The Proof of a Conjecture of Bouabdallah and Sotteau

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Let G be a connected graph of order n. A routing in G is a set of $n(n-1)$ fixed paths for all ordered pairs of vertices of G. The edge-forwarding index of G, $\pi(G)$, is the minimum of the maximum number of paths specified by a routing passing through any edge of G taken over all routings in G, and $\pi_{\Delta,n}$ is the minimum of $\pi(G)$ taken over all graphs of order n with maximum degree at most Δ. To determine $\pi_{n-2p-1,n}$ for $4p + 2\lceil p/3\rceil + 1 \le n \le 6p$, A. Bouabdallah and D. Sotteau proposed the following conjecture in [On the edge forwarding index problem for small graphs, Networks 23 (1993), 249–255]. The set $3 \times \{1, 2, \ldots, \lceil (4p)/3\rceil\}$ can be partitioned into $2p$ pairs plus singletons such that the set of differences of the pairs is the set $2 \times \{1, 2, \ldots, p\}$. This article gives a proof of this conjecture and determines that $\pi_{n-2p-1,n}$ is equal to 5 if $4p + 2\lceil p/3\rceil + 1 \le n \le 6p$ and to 8 if $3p + \lceil p/3\rceil + 1 \le n \le 3p + \lceil (3p)/5\rceil$ for any $p \ge 2$. © 2004 Wiley Periodicals, Inc. NETWORKS, Vol. 44(4), 292–296 2004

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1. INTRODUCTION

Let $G = (V, E)$ be a connected graph of order n. A routing R in G is a set of $n(n-1)$ fixed paths for all ordered pairs (x, y) of vertices of G. The path $R(x, y)$ specified by R carries the data transmitted from the source x to the destination y. It is possible that the fixed paths specified by a given routing R pass too frequently through certain vertices or edges, which means that the routing R overloads the vertex or the edge. The load of any vertex or edge is limited by the capacity of the vertex or the edge; otherwise, it would affect the efficiency of transmission, even resulting in malfunction of the network. To measure the load of a vertex or an edge, Chung, Coffman, Reiman, and Simon [2] proposed the notion of the forwarding index.

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Let G be a graph with a given routing R, x a vertex of G and e an edge of G. The load of x with respect to R, denoted by $\xi(G, R, x)$ [resp. the load of e with respect to R, denoted by $\pi(G, R, e)$], is defined as the number of paths specified by R passing through x [resp. e].

The vertex-forwarding index and the edge-forwarding index of G with respect to R are, respectively, defined as

$$\xi(G, R) = \max\{\xi(G, R, x) : x \in V(G)\}$$

and

$$\pi(G, R) = \max\{\pi(G, R, e) : e \in E(G)\}.$$

The vertex-forwarding index and the edge-forwarding index of G are, respectively, defined as

$$\xi(G) = \min\{\xi(G, R) : R \text{ is a routing of } G\}$$

and

$$\pi(G) = \min\{\pi(G, R) : R \text{ is a routing of } G\}.$$

The original study of forwarding indices is motivated by the problem of maximizing network capacity (see [2]). Minimizing the forwarding indices of a routing will result in maximizing the network capacity. Thus, it becomes very significant to determine the vertex and edge-forwarding indices of a given graph.

Many researchers have been interested in the forwarding indices of a graph (see, e.g., [1–5]). However, Saad [5] found that for an arbitrary graph determining its vertex-forwarding index is NP-complete even if the diameter of the graph is 2. It is still of interest to determine the forwarding indices subject to some graph-theoretical variables. For example, Chung et al. [2] proposed the following problem: Given Δ and n, determine $\xi_{\Delta,n}$, the minimum of $\xi(G)$ taken over all graphs of order n with maximum degree at most Δ. This problem was solved asymptotically in [2], and $\xi_{\Delta,n}$ has been determined for $n \le 15$ and $(n+4)/3 \le \Delta \le n-1$ by Heydemann et al. [3].

