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Fault diameter of Cartesian product graphs ☆

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Abstract

The $(k-1)$-fault diameter $D_k(G)$ of a k-connected graph G is the maximum diameter of $G-F$ for any $F \subset V(G)$ with $|F| < k$. Krishnamoorthy and Krishnamurthy first proposed this concept and gave $D_{\kappa(G_1)+\kappa(G_2)}(G_1 \times G_2) \leqslant D_{\kappa(G_1)}(G_1) + D_{\kappa(G_2)}(G_2)$ when $\kappa(G_1 \times G_2) = \kappa(G_1) + \kappa(G_2)$, where $\kappa(G)$ is the connectivity of G. This paper gives a counterexample to this bound and establishes $D_{k_1+k_2}(G_1 \times G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$ for any k_i-connected graph G_i and $k_i \geqslant 1$ for $i = 1, 2$.

Keywords: Diameter; Connectivity; Fault diameter; Cartesian product graph; Interconnection networks

1. Introduction

We follow [1] for graph-theoretical terminology and notation not defined here. Throughout this paper, a graph $G = (V, E)$ always means a simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The length of a path P is the number of edges in P, denoted by $\ell(P)$. The diameter of G, $d(G)$, is the maximum length over all shortest paths between any two vertices in G. Use $N_G(x)$ to denote the set of neighbors of node x in G. The connectivity of G, $\kappa(G)$, is the minimum cardinality over all vertex-separating sets in G if G is not a complete graph K_n, otherwise $\kappa(K_n) = n-1$. A graph G is said to be k-connected if $\kappa(G) \geqslant k$.

Since nodes of a network do not always work, if some nodes are faulty, the information cannot be transmitted by these nodes and the efficiency of network must be affected. A number of researchers have investigated the design of fault tolerant interconnection networks. A common notion of fault tolerance is based on the connectivity of the underlying graph G. The connectivity $\kappa(G) \geqslant k$ implies that the resulting graph remains connected when at most $k-1$ faulty vertices

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occur. However, the diameter of the resulting graph might increase.

The $(k-1)$-fault diameter of a k-connected graph G, $D_k(G)$, is defined as

$$D_k(G) = \max\{d(G-F)\colon\ F \subset V(G),\ |F| < k\}.$$

It is clear that $D_k(G) = \max\{d(G-F)\colon\ F \subset V(G),\ |F| = k-1\}$, and for any k-connected graph G we have $d(G) = D_1(G) \leqslant D_2(G) \leqslant \cdots \leqslant D_{k-1}(G) \leqslant D_k(G)$.

The fault diameter of many well-known networks have been determined by several researchers, see, for example, [4–8]. The concept of fault diameter was first proposed by Krishnamoorthy and Krishnamurthy [6] who gave $D_{\kappa(G_1)+\kappa(G_2)}(G_1 \times G_2) \leqslant D_{\kappa(G_1)}(G_1) + D_{\kappa(G_2)}(G_2)$ when $\kappa(G_1 \times G_2) = \kappa(G_1) + \kappa(G_2)$. However, this upper bound seems false. For example, consider $G = C_4 \times C_4$, where C_4 is a cycle with vertex set $\{1, 2, 3, 4\}$. Then $\kappa(G) = 4 = \kappa(C_4) + \kappa(C_4)$, $\kappa(C_4) = 2 = d(C_4) = D_2(C_4)$. If we choose $F = \{12, 14, 41\}$, the distance between the two vertices 11 and 43 in $C_4 \times C_4 - F$ is 5 and, hence, $D_4(C_4 \times C_4) \geqslant 5 > 4 = D_2(C_4) + D_2(C_4)$. In the present paper, we give the following results.

Theorem 1. $D_{k_1+k_2}(G_1 \times G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$ *for any* k_i*-connected graph* G_i *and* $k_i \geqslant 1$ *for* $i = 1, 2$.

The proof is in Section 3. Section 2 gives some preliminaries about the Cartesian product of graphs. Section 4 gives conclusions that conclude two opened problems.

2. Preliminaries

The Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, and an edge joining a vertex $x = x_1x_2$ to another $y = y_1y_2$ $(x_j, y_j \in V(G_j),\ j = 1, 2)$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$ or $x_2 = y_2$ and $x_1y_1 \in E(G_1)$.

As an operation of graph theory, the Cartesian product method is a very effective method for constructing a large graph from several specified small graphs. The graph constructed by this way can contain the factor graphs as its subgraphs and preserve many desirable properties of the factor graphs, such as regularity, vertex-transitivity, eulericity, hamiltonicity, and so forth. A number of important graph-theoretic parameters, such as degree, diameter and connectivity, can be easily calculated from the factor graphs. Thus the Cartesian product method is an important method for designing large-scale interconnection networks [9]. For example, the hypercube is one of the most popular, versatile and efficient topological structures of connection networks and it is defined as $Q_n = Q_{n-1} \times K_2 = K_2 \times K_2 \times \cdots \times K_2$ where K_2 is an edge. The following result can be found in [2,3,10].

