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Bounds for Distance Domination Numbers of Graphs *

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Abstract: For any positive integer l and any graph $G=(V,E)$, a set D of vertices of G is said to be an l-dominating set if every vertex in $V(G)$-D is at distance at most l from some vertex in D. The l-domination number, $\gamma_l(G)$, is the minimum cardinality of an l-dominating set of G. For a given graph G and an integer l, to determine $\gamma_l(G)$ is an NP-hard problem. It is proved in this paper that if G is a connected graph of order p with $p \geqslant l+1$, then $\gamma_l(G) \leqslant \left\lfloor \frac{p-\Delta+l-1}{l} \right\rfloor$ for any positive integer l. This bound is the best possible for some Δ and l and is an improvement on some known results.

Keywords: domination; distance l-domination number, diameter.

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0 Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The order, i. e., the number of vertices, and the maximum degree of vertices of G are denoted by $v(G)$ and $\Delta(G)$, respectively. The distance $d_G(x,y)$ between two vertices x and y is the length of a shortest xy-path in G. The maximum distance between any two vertices in G is called the diameter, denoted by $d(G)$. For every vertex $x \in V(G)$, the kth neighborhood $N_k(x)$ of x is defined by $N_k(x)=\{y \in V(G): d_G(x,y)=k\}$, and $N_1(x)$ is usually called the neighborhood of x in G. Let H be a proper subgraph of G or a nonempty subset of $V(G)$. We use the symbol $N_1(H)$ to denote the neighborhood of H, that is, $N_1(H)=\{x \in V(G-H): xy \in E(G) \text{ for some } y \in H\}$. For terminology and notation not given

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here, the reader is referred to [1] or [8].

Let D be a subset of $V(G)$ and l be a positive integer. A vertex v of G is distance l-dominated by D in G if v is at most l from some vertex in D. The set D is a distance l-dominating set of G if each vertex in $V(G) - D$ is distance l-dominated by D. The distance l-domination number, $\gamma_l(G)$, is the minimum cardinality of distance l-dominating sets of G. A distance l-dominating set D of G is called a γ_l -set if its cardinality is equal to $\gamma_l(G)$.

Throughout this paper, we will omit the term "distance", that is, we will write "an l-dominating set" instead of "a distance l-dominating set" and so on. Slater [5] termed a distance l-dominating set as an l-basis and also gave an interpretation for an l-basis in terms of communication networks. Since then many researchers have paid much attention to this subject[2~7].

From the definition above, it is clear that $\gamma_l(G) = 1$ for any integer l not less than $d(G)$, the diameter of G. Thus, it is of great significance to discuss $\gamma_l(G)$ for a connected graph G only when $1 \leqslant l < d(G)$. For the special case of $l = 1$, $\gamma_1(G)$ is the classic domination number of G. As a result, for any graph G and any positive integer l, to determine $\gamma_l(G)$ is an NP-hard problem. Thus, it is of interest to establish a tight lower or upper bound of $\gamma_l(G)$ for $1 \leqslant l < d(G)$. Meir and Moon [4] established an upper bound for $\gamma_l(G)$, that is, $\gamma_l(G) \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$ for any connected graph of order p with $p \geqslant l + 1$. Sridharan, Subramanian and Elias[6] considered the special case of $l = 2$, independently, and also established $\gamma_2(G) \leqslant \left\lfloor \frac{p}{3} \right\rfloor$ for $p \geqslant 3$.

In this paper, we first generalize Meir and Moon's bound to disconnected graphs, then establish a new upper bound of $\gamma_l(G)$ in terms of Δ for any connected graph G of order p with $p \geqslant l + 1$, that is, $\gamma_l(G) \leqslant \left\lfloor \frac{p - \Delta + l - 1}{l} \right\rfloor$ for any positive integer l. This bound is the best possible and is an improvement on Meir and Moon's bound for some Δ and l.

1 Upper Bounds on γ_l

Meir and Moon [4] established $\gamma_l(G) \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$ for any connected graph of order p with $p \geqslant l + 1$. This result will be referred to a consequence of the following theorem, a fundamental result in this paper, and the proof given here is much shorter than Meir and Moon's.

Theorem 1. If each component of G with order p contains at least $l + 1$ vertices, then $\gamma_l(G) \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$ for any positive integer l. The upper bound is the best possible.

