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SUPER EDGE-CONNECTIVITY OF DE BRUIJN AND KAUTZ UNDIRECTED GRAPHS

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Abstract. The super edge-connectivity of a graph is an important parameter to measure fault-tolerance of interconnection networks. This note shows that the Kautz undirected graph is super edge-connected, and provides a short proof of Lü and Zhang's result on super edge-connectivity of the de Bruijn undirected graph.

§ 1 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a graph G, the connectivity of G is an important measure for fault-tolerance of the network. We are, in this note, interested in the edge-failures, not vertex-failures, that is, we consider the edge-connectivity $\lambda(G)$ as a measurement for fault-tolerance of G.

Suppose that the edge-failure probabilities are equal to p and independent. The parameter

$$R(G,p) = 1 - \sum_{i=\lambda}^{\varepsilon} C_i p^i (1-p)^{\varepsilon - i} \qquad (1)$$

is an important measurement of global reliability of G, where $\lambda=\lambda(G)$ and C_i is the number of edge-cuts of cardinality i in G, ε is the number of edges of G. It has been proved by Ball[1] that the computation of $R(G,p)$ is NP-hard for general connected graph G. To minimize C_λ in (1), Bauer, et al.[2] suggested to study the concept of super edge-connectivity.

Definition 1. A connected graph G is said to be super edge-connected, if every minimum edge-cut isolates one vertex of G.

Since then one has found many well-known graphs that are super edge-connected, see,

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for example, [7,8,10]. In particular, Fàbrega and Fiol[6] proved that the Kautz digraph $K(d,n)$ is super edge-connected for any $d \geqslant 3$ and $n \geqslant 2$. [10] showed that the de Bruijn digraph $B(d,n)$ is super edge-connected for any $d \geqslant 2$ and $n \geqslant 1$. Recently, Lü and Zhang[9] have shown that the de Bruijn undirected graph is super edge-connected. Their proofs, however, are somewhat cumbersome.

In this note, we will provide a short proof of Lü and Zhang's result by making use of Soneoka's result and give a self-contained proof of that the Kautz undirected graph is super edge-connected.

§ 2 Properties of de Bruijn and Kautz graphs

The well-known de Bruijn digraph and Kautz digraph are two important classes of graphs, which are widely used in the design and analysis of interconnection networks. We first give the definitions of the de Bruijn digraph $B(d,n)$ and the Kautz digraph $K(d,n)$ for given integers d and n with $n \geqslant 1$ and $d \geqslant 2$.

Definition 2. The vertex-set of the de Bruijn digraph $B(d,n)$,

$$V = \{x_1x_2\ldots x_n : x_i \in \{0,1,\ldots,d-1\}, i=1,2,\ldots,n\},$$

and the edge-set E, where for $x,y \in V$, if $x = x_1x_2\ldots x_n$, then

$$(x,y) \in E \Leftrightarrow y = x_2x_3\ldots x_n\alpha, \alpha \in \{0,1,\ldots,d-1\}.$$

Definition 3. The vertex-set of the Kautz digraph $K(d,n)$,

$$V = \{x_1x_2\ldots x_n : x_i, x_n \in \{0,1,\ldots,d\}, x_{i+1} \neq x_i, i=1,2,\ldots,n-1\},$$

and the edge-set E, where for $x,y \in V$, if $x = x_1x_2\ldots x_n$, then

$$(x,y) \in E \Leftrightarrow y = x_2x_3\ldots x_n\alpha, \alpha \in \{0,1,\ldots,d\}\backslash\{x_n\}.$$

Clearly, $K(d,1)$ is a complete digraph of order $d+1$, while $B(d,1)$ is a complete digraph of order d plus a self-loop at every vertex. It has been shown that $B(d,n)$ and $K(d,n)$ are d-regular, $B(d,n)$ has connectivity $d-1$ while $K(d,n)$ has connectivity d. For more properties of de Bruijn and Kautz digraphs, readers can refer to Section 3.2 and Section 3.3, respectively, in the new book by Xu[11].

