of a graph obtained by adding t extra edges to a graph and a path with diameter d, respectively, and $f(t, d)$ denote the maximum diameter of a graph obtained after deleting t edges from a graph with diameter d. It is known that $P(1, d)=\left\lceil\frac{d+1}{2}\right\rceil$ for $d\geq 2$, $P(2, d)=\left\lceil\frac{d+1}{3}\right\rceil$ for $d\geq 3$, $P(3, d)=\left\lceil\frac{d+2}{4}\right\rceil$ if $d\geq 5$, and, in general, $\frac{d+1}{t+1}-1\leq P(t,d)<\frac{d+1}{t+1}+3$ for $t, d\geq 4$. In the present paper, we establish $F(t, f(t, d))\leq d\leq f(t, F(t, d))$ and prove $\left\lceil\frac{d}{t+1}\right\rceil\leq F(t, d)=P(t, d)\leq\left\lceil\frac{d-2}{t+1}\right\rceil+3$. Moreover, $F(t, d)\leq\left\lceil\frac{d}{t}\right\rceil+1$ if d is large enough. In particular, we derive the exact values $P(t, (2k-1)(t+1)+1)=2k$ for any positive integer k, and $\left\lceil\frac{d}{t+1}\right\rceil\leq P(t, d)\leq\left\lceil\frac{d}{t+1}\right\rceil+1$ for $t=4$ or 5 and $d\geq 4$.

Keywords Diameter; Altered graph; Edge addition; Edge deletion
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1 Introduction

In this note, a graph $G=(V, E)$ always means a simple undirected graph (with-out loops and multiple edges) with vertex-set V. We follow [1] for graph-theoretical terminology and notation not defined here.

It is well-known that when the underlying topology of an interconnection network of a system is modelled by a graph G, the diameter of G is an important measure for communication efficiency and message delay of the system [5]. In a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. If the message delay exceeds a given time-bound in a network, one often adds some links to the network to ensure that the reach of a message can be within a required time. This situation motivates Chung and Garey [2] to propose the following well-known "edge-addition problem" in graph theory: given positive integers t and d, what is the minimum diameter of the graph ob-

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tained by adding t edges to a graph with diameter d?

Let $F(t, d)$ denote the minimum diameter of an altered graph obtained by adding t extra edges to a graph with diameter d. Determining the exact value of $F(t, d)$ is fairly difficult in general since it has been proved by Schoone, Bodlaender and van Leeuwen [4] that the problem, for given integers t, d and a connected graph G, constructing an altered graph G' of G by adding t extra edges to G such that G' has diameter of at most d is NP-complete. Thus, the problem of determining sharp upper bounds of $F(t, d)$ is of interesting.

Let $P(t, d)$ denote the minimum diameter of an altered graph G obtained from a single path of diameter d plus t extra edges. Clearly, $F(t, d) \leq P(t, d)$ for any integers t and d. It is easy to verify that $P(1, d) = \left\lceil \frac{d+1}{2} \right\rceil$ for $d \geq 2$. The results of Schoone et al in [4] showed $P(2, d) = \left\lceil \frac{d+1}{3} \right\rceil$ for $d \geq 3$, and $P(3, d) = \left\lceil \frac{d+2}{4} \right\rceil$ for $d \geq 5$. In general, Chung and Garey [2] obtained that $\frac{d+1}{t+1} - 1 \leq P(t, d) < \frac{d+1}{t+1} + 3$ for t, $d \geq 4$.

That is converse to "edge-addition problem" is the so-called "edge-deletion problem". Let $f(t, d)$ denote the maximum diameter of a connected graph obtained after deleting t edges from a connected graph with diameter d. The exact values of $f(t, d)$ have been obtained for some small t or d. For example, Plesnik [3] decided $f(1, d) = 2d$, Schoone et al [4] obtained $f(2, d) = 3d - 1$ and $f(3, d) = 4d - 2$ for $d > 2$, and proved that $f(t, 2)$ is equal to $t+3$ for $t = 1, 2, 3, 4, 6$ and to $t+3$ otherwise. In the same paper, Schoone et al have ever pointed out that in order to prove an upper bound for $f(t, d)$, it is sufficient to consider graphs with diameter $d\,(\geq 2)$ that form a single path plus t extra edges. However, they have not explained any reason why it is so.

