# 2-restricted edge connectivity of vertex-transitive graphs* 

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#### Abstract

The 2-restricted edge-connectivity $\lambda^{\prime \prime}$ of a graph $G$ is defined to be the minimum cardinality $|S|$ of a set $S$ of edges such that $G-S$ is disconnected and is of minimum degree at least two. It is known that $\lambda^{\prime \prime} \leq g(k-2)$ for any connected $k$-regular graph $G$ of girth $g$ other than $K_{4}, K_{5}$ and $K_{3,3}$, where $k \geq 3$. In this paper, we prove the following result: For a connected vertex-transitive graph of order $n \geq 7$, degree $k \geq 6$ and girth $g \geq 5$, we have $\lambda^{\prime \prime}=g(k-2)$. Moreover, if $k \geq 6$ and $\lambda^{\prime \prime}<g(k-2)$, then $\lambda^{\prime \prime} \mid n$ or $\lambda^{\prime \prime} \mid 2 n$.


## 1 Introduction

In this paper, a graph $G=(V, E)$ always means a simple undirected graph (without loops and multiple edges) with vertex-set $V$ and edge-set $E$. We follow Bondy and Murty [1] or Xu [18] for graph-theoretical terminology and notation not defined here.

It is well-known that when the underlying topology of an interconnection network is modelled by a graph $G$, the connectivity of $G$ is an important measure for faulttolerance of the network [17]. However, this measure has many deficiencies (see [2]). Motivated by the shortcomings of the traditional connectivity, Harary [5] introduced the concept of conditional connectivity by requiring some specific conditions to be satisfied by every connected component of $G-S$, where $S$ is a minimum cut of $G$. Certain properties of connected components are particularly important for applications in which parallel algorithms can run on subnetworks with a given topological structure [2, 6]. In [2, 3], Esfahanian and Hakim proposed the concept of restricted connectivity by requiring that very connected component must contain no isolated

[^0]vertex. The restricted connectivity can provide a more accurate fault-tolerance measure of networks and have received much attention recently. (For example, see [2, 3], [6]-[10], [14]-[19].) For regular graphs Latifi et al [6] generalized the restricted connectivity to $h$-restricted connectivity for the case of vertices by requiring that every connected component contains no vertex of degree less than $h$. In this paper we are interested in similar kind of connectivity for the case of edges.

Let $h$ be a nonnegative integer. Let $G$ be a connected graph with minimum degree $k \geq h+1$. A set $S$ of edges of $G$ is called an $h$-restricted edge-cut if $G-S$ is disconnected and is of minimum degree at least $h$. If such an edge-cut exists, then the $h$-restricted edge-connectivity of $G$, denoted by $\lambda^{(h)}(G)$, is defined to be the minimum cardinality over all $h$-restricted edge-cuts of $G$. From this definition, it is clear that if $\lambda^{(h)}$ exists, then for any $l$ with $0 \leq l \leq h, \lambda^{(l)}$ exists and

$$
\lambda^{(0)} \leq \lambda^{(1)} \leq \cdots \leq \lambda^{(l)} \leq \cdots \leq \lambda^{(h)}
$$

It is clear that $\lambda^{(0)}$ is the traditional edge-connectivity and $\lambda^{(1)}$ is the restricted edge-connectivity defined in $[2,3]$. In this paper, we restrict ourselves to $h=2$. For the sake of convenience, we write $\lambda^{\prime \prime}$ for $\lambda^{(2)}$. We use $g=g(G)$ to denote the girth of $G$, that is, the length of a shortest cycle in $G$. The following result ensures the existence of $\lambda^{\prime \prime}(G)$ if $G$ is regular.

Theorem $1(\mathrm{Xu}[15]) \quad$ Let $G$ be a connected $k$-regular graph with girth $g$ other than $K_{4}, K_{5}$ and $K_{3,3}$, where $k \geq 3$. Then $\lambda^{\prime \prime}(G)$ exists and $\lambda^{\prime \prime}(G) \leq g(k-2)$.