A pair of directed edges is said to be symmetric if they have the same endvertices but different orientations. de Bruijn and Kautz digraphs contain pairs of symmetric edges. If there is a pair of symmetric edges between two vertices x and y, then it is not difficult to see that the coordinates of x are alternately in two different components a and b. It follows that the de Bruijn digraph $B(d,n)$ contains exactly $\binom{d}{2}$ pairs of symmetric edges, while the Kautz digraph $K(d,n)$ contains exactly $\binom{d+1}{2}$ pairs of symmetric edges. The following fact is a simple observation.

Lemma 1. The directed distance between two end-vertices in different pairs of symmetric edges in $B(d,n)$ or $K(d,n)$ is equal to either $n-1$ or n. Moreover, two end-vertices in different pairs of symmetric edges have no vertex in common if and only if $n \geqslant 2$.

Definition 4. The de Bruijn undirected graph, denoted by $UB(d,n)$, is a simple undirected graph obtained from $B(d,n)$ by deleting the orientation of all edges and omitting multiple edges and self-loops. Similarly define the Kautz undirected graph $UK(d,n)$. An edge in $UB(d,n)$ [resp. $UK(d,n)$] is said to be singular if it corresponds to a pair of symmetric edges in $B(d,n)$ [resp. $K(d,n)$].

It is clear that $UB(d,1)=K_d$ and $UK(d,1)=K_{d+1}$. Moreover, $UB(d,n)$ and $UK(d,n)$ have the minimum degree $2d-2$ and $2d-1$, respectively, for $n \geqslant 2$. Furthermore, Esfahanian and Hakimi[4] have proved that $UB(d,n)$ had connectivity $2d-2$; Bermond, Homobono and Peyrat[3] have proved that $UK(d,n)$ has connectivity $2d-1$.

Lemma 2. Let $e=xy$ be any singular edge in $UB(d,n)$ or $UK(d,n)$. If $n \geqslant 2$, then there exist $2d-1$ internally disjoint xy-paths in $UB(d,n)$ [resp. $UK(d,n)$], one of length one, and $2d-2$ of length three except $UB(d,2)$, in which one of length one, two of length two, and $2d-4$ of length three.

Proof. Since the edge $e=xy$ is an xy-path of length exactly one, we need only prove that there exist $2d-2$ internally disjoint xy-paths in $UB(d,n)$ [resp. $UK(d,n)$] of desired length, all of which do not contain the edge $e=xy$. Without loss of generality, suppose n is even. Let $x=abab\ldots ab$, then $y=baba\ldots ba$, where $a \neq b$ and $a,b \in \{0,1,2,\ldots,d-1\}$ in $UB(d,n)$, and $a,b \in \{0,1,2,\ldots,d\}$ in $UK(d,n)$.

We first prove this result for $UB(d,n)$. Consider the vertices $u_i=abab\ldots abai$, $w_i=iaba\ldots ba$, $s_j=baba\ldots abj$, $t_j=jbab\ldots ab$, where $i \neq b$ and $j \neq a$. It is clear that if $n \geqslant 3$ then $P_i=(y,u_i,w_i,x)$ and $P'_j=(x,s_j,t_j,y)$ can form $2d-2$ internally disjoint xy-paths of length exactly three for each $i \in \{0,1,2,\ldots,d-1\} \backslash \{b\}$ and $j \in \{0,1,2,\ldots,d-1\} \backslash \{a\}$; if $n=2$ then $s_b=bb=t_b$ and $u_a=aa=w_a$, and so one of length one, two of length two, and $2d-4$ of length three, as desired.

We now prove this result for $UK(d,n)$. Consider the vertices $u_i=abab\ldots abai$, $w_i=iaba\ldots ba$, $s_i=baba\ldots abi$, $t_i=ibab\ldots ab$, where $i \neq a,b$. It is clear that $P_i=(y,u_i,w_i,x)$ and $P'_i=(x,s_i,t_i,y)$ can form $2d-2$ internally disjoint xy-paths of length exactly three for each $i \in \{0,1,2,\ldots,d\} \backslash \{a,b\}$ and for $n \geqslant 2$, as desired. Thus, the lemma follows.

Lemma 3. Let F be an edge-cut of $UB(d,n)$ or $UK(d,n)$. If $n \geqslant 2$ and F contains a singular edge, then $|F| \geqslant 2d-1$.