In this note, we will answer this question by establishing a relationship between $F(t, d)$ and $P(t, d)$. For given positive integers t and $d\,(\geq 2)$, we prove that

$$\left\lceil \frac{d}{t+1} \right\rceil \leq F(t, d) = P(t, d) \leq \left\lceil \frac{d-2}{t+1} \right\rceil + 3.$$

Moreover, $F(t, d) \leq \left\lceil \frac{d}{t} \right\rceil + 1$ if d is large enough. In particular, $P(t, (2k-1)(t-1)+1) = 2k$ for any positive integer k. Furthermore,

$$\left\lceil \frac{d}{5} \right\rceil \leq P(4, d) \leq \left\lceil \frac{d}{5} \right\rceil + 1, \quad \text{and} \quad \left\lceil \frac{d}{6} \right\rceil \leq P(5, d) \leq \left\lceil \frac{d}{6} \right\rceil + 1.$$

2 Several Lemmas

Lemma 1[4] $f(t, d) \leq (t+1)d$ for any positive integers t and d.

The following lemma is simple, but useful, obtained by a direct observation from the definitions.

Lemma 2 $F(t, d) \leq F(t, d')$ and $f(t, d) \leq f(t, d')$ for $d \leq d'$.

The following lemma explores a relationship between parameters $F(t, d)$ and $f(t, d)$ for given t and d.

Lemma 3 $F(t, f(t, d)) \le d \le f(t, F(t, d))$.

Proof Suppose that G' is a connected graph with diameter $f(t, d)$ obtained by deleting t edges from a graph G with diameter d, and let B be the set of the deleted edges of G. Conversely, the graph G can be thought of as a graph obtained by adding t edges in B to G' with diameter $f(t, d)$. It follows that the diameter d of G gives an upper bound on $F(t, f(t, d))$, that is, $F(t, f(t, d)) \le d$. Similarly, we can prove another inequality.

Lemma 4 $P(t, (2k-1)t+h+1) \le 2k$ for any integers t, k and h with $0 \le h \le 2k-1$.

Proof To prove the lemma, we construct an altered graph G from a single path of diameter d plus t extra edges. Let $d=(2k-1)t+h+1$ and $P=x_0x_1x_2\cdots x_d$ be a single path, and add t extra edges $x_mx_{m+1+i(2k-1)}$ for $i=1,2,\cdots,t$ to P, where $0 \le m \le k$ and $0 \le h-m \le k-1$. Now the end-vertices of these edges divide P into $t+2$ segments L_0, $L_1, \cdots, L_t$, L_{t+1}, where

$$L_0 = P(x_0, x_m),\ L_1 = P(x_m, x_{m+2k}),\ L_{t+1} = P(x_{m+1+t(2k-1)}, x_d),$$
$$L_i = P(x_{m+1+(i-1)(2k-1)}, x_{m+1+i(2k-1)}) \text{ for } i = 2,3,\cdots,t.$$

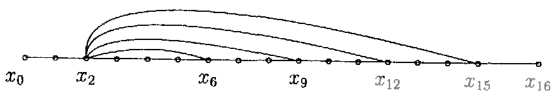

Figure 1　Construction of Lemma 4 for $k=2$, $t=4$, $m=2$ and $h=3$

We now prove that the diameter of G is at most $2k$. Note that the length of L_{t+1} is $d-(m+1+t(2k-1))=h-m \le h-m \le k-1$, and that the distance of two vertices in the same segment is at most $2k$. Thus, we need only to consider the distance between two vertices in different segments. Suppose that x is a vertex in the segment L_i and y is a vertex in the segment L_j with $0 \le i < j \le t+1$. Since x and y can reach the vertex x_m within k steps, the distance between x and y is at most $2k$. Thus, the diameter of G is at most $2k$, that is, $P(t, (2k-1)t+h+1) \le 2k$, and so the lemma follows.

Lemma 5 $P(4, 5(2k-1)+h) \le 2k+1$ for any integers k and h with $2 \le h \le 5$.

Proof Let $P=x_0x_1\cdots x_d$ be a single path, where $d=5(2k-1)+h$. The four vertices x_{2k-1}, x_{4k-1}, x_{6k}, x_{8k} partition P into segments P_1, P_2, P_3, P_4, P_5, where

$$P_1 = P(x_0, x_{2k-1}), \qquad P_2 = P(x_{2k-1}, x_{4k-1}),$$
$$P_3 = P(x_{4k-1}, x_{6k}), \qquad P_4 = P(x_{6k}, x_{8k}), \qquad P_5 = P(x_{8k}, x_d).$$

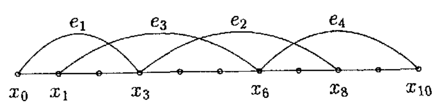

Figure 2　Construction of Lemma 5 for $k=1$ and $h=5$

We construct a graph G from P plus four edges $e_1 = x_0x_{4k-1}$, $e_2 = x_{4k-1}x_{8k}$, $e_3 = x_{6k}x_{2k-1}$, $e_4 = x_{6k}x_d$, and define 10 cycles as follows.