A graph $G$ is called vertex-transitive if there is an element $\pi$ of the automorphism group $\Gamma(G)$ of $G$ such that $\pi(x)=y$ for any two vertices $x$ and $y$ of $G$. It is wellknown $[12,13]$ that the edge-connectivity of a vertex-transitive graph is equal to its degree. The restricted edge-connectivity of vertex-transitive graphs has been studied in [16, 19].

For a special class of vertex-transitive graphs, circulant graphs, its 2-restricted edge-connectivity has been determined by Li [9]. In [14] Xu proved that $\lambda^{\prime \prime}(G)=$ $g(k-2)$ for a vertex-transitive graph $G\left(\neq K_{5}\right)$ with even degree $k$ and girth $g \geq 5$. In this paper, we prove the following result by making good use of the technique proposed by Mader [12] and Watkins [13], independently.

Theorem 2 For a connected vertex-transitive graph of order $n \geq 7$, girth $g$ and degree $k(\geq 4$ and $\neq 5)$, if $g \geq 5$ we have $\lambda^{\prime \prime}=g(k-2)$. Moreover, if $\lambda^{\prime \prime}<g(k-2)$, then $\lambda^{\prime \prime} \mid n$ or $\lambda^{\prime \prime} \mid 2 n$.

Note that in this theorem $k$ is not required to be even. The proof of Theorem 2 will be given in Section 3, and this follows the proof of two lemmas in the next section.

## 2 Notation and Lemmas

Let $G$ be a $k$-regular graph, where $k \geq 2$. Then $G$ contains a cycle and hence its girth is finite. It is known (see [11, Problem 10.11]) that

$$
|V(G)| \geq f(k, g)= \begin{cases}1+k+k(k-1)+\cdots+k(k-1)^{(g-3) / 2}, & \text { if } g \text { is odd }  \tag{1}\\ 2\left[1+(k-1)+\cdots+(k-1)^{(g-2) / 2}\right], & \text { if } g \text { is even }\end{cases}
$$

A vertex $x$ of $G$ is called singular if it is of degree zero or one. Let $X$ and $Y$ be two distinct nonempty proper subsets of $V$. The symbol $(X, Y)$ denotes the set of edges between $X$ and $Y$ in $G$. If $Y=\bar{X}=V \backslash X$, then we write $\partial(X)$ for $(X, \bar{X})$ and $d(X)$ for $|\partial(X)|$. The following inequality is well-known (see [11, Problem 6.48]).

$$
\begin{equation*}
d(X \cap Y)+d(X \cup Y) \leq d(X)+d(Y) \tag{2}
\end{equation*}
$$

A 2-restricted edge-cut $S$ of $G$ is called a $\lambda^{\prime \prime}$-cut if $|S|=\lambda^{\prime \prime}(G)>0$. Let $X$ be a proper subset of $V$. If $\partial(X)$ is a $\lambda^{\prime \prime}$-cut of $G$, then $X$ is called a $\lambda^{\prime \prime}$-fragment of $G$. It is clear that if $X$ is a $\lambda^{\prime \prime}$-fragment of $G$, then so is $\bar{X}$ and both $G[X]$ and $G[\bar{X}]$ are connected. A $\lambda^{\prime \prime}$-fragment $X$ is called a $\lambda^{\prime \prime}$-atom of $G$ if it has the minimum cardinality. It is clear that $G$ certainly contains $\lambda^{\prime \prime}$-atoms if $\lambda^{\prime \prime}(G)$ exists. For a given $\lambda^{\prime \prime}$-atom $X$ of $G$, since $G[X]$ is connected and contains no singular vertices, it contains a cycle. Thus $g(G) \leq|X| \leq|V(G)| / 2$.
Lemma 3 Let $G$ be a connected $k$-regular graph, where $k \geq 3$. Let $R$ be a proper subset of $V(G)$ and $U$ be the set of singular vertices in $G-\partial(R)$. If $\lambda^{\prime \prime}(G)$ exists and $U \subseteq R$, then $|R|<g(G)$ provided that one of the following three conditions is satisfied:
(a) $d(R) \leq \lambda^{\prime \prime}(G)$;
(b) $d(R) \leq \lambda^{\prime \prime}(G)+1$ and $|U| \geq 2$ or $k \geq 4$;
(c) $d(R) \leq \lambda^{\prime \prime}(G)+1$ and $|U|=1, k=3$, and $R$ contains no $\lambda^{\prime \prime}$-fragments of $G$.