Proof. Let G be $UB(d,n)$ or $UK(d,n)$, and F an edge-cut in G that contains a singular edge $e=xy$. Then x and y are in different components of $G-F$. By Lemma 2, there exist $2d-1$ internally disjoint xy-paths of length at most three in G, each one of which must go through F. Thus $|F| \geqslant 2d-1$.

Note that de Bruijn digraph $B(d,n)$ has the minimum degree $\delta=d=\lambda$ and is super edge-connected. Then any λ-cut of $B(d,n)$ isolates a vertex having a self-loop. By this fact and Lemma 3, we can immediately give a short proof of the following result.

Corollary[9]. de Bruijn undirected graph $UB(d,n)$ is super edge-connected for $n\geqslant 1$.

Proof. It is sufficient to prove this result for $n\geqslant 2$. Let F be an edge-cut of $UB(d,n)$ with $|F|=2d-2$, X and Y be sets of the end-vertices of edges in F such that $F=[X,Y]$. We can, without loss of generality, suppose that $|X|\leqslant|Y|$. Thus, $|X|\leqslant|Y|\leqslant|F|=2d-2$. In order to prove the result, we need only to show $|X|=1$.

Since de Bruijn digraph $B(d,n)$ is regular, the number of edges from X to Y is equal to the number of edges from Y to X in $B(d,n)$, that is, $|(X,Y)|=|(Y,X)|$. If F contains a singular edge, then $|F|\geqslant 2d-1$ by Lemma 3, which contradicts the assumption that $|F|=2d-2$. Therefore F contains no singular edge. Thus, $|(X,Y)|=|(Y,X)|=d-1=\lambda(B(d,n))$. That $B(d,n)$is super edge-connected and any λ-cut of $B(d,n)$ isolates a vertex having a self-loop implies $|X|=1$. For otherwise, without loss of generality, suppose that the λ-cut (X,Y) isolates a vertex $x=(aa\ldots a)\in X$ in $B(d,n)$, then there are an in-neighbor $u=baa\ldots a\in X$ of x and an out-neighbor $v=aa\ldots ab\in Y$ of x. However the edge $(u,v)\in(X,Y)$ is not incident with x, which implies that the λ-cut (X,Y) does not isolate a vertex in $B(d,n)$, a contradiction. It follows that $UB(d,n)$ is super edge-connected.

§ 3 Super edge-connectivity of Kautz undirected graphs

Since $UK(2,2)=K_3\times K_2$, it is not super edge-connected clearly. The following result implies that it is the only exception.

Theorem. The Kautz undirected graph $UK(d,n)$ is super edge-connected except $UK(2,2)$.

Proof. If $n=1$, then the conclusion holds clearly since $UK(d,1)$ is a complete graph of order $d+1$. Suppose $n\geqslant 2$ below. Note that the edge-connectivity of $UK(d,n)$ is equal to $2d-1$. Let F be an edge-cut of $UK(d,n)$ with $|F|=\lambda=2d-1$, X and Y be the sets of the end-vertices of edges in F such that $F=[X,Y]$. We can, without loss of generality, suppose that $|X|\leqslant|Y|$. Thus, $|X|\leqslant|Y|\leqslant|F|=2d-1$. In order to prove the theorem, we need only to show either $|X|=1$ or $n=d=2$.

Suppose that $|X|>1$. We need to prove $n=d=2$. Since $K(d,n)$ is regular, the number of edges between X and Y in $K(d,n)$ is even. Note that $|F|=2d-1$ is odd, thus there must be at least one singular edge in F. Let $e=xy\in F$ be a singular edge, $x\in X$ and $y\in Y$, then both x and y are of degree $2d-1$ in $UK(d,n)$. Let H be a connected component of $UK(d,n)-F$ that contains X. By Lemma 2, there are $2d-1$ internally disjoint xy-paths in $UK(d,n)$, each of which is of length exactly three apart from the edge e and must go through F.

First we show that any neighbor of x is not in $V(H)\backslash X$. Suppose to the contrary that some neighbor of x is contained in H not in X, then at least two neighbors of y are in X since all neighbors of x, except y, are in paths of length exact three by Lemma 2, which implies that $|Y|\leqslant 2d-2$. Thus, X is a separating set of $UK(d,n)$, but $|X|\leqslant|Y|\leqslant 2d-2$, which contradicts the known fact that the connectivity of $UK(d,n)$ is $2d-1$. It follows that any neighbor of x is not in $V(H)\backslash X$.