$C^1 = P_1 \cup + P_2 + e_1$, $C^2 = P_1 \cup + P_3 + e_1 + e_3$,

$C^3 = P_1 \cup + P_4 + e_1 + e_2 + e_3$, $C^4 = P_1 \cup + P_5 + e_1 + e_2 + e_3 + e_4$,

$C^5 = P_2 \cup + P_3 + e_3$, $C^6 = P_2 \cup + P_4 + e_2 + e_3$,

$C^7 = P_2 \cup + P_5 + e_2 + e_3 + e_4$, $C^8 = P_3 \cup + P_4 + e_2$,

$C^9 = P_3 \cup + P_5 + e_2 + e_4$, $C^{10} = P_4 \cup + P_5 + e_4$,

Their lengths are, respectively,

$$\varepsilon(C^1) = 4k; \qquad \varepsilon(C^i) = 4k + 2, \text{ for } i = 2,3,5,6,8;$$
$$\varepsilon(C^{10}) \leq 4k + 1; \qquad \varepsilon(C^i) \leq 4k + 3, \text{ for } i = 4, 7, 9.$$

It is easy to see that any two vertices x and y of G are contained some cycle C^i defined above. The fact

$$\max\{d(C^i): 1 \leq i \leq 10\} \leq \left\lceil \frac{4k+3}{2} \right\rceil = 2k + 1$$

means $P(4, 5(2k-1+h) \leq d(G) \leq 2k+1$, and so the lemma follows.

Lemma 6 $P(5, 6(2k-1)+h) \leq 2k+1$ for any integers k and h with $2 \leq h \leq 6$.

Proof Let $P = x_0x_1 \cdots x_d$ be a single path, where $d = 6(2k-1)+h$. The five vertices x_{2k-1}, x_{4k-1}, x_{6k}, x_{8k}, x_{10k} partition P into six segments P_1, P_2, P_3, P_4, P_5, P_6, where

$$P_1 = P(x_0, x_{2k-1}),\ P_2 = P(x_{2k-1}, x_{4k-1}),\ P_3 = P(x_{4k-1}, x_{6k}),$$
$$P_4 = P(x_{6k}, x_{8k}),\ P_5 = P(x_{8k}, x_{10k}),\ P_6 = P(x_{10k}, x_d).$$

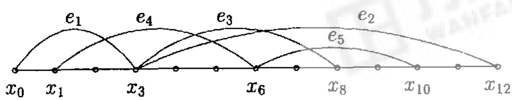

Figure 3 Construction of Lemma 6 for $k=1$ and $h=6$

We construct a graph G by adding five extra edges $e_1 = x_0x_{4k-1}$, $e_2 = x_{4k-1}x_d$, $e_3 = x_{4k-1}x_{8k}$, $e_4 = x_{6k}x_{2k-1}$, $e_5 = x_{6k}x_{10k}$ to P, and define 15 cycles as follows.

$C^1 = P_1 \cup + P_2 + e_1$, $C^2 = P_1 \cup + P_3 + e_1 + e_4$,

$C^3 = P_1 \cup + P_4 + e_1 + e_3 + e_4$, $C^4 = P_1 \cup + P_5 + e_1 + e_3 + 4_4 + e_5$,

$C^5 = P_1 \cup + P_6 + e_1 + e_2 + e_4 + e_5$, $C^6 = P_2 \cup + P_3 + e_4$,

$C^7 = P_2 \cup + P_4 + e_3 + e_4$, $C^8 = P_2 \cup + P_5 + e_3 + e_4 + e_5$,

$C^9 = P_2 \cup + P_6 + e_2 + e_4 + e_5$, $C^{10} = P_3 \cup + P_4 + e_3$,

$C^{11} = P_3 \cup + P_5 + e_3 + e_5$, $C^{12} = P_3 \cup + P_6 + e_2 + e_5$,

$C^{13} = P_4 \cup + P_5 + e_5$, $C^{14} = P_4 \cup + P_6 + e_2 + e_3 + e_5$,

$C^{15} = P_5 \cup + P_6 + e_2 + e_3$.