Proof Let $g=g(G)$. Since $\lambda^{\prime \prime}(G)$ exists, $\lambda^{\prime \prime}(G) \leq g(k-2)$ by Theorem 1. Suppose to the contrary that $|R| \geq g$. We will derive contradictions.

If $G[R]$ contains no cycles, then $|E(G[R])| \leq|R|-1$ and

$$
\begin{aligned}
g(k-2)+1 & \geq \lambda^{\prime \prime}(G)+1 \geq d(R)=|R| k-2|E(G[R])| \\
& \geq|R| k-2(|R|-1)=|R|(k-2)+2 \\
& \geq g(k-2)+2,
\end{aligned}
$$

which is a contradiction.
In the following we assume that $G[R]$ contains cycles. Let $R^{\prime}$ be the vertex-set of the union of all maximal 2-connected subgraphs of $G[R]$. Then $U \subseteq R \backslash R^{\prime}$. Note that for any two distinct vertices $u$ and $v$ in $R^{\prime}$, any neighbor of $u$ and any neighbor of $v$ in $R \backslash R^{\prime}$ are not joined by a path. This implies that $G-\partial\left(R^{\prime}\right)$ contains no singular vertices. So $\partial\left(R^{\prime}\right)$ is a $\lambda^{\prime \prime}$-restricted edge-cut of $G$ for which $d\left(R^{\prime}\right) \geq \lambda^{\prime \prime}(G)$. Also note that for any edge $e \in\left(R^{\prime}, R \backslash R^{\prime}\right)$, either $e$ is incident with some vertex $z \in U$ or there is a path in $G\left[R \backslash R^{\prime}\right]$ connecting $e$ to some vertex $z \in U$. Furthermore, if two edges $e, e^{\prime} \in\left(R^{\prime}, R \backslash R^{\prime}\right)$ are distinct, then the corresponding two vertices $z, z^{\prime} \in U$
are distinct too. Thus $\left|\left(R^{\prime}, R \backslash R^{\prime}\right)\right| \leq|U|$, and $\left|\left(R \backslash R^{\prime}, \bar{R}\right)\right| \geq|U|(k-1)$ since $U \subseteq R \backslash R^{\prime}$. It follows that

$$
\begin{aligned}
d\left(R^{\prime}\right) & =d(R)-\left|\left(R \backslash R^{\prime}, \bar{R}\right)\right|+\left|\left(R^{\prime}, R \backslash R^{\prime}\right)\right| \\
& \leq d(R)-|U|(k-1)+|U| \\
& =d(R)-|U|(k-2)
\end{aligned}
$$

from which we have

$$
\begin{equation*}
\lambda^{\prime \prime}(G) \leq d\left(R^{\prime}\right) \leq d(R)-|U|(k-2) \tag{3}
\end{equation*}
$$

If $d(R) \leq \lambda^{\prime \prime}(G)$, then from (3) we have $\lambda^{\prime \prime}(G) \leq d(R)-1 \leq \lambda^{\prime \prime}(G)-1$, which is a contradiction.

If $d(R) \leq \lambda^{\prime \prime}(G)+1$ and $|U| \geq 2$ or $k \geq 4$, then from (3) we have $\lambda^{\prime \prime}(G) \leq$ $d(R)-2=\lambda^{\prime \prime}(G)-1$, again a contradiction.