Secondly, we show $X=V(H)$. In fact, if $V(H)\backslash X$ is nonempty, then the set $X\backslash\{x\}$ is a separating set of $UK(d,n)$. However, $|X\backslash\{x\}|\leqslant 2d-2$, which contradicts the connectivity of $UK(d,n)$ being $2d-1$. Therefore, $X=V(H)$.

Lastly, we show $n=d=2$. To the end, considering the sum of degrees of all vertices in H, we should have

$$|X|(|X|-1)\geqslant 2\varepsilon(H)\geqslant(2d-1)|X|-|F|=(2d-1)(|X|-1).$$

By our assumption $|X|>1$, it follows that $|X|=2d-1$ and the subgraph of $UK(d,n)$ induced by X is a complete graph K_{2d-1}. Thus, every vertex in X is of degree $2d-1$ in $UK(d,n)$. This implies that every vertex in X is an end-vertex of some pair of symmetric edge in $K(d,n)$. Thus, $n=2$ by Lemma 1. Since $UK(d,n)$ does not contain a complete K_m for $m\geqslant 4$, the subgraph of $UK(d,n)$ induced by X is a K_3, That is, $d=2$. The theorem follows.

Remarks. In our proof of super edge-connectivity of $UB(d,n)$, we make use of super edge-connectivity of $B(d,n)$. However, in the proof of super edge-connectivity of $UK(d,n)$, we make no use of super edge-connectivity of $K(d,n)$. In fact, $K(2,n)$ is not super edge-connected for $n\geqslant 2$, while $UK(2,n)$ is super edge-connected for $n\geqslant 3$.

§ 4 Problem and discussion

From definition, if a graph G is super edge-connected, then removal of any minimum edge-cut of G will isolate a vertex. It is a quite interesting problem that for a super edge-connected graph how many edges at least can be removed to disconnect the graph such that the resulting graph contains no isolated vertex. In order to evaluate fault-tolerance of a super edge-connected graph further, Esfahanian and Hakimi[5] introduced the concept of restricted edge-connectivity of G.

Definition 5. The restricted edge-connectivity of a connected G, denoted by $\lambda'(G)$, is the minimum number λ' for which removal of λ' edges from G makes the resulting graph disconnected and contain no isolated vertex.

Clearly, $\lambda'(G)\geqslant\lambda(G)$, and, furthermore,

$$\lambda'(G)>\lambda(G) \text{ if } G \text{ is super edge-connected.} \tag{2}$$

Determining the parameter $\lambda'(G)$ not only provides a more accurate measurement for fault-tolerance of networks, but also is of importance if G is a super edge-connected regular graph since, in that case, $C_i=0$ in (1) for each $i=\lambda+1,\ldots,\lambda'-1$. Study on the restricted connectivity has attracted much attention recently (see, for example, Section 4.6 of [11]).

We show in this paper that both de Bruijn and Kautz undirected graphs are super edge-connected except a few small graphs. Thus, as a topological structure of an interconnection network, de Bruijn or Kautz undirected graph has high fault-tolerance.

It has been shown[5] that if a connected graph G is neither K_3 nor a star, then $\lambda'(G)\leqslant$

$\xi(G)$, where $\xi(G)$ is the edge-degree of G, the minimum number of edges incident with an edge of G. From (2), we immediately have $2d-1 \leqslant \lambda'(UB(d,n)) \leqslant 4d-4$ and $2d \leqslant \lambda'(UK(d,n)) \leqslant 4d-4$. In particular, $\lambda'(UK(2,2))=3$ and $\lambda'(UK(2,n))=4$ for $n \geqslant 3$. Recently, [9] has shown $\lambda'(UB(2,3))=3$ and $\lambda'(UB(2,n))=4$ for $n \geqslant 4$. Thus, we suggest a conjecture as follows.

Conjecture. $\lambda'(UK(d,n))=\lambda'(UB(d,n))=4d-4$ for $d \geqslant 2$ and $n \geqslant 4$.

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