Their lengths are, respectively,

$$\varepsilon(C^2) = 4k; \qquad \varepsilon(C^i) \leq 4k + 2, \text{ for } i = 2,3,6,7,10,15;$$
$$\varepsilon(C^{13}) = 4k + 1; \quad \varepsilon(C^i) \leq 4k + 3, \text{ for } i = 4,5,8,9,11,12,14.$$

It is easy to see that any two vertices x and y of G are contained in some cycle C^i defined above. The fact

$$\max\{d(C^i): 1 \leq i \leq 15\} \leq \left\lceil \frac{4k+3}{2} \right\rceil = 2k + 1$$

means $P(4, 6(2k-1)+h) \leq d(G) \leq 2k+1$ for any k and h with $2 \leq h \leq 6$.

3 Proofs of Main Results

Theorem 1 $F(t, d) = P(t, d)$ for any positive integers t and d.

Proof Clearly, it is sufficient to prove $F(t, d) \geq P(t, d)$. To this end, suppose that G is a graph with diameter d and G' is an altered graph with diameter $F(t, d)$ obtained by adding a set B of t extra edges to G. Let $P = x_0x_1x_2\cdots x_d$ be a shortest path of length d in G and let V_i be a set of vertices in G whose distances to x_0 are equal to i. Then $x_i \in V_i$ for each $i = 0, 1, \cdots, d$. Let H be a subgraph of G' induced by P together with the edges in B whose end-vertices are all in P, and let B_1 be a set of edges in both of B and H, $B_2 = B \backslash B_1$. If $B_2 = \emptyset$, then $F(t, d) = d(G') \geq d(H) \geq P(t, d)$.

Now assume $B_2 \neq \emptyset$. For $e = xy \in B_2$, without loss of generality, we can assume $x \in V_i$ and $y \in V_j$ with $i \neq j$. Let H' be a graph obtained by adding $|B_2|$ edges x_ix_j to H for each $e = xy \in B_2$ with $x \in V_i$ and $y \in V_j$. Then H is a spanning subgraphs of H', and so $F(t, d) = d(G') \geq d(H') \geq P(t, d)$. The theorem follows.

Theorem 2 For given positive integers t and d,

$$\left\lceil \frac{d}{t+1} \right\rceil \leq P(t, d) \leq \left\lceil \frac{d-2}{t+1} \right\rceil + 3.$$

In particular, $P(t, (2k-1)(t+1)+1) = 2k$ for any positive integer k and $F(t, d) \leq \left\lceil \frac{d}{t} \right\rceil + 1$ if k is large enough.

Proof We first show $P(t, d) \geq \frac{d}{t+1}$. Note that $d \leq f(t, F(t, d))$ by Lemma 3 and $f(t, F(t, d)) \leq (t+1)F(t, d)$ by Lemma 1. Immediately, we have $d \leq (t+1)F(t, d)$, that is, $P(t, d) = F(t, d) \geq \frac{d}{t+1}$.

Let $d = (2k-1)(t+1)+1$. In this case, $P(t, d) \geq \left\lceil \frac{d}{t+1} \right\rceil = 2k$. On the other hand, choosing $m = k$ and $h = k-1$ in Lemma 4, we have $P(t, d) \leq 2k$ immediately. Therefore, $P(t, (2k-1)(t+1)+1) = 2k$ for any positive integers t and k.

We now show $F(t, d) \leq \frac{d-2}{t+1} + 3$. To this end, for a fixed t, let $d(k) = (2k-1)(t+1)+1$. Since $d(1) = t+2$ and $d(k+1) - d(k) = 2t+2$, for a given positive integer d, there are positive integers k and r such that $d = (2k-1)(t+1)+1-r$ with $0 \leq r \leq 2t+1$. Then $d + r = (2k-$

$1)(t+1)+1$. It follows from Lemma 2 and the above result just proved that

$$F(t,d) \leq F(t,d+r) = 2k = \frac{d+r+t}{t+1} \leq \frac{d-2}{t+1} + 3.$$

Finally, we prove $F(t,d) \leq \left[\frac{d}{t}\right]+1$ if k is enough large. By Lemma 4, we have $P(t, (2k-1)t+r+1) \leq 2k$ for $0 \leq r \leq 2k-1$. For the same reason as the above, for a given positive integer d, there are positive integers k and r such that $d=(2k-1)t+1+r$ with $0 \leq r \leq 2t-1$. It follows that if $t \leq k$, then

$$P(t,d) = P(t, (2k-1)t+r+1) \leq 2k = \frac{d-r-1}{t} + 1 \leq \frac{d}{t} + 1$$

as desired.

As applications of Theorem 2, we establish tighter upper and lower bounds of $P(4,d)$ and $P(5,d)$, whose difference is only one.