If $d(R) \leq \lambda^{\prime \prime}(G)+1,|U|=1, k=3$, then from (3), we have $d\left(R^{\prime}\right)=\lambda^{\prime \prime}(G)$. Thus $R^{\prime}$ is a $\lambda^{\prime \prime}$-fragment of $G$ contained in $R$, which contradicts our condition (c). The proof of the lemma is complete.
Lemma 4 Let $G$ be a connected $k$-regular graph with $\lambda^{\prime \prime}(G)<g(k-2)$, where $k \geq 3$. If $X$ and $X^{\prime}$ are two distinct $\lambda^{\prime \prime}$-atoms of $G$, then $\left|X \cap X^{\prime}\right|<g$. Moreover, $X \cap X^{\prime}=\emptyset$ for any $k$ with $k \geq 4$ and $k \neq 5$.
Proof Note that $|X| \geq g$ since $X$ is a $\lambda^{\prime \prime}$-atom of $G$. If $|X|=g$, then $G[X]$ is a cycle of length $g$. Thus $g(k-2)=d(X)=\lambda^{\prime \prime}(G)<g(k-2)$, a contradiction. So we have $|X|>g$. Let

$$
A=X \cap X^{\prime}, \quad B=X \cap \overline{X^{\prime}}, \quad C=\bar{X} \cap X^{\prime} \quad D=\bar{X} \cap \overline{X^{\prime}} .
$$

Then $|D| \geq|A|$ and $|B|=|C|=|X|-|A| \geq 1$ since $X$ and $X^{\prime}$ are two distinct $\lambda^{\prime \prime}$-atoms of $G$.

We first show $|A|<g$. In fact, if $d(A) \leq \lambda^{\prime \prime}(G)$, then $G-\partial(A)$ contains singular vertices (for otherwise, $A$ is a $\lambda^{\prime \prime}$-fragment whose cardinality is smaller than $|X|$ ), and all of them are contained in $A$. Thus, $|A|<g$ by Lemma 3. If $d(A)>\lambda^{\prime \prime}(G)$, then

$$
d(D)=d\left(X \cup X^{\prime}\right) \leq d(X)+d\left(X^{\prime}\right)-d\left(X \cap X^{\prime}\right)<\lambda^{\prime \prime}(G)
$$

which implies that $G-\partial(D)$ contains singular vertices (for otherwise, $D$ is a 2 restricted edge-cut whose cardinality is smaller than $\lambda^{\prime \prime}$ ), and all of them are contained in $D$. Thus, $|D|<g$ by Lemma 3, and so $|A| \leq|D|<g$.

We now show $|A|=0$ for any $k$ with $k \geq 4$ and $k \neq 5$. Suppose to the contrary that $|A|>0$. Since $|A|<g, G[A]$ contains no cycle, that is, $G-\partial(A)$ contains at least one singular vertex. Let $y$ be a singular vertex in $G-\partial(A)$. Then $y \in A$. Consider the set $X \backslash\{y\}$ if $|(y, C)|>|(y, B)|$, and the set $X^{\prime} \backslash\{y\}$ if $|(y, C)|<|(y, B)|$. Then

$$
\begin{equation*}
d(X \backslash\{y\}) \leq d(X)-|(y, D)|-|(y, C)|+|(y, B)|+1 \leq d(X)=\lambda^{\prime \prime}(G) \tag{4}
\end{equation*}
$$

So there are singular vertices in $G-\partial(X \backslash\{y\})$, and all of them are in $X \backslash\{y\}$. By Lemma 3, $|X \backslash\{y\}|<g$, and so $g<|X|=|X \backslash\{y\}|+1 \leq g$, a contradiction. Thus,
we need only to consider the case where $|(y, C)|=|(y, B)|$. Note that in this case the inequality (4) does not hold only when $|(y, D)|=0$ and $y$ is a vertex of degree one in $G-\partial(A)$. It follows that $k=d_{G}(y)=|(y, C)|+|(y, B)|+1$. Thus, we need only to consider the case where $k$ is odd.