Proof. We first prove this theorem for trees. The theorem is clearly true for a tree of order no greater than $2l + 1$. Suppose that the theorem is true for all trees of order at most n. Let T be a tree of order $n + 1$, where $n \geqslant 2l + 1$, and $P = u_0, u_1, u_2, \cdots, u_m$ be a longest path in T. If $m \leqslant l$, then any vertex of T itself is an l-dominating set of T, that is, $\gamma_l(T) = 1 \leqslant \left\lfloor \frac{n+1}{l+1} \right\rfloor$.

Assume that $m \geqslant l+1$ below, let T' be the component of $T - \{u_{m-l}, \cdots, u_{m-1}, u_m\}$ containing the vertex u_0. If T' has order at most l, then the vertex u_{m-l} itself is a γ_l-set in T. If T' has order at least $l+1$, then, by the induction hypothesis, we get $\gamma_l(T') \leqslant \left\lfloor \frac{n-l}{l+1} \right\rfloor$. Let D be a γ_l-set in T', then $D \cup \{u_{m-l}\}$ is an l-dominating set in T and, hence, $\gamma_l(T) \leqslant \left\lfloor \frac{n-l}{l+1} \right\rfloor + 1 = \left\lfloor \frac{n+1}{l+1} \right\rfloor$. Therefore, $\gamma_l(T) \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$ for any tree T of order p.

Let $\mathscr{H}$ be the set of components of G, $H \in \mathscr{H}$ and T be a spanning tree in H. Clearly, $\gamma_l(H) \leqslant \gamma_l(T)$. By the hypothesis, the order $v(H) \geqslant l+1$. Thus, we have that

$$\gamma_l(G) = \sum_{H \in \mathscr{H}} \gamma_l(H) \leqslant \sum_{H \in \mathscr{H}} \left\lfloor \frac{v(H)}{l+1} \right\rfloor \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$$

as desired.

To complete the proof of the theorem, for any positive integer l, we need to construct a graph G of order p with $\gamma_l(G) = \frac{p}{l+1}$.

For any positive integer m, let H be a graph of order m and let G be the graph obtained from H by attaching a free path of length l at each vertex of H. The order of G is $p = m(l+1)$. On the one hand, since G has m free paths of length l, any l-dominating set in G contains at least one vertex of each path, which implies $\gamma_l(G) \geqslant m = \frac{p}{l+1}$. On the other hand, the vertex-set $V(H)$ of H is an l-dominating set in G, which implies $\gamma_l(G) \leqslant v(H) = m = \frac{p}{l+1}$. The theorem follows.

Remarks. The condition that each component of G contains at least $l+1$ vertices in Theorem 1 is necessary. For example, consider the graph G consisting of m copies of a path with l vertices. Clearly, $v(G) = ml$ and $\gamma_l(G) = m > \left\lfloor \frac{ml}{l+1} \right\rfloor$. The bound given in Theorem 1 is best in view of $G = \frac{p}{2}K_2$. For thus a graph G, $l = 1$ and $\gamma_1(G) = \frac{p}{2}$.

Corollary 1. (Meir and Moon [4]) $\gamma_e(G) \leqslant \left\lfloor \frac{p}{l+1} \right\rfloor$ for any connected graph G of order p with $p \geqslant l+1$.

In the following theorem, we give a new upper bound of $\gamma_l(G)$ in terms of the maximum degree Δ for a connected graph G.

Theorem 2. Let G be a connected graph of order p with $p \geqslant l+1$. Then $\gamma_l(G) \leqslant \left\lfloor \frac{p-\Delta+l-1}{l} \right\rfloor$ for any positive integer l.

Proof. Note that if $l \geqslant d$, where d is the diameter of G, then $\gamma_l(G) = 1$. Thus, the theorem is true. If $\Delta = p-1$, then $d = 1$, $l = 1$ and $\gamma_1(G) = 1$. Thus, the theorem is also true. Now suppose that $l < d$ and $\Delta < p-1$ below. Let u be a vertex in G with $d_G(u) = \Delta$. If $N_l(u)$

$=\emptyset$, then $\{u\}$ is the l-dominating set of G, that is $\gamma_l(G)=1$, the theorem follows. Therefore, we have $N_l(u)\neq\emptyset$ below. Let $N(u)=\{u\}\cup N_1(u)\cup\cdots\cup N_l(u)$, and $X=V(G)\setminus N(u)$. If $X\neq\emptyset$, then $\gamma_l(G)=1$, the theorem is true. Assume $X\neq\emptyset$ below. Then any ux-path in G must pass through $N_l(u)$ for any vertex x in X.

