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Forwarding indices of folded n-cubes☆

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Abstract

For a given connected graph G of order n, a routing R is a set of $n(n-1)$ elementary paths specified for every ordered pair of vertices in G. The vertex (resp. edge) forwarding index of a graph is the maximum number of paths of R passing through any vertex (resp. edge) in the graph. In this paper, the authors determine the vertex and the edge forwarding indices of a folded n-cube as $(n-1)2^{n-1}+1-((n+1)/2)\binom{n}{\lfloor \frac{n+1}{2} \rfloor}$ and $2^n - \binom{n}{\lfloor \frac{n+1}{2} \rfloor}$, respectively.

Keywords: Vertex forwarding index; Edge forwarding index; Folded n-cubes

1. Introduction

To measure the efficiency of a routing deterministically, Chung et al. [2] introduced the concept of forwarding index of a routing. A routing R of a simple connected graph G of order n is a set of $n(n-1)$ elementary paths $R(u,v)$ specified for all (ordered) pairs u, v of vertices of G. The vertex forwarding index $\xi(G,R)$ (resp. edge forwarding index $\pi(G,R)$) is the maximum number of paths of R going through a vertex (resp. an edge) of G. The minimum of $\xi(G,R)$ (resp. $\pi(G,R)$) over all possible routings R of G, denoted by $\xi(G)$ (resp. $\pi(G)$), is called the vertex forwarding (resp. edge forwarding) index of G.

Saad [11] proved that the problem determining the vertex forwarding index is NP-hard. However, Heydemann et al. [7] proved that for a Cayley graph $G=(V,E)$

$$\xi(G)=\sum_{v\in V} d(u,v)-(n-1), \quad \forall u \in V(G) \tag{1}$$

and for any connected graph $G=(V,E)$

$$\pi(G) \geqslant \frac{1}{|E(G)|}\sum_{(u,v)\in V\times V} d(u,v), \tag{2}$$

where equality (2) holds if and only if there exists a routing of shortest paths in G for which the number of paths going through every edge is the same. As applications of the two formulae, the forwarding indices of many well-known graphs have been determined by different authors (see, for example, [1,2,4–12]).

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In this note, we determine the forwarding indices of the folded n-cube. The folded n-cube $FH(n)$, proposed by El-Amawy and Latifi [3], is the graph of order 2^n whose vertices can be labelled as the n-length sequences of 0's and 1's, two vertices being adjacent whenever their labels differ in just one digit or all digits. We call the former a complementary edge and the latter a normal edge.

It has been known that $FH(n)$ is a Cayley graph with degree $(n+1)$ and diameter $\lfloor (n+1)/2 \rfloor$. Our main results are stated in the following theorem.

Theorem. $\xi(FH(n)) = (n-1)2^{n-1} - ((n+1)/2)\binom{n}{\lfloor \frac{n+1}{2} \rfloor} + 1$ *and* $\pi(FH(n)) = 2^n - \binom{n}{\lfloor \frac{n+1}{2} \rfloor}$.

2. Proof of Theorem

To determine $\xi(FH(n))$, from (1) we need only to compute the sum of all distances from a given vertex u in $FH(n)$. The computation of this sum for $FH(n)$ is easy, and the detail is left to the readers.

To determine $\pi(FH(n))$, we need to construct a routing R of shortest paths in $FH(n)$ such that the number of paths going through every edge is the same. Let $x = (x_1 x_2 \cdots x_n)$ and $y = (y_1 y_2 \cdots y_n)$ be any two different vertices in $FH(n)$, and let $H(x, y) = \sum_{i=1}^{n} |x_i - y_i|$. We define a directed path $R(x, y)$ in R as follows. If $H(x, y) \leqslant \lfloor n/2 \rfloor$,

$$R(x, y):\ (x_1 x_2 \cdots x_n) \to (y_1 x_2 \cdots x_n) \to (y_1 y_2 x_3 \cdots x_n) \to \cdots \to (y_1 y_2 \cdots y_n).$$

If n is even and $H(x, y) > n/2$, or n is odd and $H(x, y) > (n+1)/2$,

$$R(x, y):\ (x_1 x_2 \cdots x_n) \to (\bar{x_1} \bar{x_2} \cdots \bar{x_n}) \to (y_1 \bar{x_2} \cdots \bar{x_n}) \to \cdots \to (y_1 y_2 \cdots y_n).$$

If n is odd and $H(x, y) = (n+1)/2$,

$$R(x, y):\ (x_1 x_2 \cdots x_n) \to (y_1 x_2 \cdots x_n) \to (y_1 y_2 x_3 \cdots x_n) \to \cdots \to (y_1 y_2 \cdots y_n)$$

if $w(x) < w(\bar{x})$ and

$$R(x, y):\ (x_1 x_2 \cdots x_n) \to (\bar{x_1} \bar{x_2} \cdots \bar{x_n}) \to (y_1 \bar{x_2} \cdots \bar{x_n}) \to \cdots \to (y_1 y_2 \cdots y_n)$$

otherwise, where $w(x)$ denotes the number of 1's in a vertex x, $\bar{0} = 1$ and $\bar{1} = 0$.

It is easy to see from the symmetry that normal edges are used the same number of times, and complementary edges are used the same number of times by R. The total number of usage of normal edges by R is

$$2^n \left[\sum_{i=1}^{n/2} i \binom{n}{i} + \sum_{i=n/2+1}^{n} (n-i) \binom{n}{n-i} \right] = n 2^{n-1} \left[2^n - \binom{n}{\frac{n}{2}} \right]$$

if n is even, and

$$2^n \left[\sum_{i=1}^{(n-1)/2} i \binom{n}{i} + \sum_{i=(n+3)/2}^{n} (n-i) \binom{n}{n-i} \right] + 2^{n-1} \left[\frac{n+1}{2} \binom{n}{\frac{n+1}{2}} + \frac{n-1}{2} \binom{n}{\frac{n-1}{2}} \right]$$
$$= n 2^{n-1} \left[2^n - \binom{n}{\frac{n+1}{2}} \right]$$

if n is odd. And the total number of usage of complementary edges by R is

$$2^n \sum_{i=n/2+1}^{n} \binom{n}{i} = 2^{n-1} \left[2^n - \binom{n}{\frac{n}{2}} \right]$$

if n is even, and

$$2^n \sum_{i=(n+3)/2}^{n} \binom{n}{i} + 2^{n-1} \binom{n}{\frac{n+1}{2}} = 2^{n-1} \left[2^n - \binom{n}{\frac{n+1}{2}} \right]$$

if n is odd.

Therefore, the number of usage of every normal edge (resp. every complementary edge) by R can be obtained by dividing the total number of usage of normal edges (resp. complementary edges) with the number of normal edges (resp. complementary edges) of $FH(n)$. Note that the number of normal edges (resp. complementary edges) is equal to $n2^{n-1}$ (resp. 2^{n-1}), we can easily obtain that the number of paths going through every edge of $FH(n)$ by R equals $2^n - \binom{n}{\lfloor \frac{n+1}{2} \rfloor}$. And by (2), the result follows.

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