Let $W$ be the vertex-set of the connected component of $G[A]$ that contains $y$. Note that $W$ contains at least two vertices of degree one in $G-\partial(A)$, and that $W \subseteq A$. Thus, $2 \leq|W|<g$. Let $Y=X \backslash W$ if $|(W, B)| \leq|(W, C)|$, and $Y=X^{\prime} \backslash W$ if $|(W, B)| \geq|(W, C)|$. Then $\emptyset \neq Y \subset X$. Then

$$
d(Y)=d(X)-|(W, C)|-|(W, D)|+|(W, B)| \leq d(X)=\lambda^{\prime \prime}(G)
$$

which implies $|Y|<g$ by Lemma 3.
Since $k$ is odd and is at least 7 , there are at least 3 neighbors of $y$ in $B$ and $C$, respectively. We claim that no two neighbors of $y$ are in the same component of $G[Y]$. Suppose to the contrary that some component of $G[Y]$ contains at least two neighbors of $y$. Choose two such vertices $y_{1}$ and $y_{2}$ so that their distance in $G[Y]$ is as short as possible. Let $P$ be a shortest $y_{1} y_{2}$-path in $G[Y]$. Clearly, $P$ does not contain any other neighbors of $y$ except $y_{1}$ and $y_{2}$. Thus the length of $P$ satisfies $\varepsilon(P) \leq|Y|-2 \leq g-3$, and so the length of the cycle $y y_{1}+P+y_{2} y$ is smaller than $g$, a contradiction.

Thus, all neighbors of $y$ in $Y$ are in different components of $G[Y]$. Since $|Y|<g$, we can choose such a component $H$ of $G[Y]$ so that its order is at most $\left\lfloor\frac{1}{3} g\right\rfloor$. Let $z \in V(H)$ be a neighbor of $y$. Then $z$ is in $B$. Moreover, we claim that $d_{H}(z) \geq 2$. In fact, if $z$ is a singular vertex in $G[H]$, then $d\left(X^{\prime}\right) \leq \lambda^{\prime \prime}(G)+1$ and all singular vertices of $G-\partial\left(X^{\prime}\right)$ are in $X^{\prime}$, where $X^{\prime}=X \backslash\{y\}$. By Lemma 3, $|X|-1<g$, that is, $|X| \leq g$, a contradiction.

Let $L$ be a longest path containing $z$ in $H$ with two distinct end-vertices $a$ and $b$. Then the length of $L$ is at most $\left\lfloor\frac{1}{3} g\right\rfloor-1$. Noting that $d_{H}(a)=d_{H}(b)=1$, it follows that there exist $c, d \in W \backslash\{y\}$ such that they are neighbors of $a$ and $b$, respectively. If $c=d$, then the length of the cycle $a c+c b+L$ is equal to $2+\varepsilon(L) \leq 2+\left\lfloor\frac{1}{3} g\right\rfloor-1<g$, which is impossible. Therefore, we have $c \neq d$.

Let $Q$ and $R$ be the unique $y c$-path and $y d$-path in $G[W]$ since $G[W]$ is a tree, and let $e$ be the last common vertex of $Q$ and $R$ starting with $y$. Note that $e \neq y$ and

$$
\varepsilon(Q)+\varepsilon(R)+\varepsilon(Q(c, e) \cup R(e, d))=2[\varepsilon(Q)+\varepsilon(R(e, d))] \leq 2(g-2)
$$

Therefore, at least one of $\varepsilon(Q), \varepsilon(R)$ and $\varepsilon(Q(c, e) \cup R(e, d))$ is at most $\left\lfloor\frac{2}{3}(g-2)\right\rfloor$.
If $\varepsilon(Q) \leq\left\lfloor\frac{2}{3}(g-2)\right\rfloor$, then, by considering the lengths of the cycle $C_{1}=L(a, z)+$ $y z+Q+c a$, we have

$$
g \leq \varepsilon\left(C_{1}\right) \leq\left(\left\lfloor\frac{g}{3}\right\rfloor-2\right)+2+\left\lfloor\frac{2(g-2)}{3}\right\rfloor \leq g-1
$$

a contradiction.