Let H be the set of vertices in all connected components with at least $l+1$ vertices in $G[X]$ and let B be a γ_l -set in $G[H]$. Then, by Theorem 1, we have that $|B|\leqslant\left\lfloor\frac{|H|}{l+1}\right\rfloor$. If $H=X$, then $\{u\}\cup B$ is an l-dominating set in G. Thus,

$$\gamma_l(G)\leqslant 1+|B|\leqslant 1+\frac{|H|}{l+1}\leqslant 1+\frac{p-\Delta-l}{l+1}=$$

$$\frac{p-\Delta+1}{l+1}<\frac{p-\Delta+l-1}{l}$$

as desired. The latter strict inequality holds subject to the hypothesis of $p>\Delta+1$.

Now suppose $H\subset X$ and let $S=X\setminus H$. Let I be the set of orders of connected components in $G[S]$. For each $i\in I$, let h_i be the number of connected components of order i in $G[S]$, and let H_i be the set of vertices of all connected components of order i in $G[S]$. Then

$$G[S]=\bigcup_{i\in I}G[H_i]\text{ and }|S|=\sum_{i\in I}ih_i.$$

Since G is connected, $N_l(u)\cap N_1(H_i)\neq\emptyset$ for each $i\in I$. Choose $M_l^i(H_i)\subseteq N_l(u)\cap N_1(H_i)$ is the set with minimum cardinality such that each vertex of $M_l^i(H_i)$ is adjacent to atleast one vertex of $N_1(H_i)$. For each j with $i\leqslant j\leqslant l-1$, let $M_j^i(H_i)\subseteq N_j(u)\cap N_1(M_{j+1}^i(H_i))$ be the set with minimum cardinality such that each vertex of $M_j^i(H_i)$ is adjacent to atleast one vertex of $M_{j+1}^i(H_i)$. Then, $|M_i^i(H_i)|\leqslant\cdots\leqslant|M_{l-1}^i(H_i)|\leqslant|M_l^i(H_i)|\leqslant i\,h_i$.

If $H_1=\emptyset$, let

$$M=\bigcup_{i\in I,1<i\leqslant j\leqslant l}M_j^i(H_i)\cup H_i.$$

Then each component of $G[M]$ has at least $l+1$ vertices, and by Theorem 1, we have that $\gamma_l(G[M])\leqslant\left\lfloor\frac{|M|}{l+1}\right\rfloor$. Let D' be a γ_l -dominating set of $G[M]$. And since $N_1(u)\notin M$, we have $|H|+|M|\leqslant p-|\{u\}\cup N_1(u)|=p-1-\Delta$. Clearly, the set $D=\{u\}\cup B\cup D'$ is an l-dominating set in G. Thus, we have that

$$\gamma_l(G)\leqslant|D|=1+|B|+|D'|\leqslant 1+\frac{|H|}{l+1}+\frac{|M|}{l+1}\leqslant$$

$$1+\frac{p-\Delta-1}{l+1}=\frac{p-\Delta+l}{l+1}<\frac{p-\Delta+l-1}{l}.$$

The latter strict inequality also holds subject to the hypothesis of $p>\Delta+1$.

If $H_1\neq\emptyset$, then let H_1^* be the set of vertices in H_1 which can be connected to some vertex in $M_j^i(H_i)$ $(1<i\leqslant j\leqslant l)$ by some path, and let $P_{H_1^*}$ be the set of vertices on these paths. Let $H_1^{**}=H_1-H_1^*$. Let

$$M' = \bigcup_{i\in I, 1<i\leq j\leq l} M_j^i(H_i) \cup H_i \cup H_1^* \cup P_{H_1^*}.$$

We also have that each component of $G[M']$ has at least $l+1$ vertices, and by Theorem 1, we have that $\gamma_l(G[M']) \leqslant \left\lfloor \frac{|M'|}{l+1} \right\rfloor$. Let D'' be a γ_l -dominating set of $G[M']$.

