应 用 数 学
MATHEMATICA APPLICATA
2004 ,17 (supplement) :56～60

On Connectivity of Möbius Cubes *

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Abstract : The connectivity ,super connectedness and restricted edge-connectivity of a graph are important parameters to measure fault-tolerance of an interconnection network. This paper considers the n-dimensional Möbius cube MQ_n , shows that its connectivity and edge-connectivity both are equal to n , that is ,it is super vertex-connected for any $n \geq 1$ and super edge-connected if $n \neq 2$, and that its restricted connectivity and restricted edge-connectivity both are equal to $2n - 2$, where n is subject to $n \geq 3$ for the former and $n \geq 2$ for the latter.

Key words : Connectivity ; Restricted connectivity ; Super connectivity ; Möbius cubes
CLC Number : O157.5 **AMS(2000) Subject Classification :** 05C40 ;90B10
Document code : A **Article ID :** 1001-9847 (2004) Supplement-0056-05

Throughout this paper ,a graph $G = (V, E)$ always means a simple connected graph with vertex-set V and edge-set E. We follow [1] for graph-theoretical terminology and notation not defined here. A set of vertices (resp. ,edges) S of G is called a vertex-cut (resp. ,an edge-cut) if $G - S$ disconnected. The connectivity $\kappa(G)$ (resp. ,the edge-connectivity $\lambda(G)$) of G is defined as the minimum cardinality of a vertex-cut (resp. ,an edge-cut) S. It is customary to define $\kappa(K_n) = n - 1$, where K_n is a complete graph of order n.

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network ,the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important parameters to measure the fault-tolerance of the network. The parameters ,however ,have an obvious deficiency ,that it to tacitly assume that all elements in any subset of G can potentially fail at the same time. In other words ,in the definition of $\kappa(G)$ or $\lambda(G)$, absolutely no conditions or restrictions are imposed either on the set S or on the components of $G - S$. Consequently ,to compensate for these shortcomings ,it would seem natural to generalize the notion of the classical connectivity by introducing some conditions or restrictions on the set S and/or the components of $G - S$.

* **Received date :** Jan 21 ,2004
Foundation item : Supported by ANSF(01046102) ,NNSF of China (10271114)
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Bauer et al [2] suggested the concept of the super connectedness. A connected graph G is said to be super vertex-connected (resp. ,edge-connected) ,if every minimum vertex-cut (resp. ,edge-cut) isolates a vertex of G. Esfahanian and Hakimi [3,4] introduced the concepts of the restricted cut and the restricted connectivity of a graph. A set $S \subset V(G)$ (resp. , $S \subset E(G)$) is called a restricted vertex-set (resp. , edge-set) if it does not contain the neighbor-set of any vertex in G as its subset. A restricted vertex-set (resp. ,edge-set) S is called a restricted vertex-cut (resp. ,edge-cut) if $G - S$ is disconnected. The restricted vertex-connectivity (resp. , edge-connectivity) of a graph G, denoted by $\kappa'(G)$ (resp. , $\lambda'(G)$) ,is the minimum cardinality of a restricted vertex-cut (resp. ,edge-cut) in G. From definitions , the following proposition holds ,clearly.

Proposition Let G be a k- regular graph.

(1) If $\kappa'(G)$ exists and $\kappa'(G) > \kappa(G) = k$, then G must be super vertex-connected.

(2) If $\lambda'(G)$ exists and $\lambda'(G) > \lambda(G) = k$, then G must be super edge-connected.

Thus ,the restricted connectivity is a measure for fault tolerance of a network more accurately than the classical connectivity. Since then the researchers have payed much attention to the concept and determined the restricted connectivity for many well-known graphs (see ,for example ,[3]～[14]) .