Heydemann et al. [4] considered the same problem for

the edge-forwarding index: given Δ and n, determine $\pi_{\Delta,n}$, the minimum of $\pi(G)$ taken over all graphs of order n with maximum degree at most Δ. Bouabdallah and Sotteau [1] determined $\pi_{\Delta,n}$ for some special values of n and Δ. For example, $\pi_{2,n} = \lfloor n^2/4 \rfloor$ for any $n \geq 3$, $\pi_{n-1,n} = 2$ for any $n \geq 2$, $\pi_{n-2,n} = 3$ for any $n \geq 6$, $n \neq 7$ and $\pi_{n-2,n} = 4$ for any $n = 4, 5, 7$. In particular, they obtained that for any $p \geq 1$,

$$\pi_{n-2p-1,n} = \begin{cases} 3, & \text{if } n \geq 10p + 1; \\ 4, & \text{if } 6p + 1 \leq n < 10p + 1; \\ 6, & \text{if } 4p + 1 \leq n \leq 4p + \lceil (2p)/3 \rceil. \end{cases}$$

The value of $\pi_{n-2p-1,n}$ has not been determined for $4p + \lceil (2p)/3 \rceil + 1 \leq n \leq 6p$. To determine this value, they remarked that $\pi_{n-2p-1,n} \leq 5$ for $4p + 2\lceil p/3 \rceil + 1 \leq n \leq 6p$ if the following conjecture is true (see Remark 3.14 in [1]).

To state the conjecture, we illustrate the strange notation and terminology that will be used in the conjecture. Let $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_t\}$ be two sets. We use $A \uplus B$ to denote the multi-set $\{a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t\}$. Then $2 \times A$ is defined as $A \uplus A$ and $3 \times A$ as $A \uplus A \uplus A$. In general, for a positive integer n, $n \times A$ is defined as the multi-set

$$\{\overbrace{a_1, \ldots, a_1}^{n}, \overbrace{a_2, \ldots, a_2}^{n}, \ldots, \overbrace{a_s, \ldots, a_s}^{n}\}.$$

Conjecture. *The set* $3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\}$ *can be partitioned into* $2p$ *pairs plus singletons such that the set of differences of the pairs is the set* $2 \times \{1, 2, \ldots, p\}$.

For example, when $p = 4$ in the conjecture, the set is

$$3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\} = \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6\}.$$

We partition this set into 8 pairs $\{2, 3\}$, $\{4, 5\}$, $\{1, 3\}$, $\{2, 4\}$, $\{1, 4\}$, $\{3, 6\}$, $\{1, 5\}$, $\{2, 6\}$ plus two singletons $\{5\}$ and $\{6\}$. Then the set of differences of the pairs is $\{3 - 2, 5 - 4, 3 - 1, 4 - 2, 4 - 1, 6 - 3, 5 - 1, 6 - 2\} = \{1, 1, 2, 2, 3, 3, 4, 4\} = 2 \times \{1, 2, 3, 4\}$.

In this article we will prove this conjecture and obtain the following result.

Theorem 2. *For any* $p \geq 2$, *we have*

$$\pi_{n-2p-1,n} = \begin{cases} 5, & \text{if } 4p + 2\lceil p/3 \rceil + 1 \leq n \leq 6p; \\ 8, & \text{if } 3p + \lceil p/3 \rceil + 1 \leq n \leq 3p + \lceil (3p)/5 \rceil. \end{cases}$$

The proofs of the conjecture and Theorem 2 are given in Section 2 and Section 3, respectively. In Section 4, we give some remarks.

TABLE 1. $3k + 1$ pairs of integers with odd differences.

Difference	Pairs of integers	
$3k$	$\{1, 3k + 1\}$	$\{k, 4k\}$
$3k - 2$	$\{2, 3k\}$	$\{k + 1, 4k - 1\}$
$\vdots$	$\vdots$	$\vdots$
3	$\left\{\frac{3k-1}{2}, \frac{3k+5}{2}\right\}$	$\left\{\frac{5k-3}{2}, \frac{5k+3}{2}\right\}$
1	$\left\{\frac{3k+1}{2}, \frac{3k+3}{2}\right\}$	$\left\{\frac{5k-1}{2}, \frac{5k+1}{2}\right\}$

2. PROOF OF THE CONJECTURE

We now state the conjecture as the following theorem.

Theorem 1. *The multi-set* $3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\}$ *can be partitioned into* $2p$ *pairs plus singletons such that the set of differences of the pairs is equal to* $2 \times \{1, 2, \ldots, p\}$ *for any positive integer* p.