If $H_1^{**} = \emptyset$, then since $N_1(u) \not\subseteq M'$, we have $|H| + |M'| \leqslant p - |\{u\} \cup N_1(u)| = p - 1 - \Delta$. And we could see the set $D = \{u\} \cup B \cup D''$ is an l-dominating set in G. Thus, we have that

$$\gamma_l(G) \leqslant |D| = 1 + |B| + |D''| \leqslant 1 + \frac{|H|}{l+1} + \frac{|M'|}{l+1} \leqslant$$
$$1 + \frac{p-\Delta-1}{l+1} = \frac{p-\Delta+l}{l+1} < \frac{p-\Delta+l-1}{l}.$$

The latter strict inequality also holds subject to the hypothesis of $p > \Delta + 1$.

If $H_1^{**} \neq \emptyset$, then $M_1{}^1(H_1{}^{**}) \neq \emptyset$ and $D = \{u\} \cup B \cup D'' \cup M_1{}^1(H_1{}^{**})$ is an l-dominating set of G. Note $M_j{}^1(H_1{}^{**}) \cap M_j{}^i(H_i) = \emptyset$ for any i and j with $2 \leqslant i, j \leqslant l$. Then

$$\gamma_l(G) = |D| \leqslant 1 + |B| + |D''| + |M_1{}^1(H_1^{**})| \leqslant$$
$$1 + \frac{|H|}{l+1} + \frac{|M'|}{l+1} + \frac{|H_1^{**}| + |M_2^1(H_1^{**})| + \cdots + |M_l^1(H_1^{**})|}{l} \leqslant$$
$$1 + \frac{|H| + |M'|}{l+1} + \frac{p - |H| - |M'| - \Delta - 1}{l} =$$
$$\frac{p-\Delta+l-1}{l} + \frac{|H|+|M'|}{l+1} - \frac{|H|+|M'|}{l} \leqslant \frac{p-\Delta+l-1}{l}.$$

The theorem follows.

Remarks.

(Ⅰ) Note that for given Δ and l with $\Delta - l + 1 > 0$ if $p < (l+1)(\Delta - l + 1)$, then $\frac{p-\Delta+l-1}{l} < \frac{p}{l+1}$, and hence, the upper bound for $\gamma_l(G)$ given in Theorem 2 is tighter than Meir and Moon's in Corollary 1.

(Ⅱ) For given $\Delta \geqslant 2$ and l, if $m = \frac{p-\Delta+l-1}{l} \leqslant \Delta$, then there is a graph G of order p with maximum degree Δ such that $\gamma_l(G) = m$. Such a graph G can be obtained from a star $K_{1,\Delta}$ by joining $(m-1)$ paths of length l and one path of length $(l-1)$ at each of m vertices of degree one in $K_{1,\Delta}$. Clearly, $p = \Delta + ml$ and $\gamma_l(G) = m = \left\lfloor \frac{ml+l-1}{l} \right\rfloor = \left\lfloor \frac{p-\Delta+l-1}{l} \right\rfloor$. However, if $p \leqslant (l+1)(\Delta - l)$ with $\Delta > l$, that is, $m \leqslant \Delta - l - 1$, then $\left\lfloor \frac{p}{l+1} \right\rfloor \geqslant m + 1$.

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关于图的距离控制数的上界

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摘要 对于任意的正整数 l,连通图 G 的顶点子集 D 被称为距离 l-控制集,是指对于任意顶点 $v \notin D$,D 中至少含有一个顶点 u,使得距离 $d_G(u,v) \leqslant l$. 图 G 距离 l-控制数 $\gamma_l(G)$ 是指 G 中所有距离 l-控制集的基数的最小者. 确定图 G 的距离 l-控制数 $\gamma_l(G)$ 是 NP-问题. 给出了当 G 是阶数为 p ($p \geqslant l+1$) 的连通图时,对于任意的正整数 l,都有最优上界 $\gamma_l(G) \leqslant \left\lfloor \frac{p-\Delta+l-1}{l} \right\rfloor$. 而且针对某些 Δ 和 l,是对 Meir 和 Moon 的结果的一种改进。

关键词:距离控制数,控制数,直径.