In the present paper ,we consider the n- dimensional Möbius cube MQ_n, as an attractive alternative to the n- cube Q_n. We show that both the connectivity and edge-connectivity of MQ_n are equal to n, and that MQ_n is super vertex-connected for any $n \geq 1$ and super edge-connected if $n \neq 2$, and both the restricted connectivity and restricted edge-connectivity of MQ_n are equal to $2n - 2$, where n is subject to $n \geq 3$ for the former and $n \geq 2$ for the latter.

The n- dimensional Möbius cube is such a graph ,its vertex-set is $V = \{x_1 x_2 \cdots x_n : x_i \in \{0,1\},\ 1 \leq i \leq n\}$, the vertex $X = x_1 x_2 \cdots x_n$ connects to n other vertices $Y_i, (1 \leq i \leq n)$, where each Y_i satisfies one of the following equations:

$$
\begin{aligned}
Y_i &= x_1 x_2 \cdots x_{i-1} \overline{x_i} x_{i+1} \cdots x_n \text{ if } x_{i-1} = 0, \\
&= x_1 x_2 \cdots x_{i-1} \overline{x_i}\,\overline{x_{i+1}} \cdots \overline{x_n} \text{ if } x_{i-1} = 1,
\end{aligned}
$$

where $\overline{x_i}$ is the complement of x_i in $\{0,1\}$,that is , $\overline{x_i} \in \{0,1\} \setminus x_i$.

From the above definition , X connects to Y_i by complementing the coordinate x_i if $x_{i-1} = 0$ or by complementing all coordinates of $x_i, \cdots, x_n$ if $x_{i-1} = 1$. The connection between X and Y_1 is undefined , so we can assume x_0 is either equal to 0 or equal to 1 ,which gives us slightly different graphs. We call the graph a “0-Möbius cube ”if we assume $x_0 = 0$, and call the graph a “1 - Möbius cube ”if we assume $x_0 = 1$, denoted by MQ_n^0 and MQ_n^1, respectively. Generally ,we will use MQ_n to denote MQ_n^0 or MQ_n^1.

The Möbius cubes MQ_n is first proposed by Cull and Larson [15] as an attractive alternative to the n-cube Q_n. Like Q_n, MQ_n is an n- regular graph with 2^n vertices and $n2^{n-1}$ edges. The results given in the present paper show that MQ_n and Q_n have the same connectivity ,super connectedness and restricted connectivity. Furthermore , MQ_n is superior to Q_n in having a diameter of $[n + 2/2]$ for $MQ_n^0 (n \geq 4)$ and $[n + 1/2]$ for $MQ_n^1 (n \geq 1)$, approximately a half of Q_n 's diameter. However ,for $n \geq 4$, MQ_n is neither vertex-transitive nor edge-transitive. This lack of symmetry removes the Möbius cubes from the class of Cayley graphs.

From the definition of the Möbius cubes , MQ_n has a simple recursive property ,that is , MQ_n^0 and

MQ_n^1 can be constructed from MQ_{n-1}^0 and MQ_{n-1}^1 by adding 2^{n-1} edges, called cross-edges, connecting all pairs of vertices that differ only in the first coordinate, and in the first through the nth coordinates, respectively. For convenience, we express MQ_n as $MQ_n = L \odot R$, where $L = MQ_{n-1}^0$ and $R = MQ_{n-1}^1$, and denote by $x_l x_r$ the cross-edge connecting $x_l \in L$ and $x_r \in R$. The recursive structure of MQ_n gives the following simple property.

Lemma 1 Let $MQ_n = L \odot R$ with $n \geq 2$. Then every vertex $x_l \in L$ has exactly one neighbor x_r in R connected by the cross-edge $x_l x_r$.

Using this simple observation and the recursive property of MQ_n, we can also obtain the following property easily.

Lemma 2 MQ_n contains no triangle and any two nonadjacent vertices in MQ_n have common neighbors at most two for $n \geq 2$.

Now, we investigate the connectivity, the super connectedness and the restricted connectivity of MQ_n. We first consider the connectivity of MQ_n.