Lemma. $d(G_1 \times G_2) = d(G_1) + d(G_2)$ *and* $\kappa(G_1 \times G_2) \geqslant \kappa(G_1) + \kappa(G_2)$. *In particular,* $G_1 \times G_2$ *is* k*-regular* k*-connected if* G_i *is* k_i*-regular* k_i $(\geqslant 1)$*-connected, where* $k = k_1 + k_2$.

We observe that if we identify isomorphic graphs, the operations of the Cartesian product satisfy the commutative law clearly. This simple observation can greatly simplify proofs of some results concerning the Cartesian product.

We give some notations used in the proof of the theorem. If $P(x_1, y_1) = (x_1, v_1, v_2, \ldots, v_m, y_1)$ is an (x_1, y_1)-path in G_1, then for any $b \in V(G_2)$, $(x_1b, v_1b, v_2b, \ldots, v_mb, y_1b)$, denoted by $P(x_1, y_1)b$, is an (x_1b, y_1b)-path from the vertex x_1b to the vertex y_1b in $G_1 \times G_2$. Similarly, if $Q(x_2, y_2) = (x_2, u_1, u_2, \ldots, u_l, y_2)$ is an (x_2, y_2)-path in G_2, then for any $a \in V(G_1)$, $(ax_2, au_1, au_2, \ldots, au_l, ay_2)$, denoted by $aQ(x_2, y_2)$, is an (ax_2, ay_2)-path from the vertex ax_2 to the vertex ay_2 in $G_1 \times G_2$. Let $x = x_1x_2$ and $y = y_1y_2$. If x and y are two distinct vertices in $G_1 \times G_2$, then $P(x_1, y_1)x_2 \cup y_1Q(x_2, y_2)$ is an (x, y)-path from x to y in $G_1 \times G_2$. Such a path will, in this paper, be expressed as

$$x = x_1x_2 \xrightarrow{P(x_1,y_1)x_2} y_1x_2 \xrightarrow{y_1Q(x_2,y_2)} y_1y_2 = y.$$

3. Proof of Theorem 1

Let $G = G_1 \times G_2$ and $k = k_1 + k_2$. By lemma, we have $\kappa(G) \geqslant k$ and, hence, $D_k(G)$ is well-defined. Let δ_i be the minimum degree of G_i for $i = 1, 2$ in the following discussion, then $\delta_1 \geqslant k_1$, $\delta_2 \geqslant k_2$. Let F be a subset of $V(G)$ with $|F| = k-1$, x and y be any two distinct vertices in $G - F$. We will complete the proof of Theorem 1 by constructing an (x, y)-path $R(x, y)$ in $G - F$ with required length.

Throughout this section, we use the following terminology and notation. Let H be a subgraph of G, and we say H avoids F if H contains no vertices in F. Let $x = x_1x_2$ and $y = y_1y_2$, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$. We always use $P(x_1, y_1)$ to denote a shortest (x_1, y_1)-path in G_1, and $Q(x_2, y_2)$ to denote a shortest (x_2, y_2)-path in G_2.

We first prove that

$$D_k(G) \leqslant \max\{2 + d(G_1), D_{k_1}(G_1)\} \quad \text{if } x_2 = y_2 \quad \text{or}$$

$$D_k(G) \leqslant \max\{2 + d(G_2), D_{k_2}(G_2)\} \quad \text{if } x_1 = y_1.$$

By the commutative law, we can assume $x_2 = y_2$. If $|F \cap V(G_1x_2)| < k_1$, then there is a path in $G_1x_2 - F$ with length at most $D_{k_1}(G_1)$. Otherwise, let $b_1, b_2, \ldots, b_{\delta_2} \in N_{G_2}(x_2)$, then δ_2 disjoint subgraphs $G_1b_1, G_1b_2, \ldots, G_1b_{\delta_2}$ all are isomorphic to G_1, of which at least one, say G_1b_1, avoids F. Thus,

$$R: x_1x_2 \to x_1b_1 \xrightarrow{P(x_1,y_1)b_1} y_1b_1 \to y_1y_2$$

is an (x, y)-path in $G - F$ with length at most $d(G_1) + 2$.

We assume $x_1 \neq y_1$ and $x_2 \neq y_2$ below. There are two cases depending on whether y_1x_2 and x_1y_2 are in F or not.

Case 1. $y_1x_2 \notin F$ or $x_1y_2 \notin F$. Assume without loss of generality that $y_1x_2 \notin F$.

Subcase 1.1. $|V(G_1x_2) \cap F| \geqslant k_1$ or $|V(y_1G_2) \cap F| \geqslant k_2$.

Without loss of generality assume the former holds. Subject to this condition we have $|V(y_1G_2) \cap F| \leqslant |F| - |V(G_1x_2) \cap F| \leqslant k_1 + k_2 - 1 - k_1 = k_2 - 1$. Since $b_1, b_2, \ldots, b_{\delta_2} \in N_{G_2}(x_2)$, consider δ_2 disjoint subgraphs $G_1b_1, G_1b_2, \ldots, G_1b_{\delta_2}$, of which at least one, say G_1b_1, avoids F. So, an (x, y)-path in $G - F$ can be constructed as follows:

$$R: x_1x_2 \to x_1b_1 \xrightarrow{P(x_1,y_1)b_1} y_1b_1 \xrightarrow{y_1G_2-F} y_1y_2$$

with length $\ell(R) \leqslant 1 + d(G_1) + D_{k_2}(G_2)$.