Proof. We prove this theorem by constructing the required partition according to the following three possible cases (six subcases), respectively. We detail only the first subcase here. The interested reader is referred to [6] for detailed proofs of the remaining subcases.

Case 1. $p = 3k$. In this case, $\{1, 2, \ldots, \lceil (4p)/3 \rceil\} = \{1, 2, \ldots, 4k\}$.

Subcase 1.1. k is odd. We construct $3k + 1$ pairs of integers with the set of odd differences $\{1, 3, \ldots, 3k\}$ as follows

$$\{i, 3k + 2 - i\}, \quad \{i + k - 1, 4k + 1 - i\}, \qquad 1 \leq i \leq (3k + 1)/2,$$

and $3k - 1$ pairs of integers with the set of even differences $\{2, 4, \ldots, 3k - 1\}$ as follows

$$\{i, 3k - 1 - i\}, \quad \{i + k, 4k + 1 - i\}, \quad 1 \leq i \leq k - 1;$$
$$\{i, k + 1 - i\}, \quad \{i + 3k - 2, 4k + 1 - i\}, \qquad 1 \leq i \leq (k - 1)/2;$$
$$\{(7k - 3)/2, (7k + 1)/2\}, \quad \{(k + 1)/2, (7k - 1)/2\}.$$

These pairs of integers are shown in Tables 1 and 2, respectively, for more details.

It is easy to see that all integers used in the pairs of the left column in Table 1 form the set $\{1, 2, \ldots, 3k + 1\}$ and all integers used in the pairs of the right column in Table 1 form the set $\{k, k + 1, \ldots, 4k\}$, from which it is clear that

TABLE 2. $3k - 1$ pairs of integers with even differences.

Difference	Pairs of integers	
$3k - 1$	$\left\{\frac{k+1}{2}, \frac{7k-1}{2}\right\}$	$\{k + 1, 4k\}$
$3k - 3$	$\{1, 3k - 2\}$	$\{k + 2, 4k - 1\}$
$3k - 5$	$\{2, 3k - 3\}$	$\{k + 3, 4k - 2\}$
$\vdots$		$\vdots \quad \vdots$
$k + 3$	$\{k - 2, 2k + 1\}$	$\{2k - 1, 3k + 2\}$
$k + 1$	$\{k - 1, 2k\}$	$\{3k - 1, 4k\}$
$k - 1$	$\{1, k\}$	$\{3k, 4k - 1\}$
$k - 3$	$\{2, k - 1\}$	$\{3k + 1, 4k - 2\}$
$\vdots$	$\vdots$	$\vdots$
4	$\left\{\frac{k-3}{2}, \frac{k+5}{2}\right\}$	$\left\{\frac{7k-5}{2}, \frac{7k+3}{2}\right\}$
2	$\left\{\frac{k-1}{2}, \frac{k+3}{2}\right\}$	$\left\{\frac{7k-3}{2}, \frac{7k+1}{2}\right\}$

the intersection of the two sets is the set $\{k, k + 1, \ldots, 3k + 1\}$ and, thus, all integers of pairs in Table 1 form the multi-set

$$\{1, 2, \ldots, k - 1\} \uplus 2 \times \{k, k + 1, \ldots, 3k + 1\} \uplus \{3k + 2, 3k + 3, \ldots, 4k\}. \quad (1)$$

It is also easy to see that all integers used in the pairs of the left column in Table 2 form the multi-set $\{1, 2, \ldots, k - 1\} \uplus \{1, 2, \ldots, k - 1, k\} \uplus \{2k, 2k + 1, \ldots, 3k - 2\} \uplus \{(7k - 1)/2\}$ and all integers used in the pairs of the right column in Table 2 form the multi-set $\{k + 1, \ldots, 2k - 1\} \uplus \{3k - 1, \ldots, (7k - 3)/2\} \uplus \{(7k + 1)/2, \ldots, 4k\} \uplus \{3k + 2, \ldots, 4k\}$. Thus, all integers in pairs in Table 2 form the multi-set

$$2 \times \{1, 2, \ldots, k - 1\} \uplus \{k, k + 1, \ldots, 3k + 1\} \uplus 2 \times \{3k + 2, 3k + 3, \ldots, 4k\}. \quad (2)$$

By (1) and (2), all integers in Tables 1 and 2 form the multi-set $3 \times \{1, 2, \ldots, 4k\}$. Thus, the $6k$ pairs in Tables 1 and 2 form a required partition of the set $3 \times \{1, 2, \ldots, 4k\}$ when k is odd.