If $\varepsilon(R) \leq\left\lfloor\frac{2}{3}(g-2)\right\rfloor$, then, by considering the lengths of the cycle $C_{2}=z y+R+$ $d b+L(d, z)$, we have

$$
g \leq \varepsilon\left(C_{2}\right) \leq 2+\left\lfloor\frac{2(g-2)}{3}\right\rfloor+\left(\left\lfloor\frac{g}{3}\right\rfloor-2\right) \leq g-1
$$

again a contradiction.
If each of $\varepsilon(Q)$ and $\varepsilon(R)$ is more than $\left\lfloor\frac{2}{3}(g-2)\right\rfloor$, then $\varepsilon(Q(c, e) \cup R(e, d)) \leq\left\lfloor\frac{2}{3}(g-\right.$ $2)\rfloor$ and by considering the lengths of the cycle $C_{3}=a c+Q(c, e) \cup R(e, d)+d b+L$, we have

$$
g \leq \varepsilon\left(C_{3}\right) \leq 2+\left\lfloor\frac{2(g-2)}{3}\right\rfloor+\left(\left\lfloor\frac{g}{3}\right\rfloor-1\right) \leq g-1
$$

a contradiction.
The proof of Lemma 4 is complete.

## 3 Proof of Theorem 2

Let $G$ be a connected vertex-transitive graph with order $n(\geq 7)$ and degree $k(\geq 4$ and $\neq 5)$. Then $\lambda^{\prime \prime}(G)$ exists and $\lambda^{\prime \prime}(G) \leq g(k-2)$ by Theorem 1. Suppose that $\lambda^{\prime \prime}(G)<g(k-2)$, and let $X$ be a $\lambda^{\prime \prime}$-atom of $G$. Under these assumptions we prove the following claims.
Claim $1 G[X]$ is vertex-transitive.
Proof Let $x$ and $y$ be any two vertices in $X$. Since $G$ is vertex-transitive, there is $\pi \in \Gamma(G)$ such that $\pi(x)=y$. Denote $\pi(X)=\{\pi(x): x \in X\}$. It is clear that $G[X] \cong G[\pi(X)]$ because $\pi$ induces an isomorphism between $G[X]$ and $G[\pi(X)]$. Hence $\pi(X)$ is also a $\lambda^{\prime \prime}$-atom of $G$. Since $y \in X \cap \pi(X)$, by Lemma $4, X=\pi(X)$. Thus, the setwise stabilizer

$$
\Pi=\{\pi \in \Gamma(G): \pi(X)=X\}
$$

is a subgroup of $\Gamma(G)$, and the constituent of $\Pi$ on $X$ acts transitively. This shows that $G[X]$ is vertex-transitive.
Claim 2 There exists a partition $\left\{X_{1}, X_{2}, \cdots, X_{m}\right\}$ of $V(G)$, where $m \geq 2$, such that $G\left[X_{i}\right] \cong G[X]$ and $X_{i}$ is a $\lambda^{\prime \prime}$-atom for $i=1,2, \cdots, m$.
Proof Let $x$ be a fixed vertex in $X$. Let $u$ be any element in $\bar{X}$. Since $G$ is vertextransitive, there exists $\sigma \in \Gamma(G)$ such that $\sigma(x)=u$. Moreover, $\sigma(X)$ is a $\lambda^{\prime \prime}$-atom of $G$. Let $X_{u}=\sigma(X)$. Then $X \cap X_{u}=\emptyset$ by Lemma 4 and $G[X] \cong G\left[X_{u}\right]$. Thus there are at least two $\lambda^{\prime \prime}$-atoms of $G$. It follows that for every $u$ in $G$ there is a $\lambda^{\prime \prime}$-atom $X_{u}$ that contains $u$ such that $G\left[X_{u}\right] \cong G[X]$, and either $X_{u}=X_{v}$ or $X_{u} \cap X_{v}=\emptyset$ for any two distinct vertices $u$ and $v$ of $G$. These $\lambda^{\prime \prime}$-atoms, $X_{1}, X_{2}, \cdots, X_{m}$, form a partition of $V(G)$, and $G\left[X_{i}\right] \cong G[X], i=1,2, \cdots, m$. Since $G$ has at least two distinct $\lambda^{\prime \prime}$-atoms, we have $m \geq 2$.
Claim $3 g=3$ or 4 and $\lambda^{\prime \prime} \mid n$ or $\lambda^{\prime \prime} \mid 2 n$.