Subcase 1.2. $|V(G_1x_2) \cap F| \leqslant k_1 - 1$ and $|V(y_1G_2) \cap F| \leqslant k_2 - 1$.

An (x, y)-path in $G - F$ can be constructed as follows:

$$R: x_1x_2 \xrightarrow{G_1x_2-F} y_1x_2 \xrightarrow{y_1G_2-F} y_1y_2$$

with length $\ell(R) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2)$.

Case 2. $\{y_1x_2, x_1y_2\} \subseteq F$. In this case, choose $c_1, c_2, \ldots, c_{\delta_1} \in N_{G_1}(x_1)$ and $d_1, d_2, \ldots, d_{\delta_2} \in N_{G_2}(y_2)$. We consider $(\delta_1 + \delta_2)$ pairs of vertices

$$\{c_1x_2, c_1y_2\},\ \{c_2x_2, c_2y_2\},\ \ldots,\ \{c_{\delta_1}x_2,\ c_{\delta_1}y_2\}, \quad (1)$$

$$\{x_1d_1, y_1d_1\},\ \{x_1d_2, y_1d_2\},\ \ldots,\ \{x_1d_{\delta_2},\ y_1d_{\delta_2}\}. \quad (2)$$

Since $|F| \leqslant k_1 + k_2 - 1 < \delta_1 + \delta_1$, there exists at least one pair of vertices in (1) and (2), say $\{c_1x_2, c_1y_2\}$, that is not in F (similarly if such a pair of vertices is in (2)). Then $c_1 \neq y_1$ since $y_1x_2 \in F$. We will construct an (x, y)-path in $G - F$ with required length according to the following three cases.

Subcase 2.1. $|V(G_1y_2) \cap F| \geqslant k_1$.

Subject to this condition we have $|V(x_1G_2) \cap F| \leqslant k_2 - 1$ and $|V(G - G_1y_2) \cap F| \leqslant k_2 - 1$. Consider δ_2 disjoint subgraphs $G_1d_1, G_1d_2, \ldots, G_1d_{\delta_2}$. Since $\delta_2 \geqslant k_2 > k_2 - 1$, at least one of these subgraphs, say G_1d_1, avoids F. Note that $c_1 \neq y_1$, we can construct an (x, y)-path in $G - F$ as follows:

$$R: x_1x_2 \xrightarrow{x_1G_2-F} x_1d_1 \xrightarrow{P(x_1,y_1)d_1} y_1d_1 \to y_1y_2,$$

with length $\ell(R) \leqslant 1 + d(G_1) + D_{k_2}(G_2)$.

Subcase 2.2. $|V(c_1G_2) \cap F| \geqslant k_2$.

Subject to this condition, we have $|V(G_1y_2) \cap F| \leqslant k_1 - 2$ and $|V(G - c_1G_2 - x_1G_2) \cap F| \leqslant k_1 - 2$. Consider $\delta_1 - 1$ disjoint subgraphs $c_2G_2, c_3G_2, \ldots, c_{\delta_1}G_2$. Since $\delta_1 - 1 \geqslant k_1 - 1 > k_1 - 2$, at least one of these subgraphs, say c_2G_2, avoids F. We can construct an (x, y)-path in $G - F$ as follows:

$$R: x_1x_2 \to c_2x_2 \xrightarrow{c_2Q(x_2,y_2)} c_2y_2 \xrightarrow{G_1y_2-F} y_1y_2,$$

with length $\ell(R) \leqslant 1 + D_{k_1-1}(G_1) + d(G_2)$.

Subcase 2.3. $|V(G_1y_2) \cap F| \leqslant k_1 - 1$ and $|V(c_1G_2) \cap F| \leqslant k_2 - 1$.

An (x, y)-path in $G - F$ can be constructed as follows:

$$R: x_1x_2 \to c_1x_2 \xrightarrow{c_1G_2-F} c_1y_2 \xrightarrow{G_1y_2-F} y_1y_2$$

with length $\ell(R) \leqslant 1 + D_{k_1}(G_1) + D_{k_2}(G_2)$.

Summing up all possible cases, we get the upper bound in Theorem 1.

4. Conclusions

The fault diameter is an important measurement for reliability and efficiency of an interconnection network. In the present paper, we establish $D_{k_1+k_2}(G_1 \times G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$ for any k_i-connected graph G_i and $k_i \geqslant 1$ for $i = 1, 2$.

The fault diameter of many well-known networks have been determined by several authors see, for example, [4–8]. However, there are a lot of problems that are still open so far. One of them is whether or not determining $D_k(G)$ is NP-hard for any k-connected graph G, and another is for a fixed $k \leqslant \kappa(G)$ how to choose a k_1-connected graph G_1 and a k_2-connected graph G_2 such that $k = k_1 + k_2$ and $G = G_1 \times G_2$ such that $D_k(G)$ is as small as possible.

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