Theorem 1 $\kappa(MQ_n) = \lambda(MQ_n) = n$ for $n \geq 1$.

Proof By Whitney's inequality (see Theorem 3.1 in [1]), we have $\kappa(MQ_n) \leq \lambda(MQ_n) \leq \delta(MQ_n) = n$. Thus, in order to prove the theorem, we only need to prove $\kappa(MQ_n) \geq n$. We proceed by induction on $n \geq 1$. The assertion is true if $n = 1$ since both MQ_1^0 and MQ_1^1 are a complete graph K_2. Assume the induction hypoghesis for $n - 1$ when $n \geq 2$. To prove $\kappa(MQ_n) \geq n$, we only need to show that for any subset $F \subset V(MQ_n)$, if $|F| \leq n - 1$ then $MQ_n - F$ is connected.

Let $MQ_n = L \odot R$, and let $F_l = F \cap L$, and $F_r = F \cap R$. Then at least one of $L - F_l$ and $R - F_r$ is connected by the induction hypothesis. We can, without loss of generality, suppose that $R - F_r$ is connected. We show that any vertex u_l in $L - F_l$ can be connected to $R - F_r$. Let $u_l u_r$ be the cross-edge in $MQ_n = L \odot R$. If $u_r \notin F_r$, then we are done. So we assume that $u_r \in F_r$. Consider $N = N_{MQ_n}(u_l)$, which is the neighbor-set of u_l in MQ_n. Since $|N| = n > n - 1 \geq |F|$, there is a vertex $x_l \in N$ such that both x_l and $x_r \in R$ are not in F. This implies that u_l in $L - F_l$ can be connected to $R - F_r$ via the cross-edge $x_l x_r$. Thusm, we show that $|S| \geq n$ for any vertex-cut S in MQ_n, that is, $\kappa(MQ_n) = |S| \geq n$. The theorem follows.

We now determine $\kappa'(MQ_n)$. Clearly, $\kappa'(MQ_1)$ and $\kappa'(MQ_2)$ do not exist. When $n \geq 3$, we have the following theorem.

Theorem 2 $\kappa'(MQ_n) = 2n - 2$ for $n \geq 3$.

Proof We first show $\kappa'(MQ_n) \leq 2n - 2$ for $n \geq 3$. It is easy to be verified that $\kappa'(MQ_3) \leq 4$ and $\kappa'(MQ_4) \leq 6$. Suppose $n \geq 5$ below. Let u and v be two adjacent vertices in MQ_n and $S = N_{MQ_n}(u, v)$. Then $|S| = 2n = 2$ since MQ_n is n-regular and contains no triangle by Lemma 2, and $MQ_n - S$ is disconnected since $2^n - (2n - 2) - 2 \geq 2$. Because $n \geq 5$ and any two distinct vertices have common neighbors at most two by Lemma 2, the neighbor-set $N_{MQ_n}(x)$ is not included in S for any $x \in V(MQ_n)$. This fact shows that S is a restricted vertex-cut of MQ_n. Thus $\kappa'(MQ_n) \leq |S| = 2n - 2$ for $n \geq 3$.

We now prove $\kappa'(MQ_n) \geq 2n - 2$. To the end, we only need to show that for any restricted vertex-set F in MQ_n, if $|F| \leq 2n - 3$ then $MQ_n - F$ is connected.