Proof Suppose that $\lambda^{\prime \prime}(G)<g(k-2)$ and $X$ is a $\lambda^{\prime \prime}$-atom of $G$. Then $G[X]$ is vertextransitive by Claim 1 and there exists a divisor $m(\geq 2)$ of $n$ such that $|X|=n / m$ by Claim 2. Let $t$ denote the degree of $G[X]$. Then $2 \leq t \leq k-1$ and

$$
\begin{equation*}
\lambda^{\prime \prime}(G)=d(X)=|\partial(X)|=(k-t)|X|=(k-t) n / m \tag{5}
\end{equation*}
$$

Since $G[X]$ contains a cycle of length at least $g$, it follows from (1) and (5) that

$$
\begin{equation*}
g(k-2)>\lambda^{\prime \prime}(G)=(k-t)|X| \geq(k-t) f(t, g) \tag{6}
\end{equation*}
$$

Case $1 g$ is even. In this case, from (1) and (6), we have

$$
\begin{equation*}
0<g(k-2)-(k-t) 2\left[1+(t-1)+\cdots+(t-1)^{(g-2) / 2}\right] . \tag{7}
\end{equation*}
$$

The right hand side of (7) is increasing with respect to $t$ and is decreasing with respect to $g$. It is not difficult to show that the inequality (7) can hold only when $g=4$ and $t=k-1$. So $\lambda^{\prime \prime}(G)=|X|=n / m$ by (5).

Case $2 g$ is odd. In this case, from (1) and (6), we have

$$
\begin{equation*}
0<g(k-2)-(k-t)\left[1+t+t(t-1)+\cdots+t(t-1)^{(g-3) / 2}\right] \tag{8}
\end{equation*}
$$

The right hand side of (8) is increasing with respect to $t$ and is decreasing with respect to $g$. It is not difficult to show that the inequality in (8) can hold only when $g=3$ and $t=k-2$ or $t=k-1$. If $t=k-1$, then $\lambda^{\prime \prime}(G)=|X|=n / m$ by (5). If $t=k-2$, then $\lambda^{\prime \prime}(G)=2|X|=2 n / m$ by (5).

From Claim 3, it follows that, if $g \geq 5$, then $\lambda^{\prime \prime}=g(k-2)$. Also, if $\lambda^{\prime \prime}(G)<$ $g(k-2)$, then $g=3$ or 4 , and hence $\lambda^{\prime \prime} \mid n$ or $\lambda^{\prime \prime} \mid 2 n$. The proof of Theorem 2 is complete.


Figure 1: A vertex-transitive graph of degree $k=5$ and $\lambda^{\prime \prime}=8$
Remarks The result $\lambda^{\prime \prime}(G)=g(k-2)$ is invalid for connected vertex-transitive graphs of degree $k=5$. For example, consider the lexicographical product $C_{n}\left[K_{2}\right]$ of $C_{n}$ by $K_{2}$, where $C_{n}$ is a cycle of order $n \geq 4, K_{2}$ is a complete graph of order two. The definition of lexicographical product of graphs is referred to [4, pp.21-22] and the graph shown in Figure 1 is $C_{7}\left[K_{2}\right]$. Since both $C_{n}$ and $K_{2}$ are vertex-transitive, $C_{n}\left[K_{2}\right]$ is vertex-transitive (see, [4, the exercise 14.19]). It is easy to see that $C_{n}\left[K_{2}\right]$
is of degree $k=5$, girth $g=3$ and a set of any four vertices that induce a complete graph $K_{4}$ is a $\lambda^{\prime \prime}$-atom of $C_{n}\left[K_{2}\right]$, and hence $\lambda^{\prime \prime}=8<3(5-2)$. Two distinct $\lambda^{\prime \prime}$ atoms $X$ and $X^{\prime}$ corresponding two complete graphs of order four with an edge in common satisfy $\left|X \cap X^{\prime}\right|=2<3=g$. This fact shows that the latter half of Lemma 4 is invalid for $k=5$.

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