Theorem 3 $\left[\frac{d}{5}\right] \leq P(4,d) \leq \left[\frac{d}{5}\right]+1$ for any integer $d\,(\geq 4)$.

Proof It is easy to verify that $P(4,d)=2=\left[\frac{d}{5}\right]+1$ if $d=4$ or 5. Suppose $d \geq 6$ below. By Theorem 2, we have $P(4,d) \geq \left[\frac{d}{5}\right]$ and $P(4, 5(2k-1)+1)=2k$. One the other hand, by Lemma 5, we have $P(4, 5(2k-1)+h) \leq 2k+1$ for any k and h with $2 \leq h \leq 5$. Note that for any positive integer $d\,(\geq 6)$ there are integers k and h with $k \geq 1$ and $1 \leq h \leq 10$ such that $d=5(2k-1)+h$. Thus, by Theorem 2 and Lemma 5, we have

$$P(4,d) = 2k, \qquad \text{if } d = 5(2k-1)+1;$$
$$2k \leq P(4,d) \leq 2k+1, \qquad \text{if } d = 5(2k-1)+2,3,4 \text{ or } 5;$$
$$2k+1 \leq P(4,d) \leq 2k+2, \qquad \text{if } d = 10k+1,2,3,4 \text{ or } 5.$$

These imply that $\left[\frac{d}{5}\right] \leq P(4,d) \leq \left[\frac{d}{5}\right]+1$ for $d \geq 6$, and so the theorem follows.

Theorem 4 $\left[\frac{d}{6}\right] \leq P(5,d) \leq \left[\frac{d}{6}\right]+1$ for any integer $d\,(\geq 4)$.

Proof It is easy to verify that $P(5,d)=2=\left[\frac{d}{6}\right]+1$ if $d=4,5$ or 6. Suppose $d \geq 7$ below. By Theorem 2, we have $P(5,d) \geq \left[\frac{d}{6}\right]$ and $P(5, 6(2k-1)+1)=2k$. One the other hand, by Lemma 6, we have $P(5, 6(2k-1)+h) \leq 2k+1$ for any k andn h with $2 \leq h \leq 6$. Note that for any positive integer $d\,(\geq 7)$ there are integers k and h with $k \geq 1$ and $1 \leq h \leq 12$ such that $d=6(2k-1)+h$. Thus, by Theorem 2 and Lemma 6, we have

$$P(5,d) = 2k, \qquad \text{if } d = 6(2k-1)+1;$$
$$2k \leq P(5,d) \leq 2k+1, \qquad \text{if } d = 6(2k-1)+2,3,4,5 \text{ or } 6;$$
$$2k+1 \leq P(5,d) \leq 2k+2, \qquad \text{if } d = 10k+1,2,3,4,5 \text{ or } 6.$$

These imply that $\left[\frac{d}{6}\right] \leq P(5,d) \leq \left[\frac{d}{6}\right]+1$ for $d \geq 7$, and so the theorem follows.

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变更图的直径

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摘要　对于给定的正整数 t 和 $d(\geq 2)$，用 $F(t,d)$ 和 $P(t,d)$ 分别表示在所有直径为 d 的图和路中添加 t 条边后得到的图的最小直径，用 $f(t,d)$ 表示从所有直径为 d 的图中删去 t 条边后得到的图的最大直径. 已经证明 $P(1,d)=\left\lceil\frac{d}{2}\right\rceil$，$P(2,d)=\left\lceil\frac{d+1}{3}\right\rceil$ 和 $P(3,d)=\left\lceil\frac{d+2}{4}\right\rceil$. 一般地，当 t 和 $d\geq 4$ 时有 $\frac{d+1}{t+1}-1\leq P(t,d)\leq\frac{d+1}{t+1}+3$. 在这篇文章中，我们得到 $F(t,f(t,d))\leq d\leq f(t,F(t,d))$ 和 $\left\lfloor\frac{d}{t+1}\right\rfloor\leq F(t,d)=P(t,d)\leq\left\lfloor\frac{d-2}{t+1}\right\rfloor+3$，而且当 d 充分大时，$F(t,d)\leq\left\lceil\frac{d}{t}\right\rceil+1$. 特别地，对任意正整数 k 有 $P(t,(2k-1)(t+1)+1)=2k$，当 $t=4$ 或 5，且 $d\geq 4$ 时有 $\left\lfloor\frac{d}{t+1}\right\rfloor\leq P(t,d)\leq\left\lfloor\frac{d}{t+1}\right\rfloor+1$.

关键词　直径；变更图；边增加；边减少