SUBCASE 1.2. k is even. The required partition of the set $3 \times \{1, 2, \ldots, 4k\}$ consists of the following $6k$ pairs of integers, where the $3k$ pairs with the set of odd differences $\{1, 3, \ldots, 3k - 1\}$ are

$$\{i, 3k + 1 - i\}, \quad \{i + k, 4k + 1 - i\}, \quad 1 \le i \le (3k)/2,$$

and the $3k$ pairs with the set of even differences $\{2, 4, \ldots, 3k\}$ are

$$\{i, 3k + 2 - i\}, \quad \{i + k + 1, 4k + 1 - i\}, \quad 1 \le i \le k;$$
$$\{i, k + 2 - i\}, \quad \{i + 3k + 1, 4k + 1 - i\}, \quad 1 \le i \le (k - 2)/2;$$
$$\{k/2, (k + 4)/2\}, \quad \{(k + 2)/2, (7k + 2)/2\}.$$

CASE 2. $p = 3k + 1$. In this case, $\{1, 2, \ldots, \lceil (4p)/3 \rceil\} = \{1, 2, \ldots, 4k + 2\}$.

SUBCASE 2.1. k is odd. The required partition of the set $3 \times \{1, 2, \ldots, 4k + 2\}$ consists of the following $6k + 2$ pairs of integers plus two singletons $4k + 1$ and $4k + 2$, where the $3k + 1$ pairs with the set of odd differences $\{1, 3, \ldots, 3k\}$ are

$$\{i, 3k + 2 - i\}, \quad \{i + k + 1, 4k + 3 - i\}, \quad 1 \le i \le (3k + 1)/2,$$

and the $3k + 1$ pairs with the set of even differences $\{2, 4, \ldots, 3k + 1\}$ are

$$\{i, 3k + 1 - i\}, \quad \{i + k, 4k + 1 - i\}, \quad 1 \le i \le k;$$
$$\{i + 1, k + 2 - i\}, \quad \{i + 3k + 2, 4k + 3 - i\}, \quad 1 \le i \le (k - 1)/2;$$
$$\{(k + 3)/2, (7k + 5)/2\}, \quad \{1, 3k + 2\}.$$

SUBCASE 2.2. k is even. The required partition of the set $3 \times \{1, 2, \ldots, 4k + 2\}$ consists of the following $6k + 2$ pairs of integers plus two singletons $(k + 2)/2$ and $(7k + 4)/2$, where the $3k + 2$ pairs with the set of odd differences $\{1, 3, \ldots, 3k + 1\}$ are

$$\{i, 3k + 3 - i\}, \quad \{i + k, 4k + 3 - i\}, \quad 1 \le i \le (3k + 2)/2,$$

and the $3k$ pairs with the set of even differences $\{2, 4, \ldots, 3k\}$ are

$$\{i, 3k + 2 - i\}, \quad \{i + k + 1, 4k + 3 - i\}, \quad 1 \le i \le k;$$
$$\{i, k + 2 - i\}, \quad \{i + 3k + 1, 4k + 3 - i\}, \quad 1 \le i \le k/2.$$

CASE 3. $p = 3k + 2$. In this case, $\{1, 2, \ldots, \lceil (4p)/3 \rceil\} = \{1, 2, \ldots, 4k + 3\}$.