Let $MQ_n = L \odot R$, and let $F_l = F \cap L$, and $F_r = F \cap R$. Obviously, $F_l \cap F_r = \emptyset$. Thus, either $|F_l| \leq n - 2$ or $|F_r| \leq n - 2$. We can, without loss of generality, suppose that $|F_r| \leq n - 2$. Then $R - F_r$ is connected since $\kappa(R) = \kappa(MQ^1_{n-1}) = n - 1$. We show that any vertex u_l in $L - F_l$ can be connected to the connected graph $R - F_r$. Let $u_l u_r$ be the cross-edge in $MQ_n = L \odot R$. If $u_r \notin F_r$, then we are done. So we assume that $u_r \in F_r$. Since F is a restricted vertex-set, there exist a vertex v_l adjacent to u_l in $L - F_l$. Consider $N = N_{MQ_n}(u_l, v_l)$, which is the neighbor-set of $\{u_l, v_l\}$ in MQ_n. Since $|N| - = 2n - 2 > 2n - 3$, there is a vertex $x_l \in N$ such that both x_l and $x_r \in R$ are not in F. This implies that u_l in $L - F_l$ can be connected to $R - F_r$ via the cross-edge $x_l x_r$.

Thus, we show that $|S| \geq 2n - 2$ for any restricted vertex-cut S in MQ_n, that is, $\kappa'(MQ_n) = |S| \geq 2n - 2$. The theorem follows.

Corollary MQ_n is super vertex-connected for any $n \geq 1$.

Theorem 3 $\lambda'(MQ_n) = 2n - 2$ for $n \geq 2$.

Proof Consider an edge xy in MQ_n and the set of its adjacent edges $E(xy) = \{e \in E(MQ_n) \setminus \{xy\} : e = xu$ or $e = yu \in E(MQ_n), u \in V(MQ_n)\}$. Clearly, $MQ_n - E(xy)$ is disconnected and $E(xy) = 2n - 2$ since MQ_n is n-regular. Since MQ_n contains no triangle, $MQ_n - E(xy)$ contains no isolated vertex and, thus, $E(xy)$ is a restricted edge-cut of MQ_n. This gives $\lambda'(MQ_n) \leq |E(xy)| = 2n - 2$.

We now show $\lambda'(MQ_n) \geq 2n - 2$. Clearly, $\lambda'(MQ_2) \geq 2$. We assume that $n \geq 3$ and F is a restricted edge-set of MQ_n. We need to prove that if $|F| \leq 2n - 3$ then $MQ_n - F$ is connected. Let $MQ_n = L \odot R$. Then at least one of two $(n - 1)$-dimensional Möbius cubes L and R contains elements in F at most $\left[\frac{2n - 3}{2}\right] \leq n - 2$. Without loss of generality, assume that L contains elements in F at most $n - 2$. Then, by Theorem 1, $L - F$ is a connected spanning subgraph of L. In order to prove that $MQ_n - F$ is connected, we only need to show that any vertex x_r in R can be connected to L in $MQ_n - F$.

If the cross-edge $x_l x_r$ is not in F, then there is nothing to do. Suppose that $x_l x_r \in F$. Since F is a restricted edge-set, there exists an edge $x_r y_r$ in R such that $x_r y_r \notin F$. Since MQ_n contains no triangle by Lemma 2, $|N_{MQ_n}(x_r, y_r)| = 2n - 2 > 2n - 3$, and so there exists at least one $u_r \in N_{MQ_n}(x_r, y_r)$ such that the cross-edge $u_l u_r$ is not F. Thus, x_r can be connected to L via the vertex the cross-edge $u_l u_r$. Thus, we show that $|S| \geq 2n - 2$ for any restricted edge-cut S in MQ_n, that is, $\lambda'(MQ_n) = |S| \geq 2n - 2$. The theorem follows.

Corollary MQ_n is super edge-connected if $n \neq 2$.

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关于 Möbius 立方体网格的连通度

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摘要: 图的连通度、超连通性和限制连通度是度量互连网络容错性的重要参数. 该文考虑 n 维 Möbius 立方体网络 MQ_n, 证明了它的点和边连通度都为 n, 当 n 是任何正整数时它是超连通的, 当 $n\neq 2$ 时它是超边连通的, 当 $n\geq 3$ 时它的限制点连通度和当 $n\geq 2$ 时的限制边连通度都为 $2n-2$.

关键词: 连通度; 限制连通度; 超连通性; Möbius 立方体网络