SUBCASE 3.1. k is odd. The required partition of the set $3 \times \{1, 2, \ldots, 4k + 3\}$ consists of the following $6k + 4$ pairs of integers plus the singleton $k + 1$, where the $3k + 3$

pairs with the set of odd differences $\{1, 3, \ldots, 3k + 2\}$ are

$$\{i, 3k + 4 - i\}, \quad \{i + k + 2, 4k + 4 - i\}, \qquad 1 \le i \le (3k + 1)/2;$$

$$\{(3k + 3)/2, (3k + 5)/2\}, \quad \{(k + 3)/2, (7k + 7)/2\},$$

and the $3k + 1$ pairs with the set of even differences $\{2, 4, \ldots, 3k + 1\}$ are

$$\{i, 3k + 3 - i\}, \quad 1 \le i \le k;$$
$$\{i + k + 1, 4k + 4 - i\} 1 \le i \le k + 1;$$
$$\{i, k + 3 - i\}, \quad 1 \le i \le (k + 1)/2;$$
$$\{i + 3k + 3, 4k + 4 - i\}, \quad 1 \le i \le (k - 1)/2.$$

SUBCASE 3.2. k is even. The required partition of the set $3 \times \{1, 2, \ldots, 4k + 3\}$ consists of the following $6k + 4$ pairs of integers plus the singleton $k + 2$, where the $3k + 2$ pairs with the set of odd differences $\{1, 3, \ldots, 3k + 1\}$ are

$$\{i, 3k + 3 - i\}, \quad \{i + k + 1, 4k + 4 - i\}, \qquad 1 \le i \le (3k + 2)/2,$$

and the $3k + 2$ pairs with the set of even differences $\{2, 4, \ldots, 3k + 2\}$ are

$$\{i, 3k + 4 - i\}, \quad \{i + k + 2, 4k + 4 - i\}, \quad 1 \le i \le k;$$
$$\{i, k + 2 - i\}, \quad \{i + 3k + 2, 4k + 4 - i\}, \qquad 1 \le i \le k/2;$$

$$\{k + 1, 2k + 3\}, \quad \{(k + 2)/2, (7k + 6)/2\}.$$

All possible cases are exhausted and the proof of the theorem is completed. ■

3. PROOF OF THEOREM 2

The symbol $C(n; a_1, a_2, \ldots, a_k)$, $1 \le a_1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$, denotes a circulant graph of order n with the set of vertices $\{0, 1, 2, \ldots, n - 1\}$, where vertex i is joined to $i \pm a_s \pmod n$ for every s, $1 \le s \le k$. Note that this graph is $(2k - 1)$-regular if $a_k = n/2$, and $(2k)$-regular otherwise.

Given a graph G with the vertices $\{0, 1, 2, \ldots, n - 1\}$, the difference of an edge $\{x, y\}$ is defined as $\min\{|x - y|, n - |x - y|\}$. We notice that all the edges $\{(x + i) \bmod n, (y + i) \bmod n\}$ have difference d if the edge $\{x, y\}$ has difference d.

The definitions and notation not given here can be found in [2].

To prove our theorem, we need the following lemma, the former part of which is due to Bouabdallah and Sotteau [1] while the latter part is new. For the sake of completeness, we give its proof in full.

Lemma 1. *For any $p \ge 1$,*

$$\pi_{n-2p-1,n} \le \begin{cases} 5, & \text{if } n \ge 4p + 2\lceil p/3 \rceil + 1; \\ 8, & \text{if } n \ge 3p + \lceil p/3 \rceil + 1. \end{cases}$$

Proof. Consider a graph G with the set of vertices $\{0, 1, \ldots, n - 1\}$ isomorphic to the complete graph K_n minus the circulant graph $C(n; 1, 2, \ldots, p)$, that is, $G = K_n - E(C(n; 1, 2, \ldots, p))$. It is clear that if $n > 2p + 1$ then G is $(n - 2p - 1)$-regular and $\pi_{n-2p-1,n} \le \pi(G)$.

Let us define a routing R of G as follows: for any edge $\{x, y\}$ of G,

$$R(x, y) = R(y, x) = \{x, y\}; \tag{3}$$

the path $R(x, y)$ between two nonadjacent vertices x and y of G with difference d, $1 \le d \le p$, is defined below.

Consider a partition of the multiset $3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\}$ into pairs plus singletons, so that the set of differences between the pairs is $2 \times \{1, 2, \ldots, p\}$ (such a partition exists from Theorem 1).

For any d, $1 \le d \le p$, let $\{x_d, y_d\}$ and $\{x'_d, y'_d\}$ be two pairs of this partition with difference d, for example, $y_d - x_d = d = y'_d - x'_d$.

For any i, $0 \le i \le n - 1$, and any d, $1 \le d \le p$, we define

$$R(i, i + d) = (i, i + d + p + x_d, i + d)$$

$$R(i + d, i) = (i + d, i + d + p + x'_d, i), \tag{4}$$

where values are taken modulo n.

We first consider the case that $n \ge 4p + 2\lceil p/3 \rceil + 1$. Note that $4p + 2\lceil p/3 \rceil = 2p + 2\lceil (4p)/3 \rceil$. If $n \ge 2p + 2\lceil (4p)/3 \rceil + 1$, then $n - r > r$ for $p + 1 \le r \le p + \lceil (4p)/3 \rceil$. All paths in R defined above between nonadjacent vertices use the edges of G with differences $p + y_d$, $p + x_d$, $p + x'_d$ or $p + y'_d$. Because the multiplicity of each element in $\{x_d, y_d, x'_d, y'_d : 1 \le d \le p\} \subseteq 3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\}$ is at most 3, each edge of G with difference r, $p + 1 \le r \le p + \lceil (4p)/3 \rceil$, is used at most three times by these paths of R in (4). Also, the paths between the adjacent vertices in (3) use each edge of G twice, which implies $\pi_{n-2p-1,n} \le \pi(G) \le 5$.

Similarly, we can consider the case that $3p + \lceil p/3 \rceil + 1 \le n \le 4p + 2\lceil p/3 \rceil$. Because $3p + \lceil p/3 \rceil = 2p + \lceil (4p)/3 \rceil$, we have that $2p + \lceil (4p)/3 \rceil + 1 \le n \le 2p + 2\lceil (4p)/3 \rceil$. In this case, $n - r \ge p + 1$ when r is between $p + 1$ and $p + \lceil (4p)/3 \rceil$, the above routing R is also well defined. Different from the preceding case, in this case there exist some edges with difference r such that r and

$n - r$ are in $\{p + 1, \ldots, p + \lceil (4p)/3 \rceil\}$. These edges such as $\{x, x + p + r\}$ may be used in either the form $\{x, x + p + r\}$ or the form $\{x + p + r, (x + p + r) + (n - p - r)\} = \{x + p + r, (x + p + r) + p + r'\}$ in R for $n - p - r = p + r'$, $1 \le r, r' \le \lceil (4p)/3 \rceil$. So each edge of G is used at most six times by paths in (4). Also, the paths between the adjacent vertices in (3) use each edge of G twice, which implies that $\pi_{n-2p-1,n} \le \pi(G) \le 8$. ■

Lemma 2 [1]. $\Delta\pi_{\Delta,n} \ge 2\xi_{\Delta,n} + 2(n - 1)$ *for any n and* Δ.

Lemma 3 [3]. *For any n and* $p \ge 1$ *such that* $n \ge 3p + 2$, $\xi_{n-2p-1,n} = 2p$.

Proof of Theorem 2. From Lemma 2 and Lemma 3, we have that, for $n \ge 3p + 2$ and $p \ge 1$,

$$\pi_{n-2p-1,n} \ge \frac{4p + 2n - 2}{n - 2p - 1},$$

which gives

$$\pi_{n-2p-1,n} \ge 2 + \left[\frac{8p}{n - 2p - 1}\right]. \tag{5}$$

It follows from (5) that, for any $p \ge 2$,

$$\pi_{n-2p-1,n} \ge \begin{cases} 5, & \text{if } 3p + 2 \le n \le 6p; \\ 8, & \text{if } 3p + 2 \le n \le 3p + \lceil (3p)/5 \rceil. \end{cases}$$

These inequalities, together with Lemma 1, give the proof of the theorem. ■

4. REMARKS

The value of $\pi_{n-2p-1,n}$ has not been determined for $4p + \lceil (2p)/3 \rceil + 1 \le n < 4p + 2\lceil p/3 \rceil + 1$. Note that $\lceil (2p)/3 \rceil \ne 2\lceil p/3 \rceil$ only when $p = 3k + 1$. Thus, if we could prove that the number $4k + 2$ is used only once in a partition of $3 \times \{1, 2, \ldots, \lceil (4p)/3 \rceil\}$ when $p = 3k + 1$ in Theorem 1, we could obtain that $\pi_{n-2p-1,n} = 5$ when $n = 4p + \lceil (2p)/3 \rceil + 1$.

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