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Super connectivity of line graphs ☆

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Abstract

The super connectivity κ' and the super edge-connectivity λ' are more refined network reliability indices than connectivity κ and edge-connectivity λ. This paper shows that for a connected graph G with order at least four rather than a star and its line graph $L(G)$, $\kappa'(L(G)) = \lambda'(G)$ if and only if G is not super-λ'. As a consequence, we obtain the result of Hellwig et al. [Note on the connectivity of line graphs, Inform. Process. Lett. 91 (2004) 7] that $\kappa(L(G)) = \lambda'(G)$. Furthermore, the authors show that the line graph of a super-λ' graph is super-λ if the minimum degree is at least three.

Keywords: Line graphs; Super connectivity; Super edge-connectivity; Super-λ'; λ'-connected; Combinatorial problems

1. Introduction

In general, we use a simple connected graph $G = (V, E)$ to model an interconnection network, where V is the set of processors and E is the set of communication links in the network. The connectivity $\kappa(G)$ or the edge-connectivity $\lambda(G)$ of G is an important measurement for fault-tolerance of the network, and the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. It is well known that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G. One might be interested in more refined indices of reliability. As more refined indices than connectivity and edge-connectivity, super connectivity and super edge-connectivity were proposed in [1,2]. A graph G is *super connected*, *super-κ*, for short (resp. *super edge-connected*, *super-λ*, for short) if every minimum vertex-cut (resp. edge-cut) isolates a vertex of G.

A quite natural problem is that if a connected graph G is super-κ or super-λ then how many vertices or edges must be removed to disconnect G such that every component of the resulting graph contains no isolated vertices. This problem results in the concept of the super (edge-)connectivity.

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A subset $S \subset V(G)$ (resp. $F \subset E(G)$) is called a *super vertex-cut* (resp. *super edge-cut*) if $G - S$ (resp. $G - F$) is not connected and every component contains at least two vertices. In general, super vertex-cuts or super edge-cuts do not always exist. The *super connectivity* $\kappa'(G)$ (resp. *super edge-connectivity* $\lambda'(G)$) is the minimum cardinality over all super vertex-cuts (resp. super edge-cuts) in G if any, and, by convention, is $+\infty$ otherwise. It is easy to see that G is super-κ if and only if $\kappa'(G) > \kappa(G)$, and super-λ if and only if $\lambda'(G) > \lambda(G)$.

It is easy to see from the result of Esfahanian and Hakimi [3,4] that $\lambda'(G)$ exists if G has order at least 4 and G is not a star. A connected graph G with order at least 4 is called a λ'*-graph* if G is not a star. A λ'-graph is called *super-*λ' if every minimum super edge-cut isolates an edge.

For λ', it has been widely studied by several authors, see, for example, [3–10,13], in which authors called it the restricted edge-connectivity. However, we have known little results on κ'.

We consider the relationship between the super edge-connectivity of a graph G and the super connectivity of its line graph $L(G)$. Very recently, Hellwig et al. [5] have established $\kappa(L(G)) = \lambda'(G)$ if G is a λ'-graph. As a result, $\kappa'(L(G)) \geqslant \kappa(L(G)) = \lambda'(G)$ if $\kappa'(L(G))$ exists. In this paper, we show that $\kappa'(L(G))$ exists and $\kappa'(L(G)) = \lambda'(G)$ if and only if G is not a super-λ' graph. We also show that $L(G)$ is super-λ if G is super-λ' and $\delta(G) \geqslant 3$. Our proofs are independent on the result of Hellwig et al., which is a direct consequence of our results.

2. Super edge-connectedness of line graphs

We follow [12] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a simple connected graph. The line graph of G, denoted by $L(G)$, or L shortly, is a graph with the vertex-set $V(L) = E(G)$, and a vertex xy is adjacent to a vertex wz in L if and only if they are adjacent as edges in G. Clearly, $\delta(L) = \xi(G)$, where $\xi(G) = \min\{d_G(x) + d_G(y) - 2 \mid xy \in E(G)\}$, and $d_G(x)$ is the degree of the vertex x in G.

Two edges are said to be *independent* if they are nonadjacent. A set of edges is said to be an *independent edge-set* if any two edges of it are independent. The set of edges incident with a vertex x is said to be the *incidence edge-set* of x.

Theorem 1. *Let G be a connected graph. Then*

(a) *any minimum edge-cut of G is either an incidence edge-set of some vertex of G or an independent edge-set if $\kappa(G) = \lambda(G)$;*
(b) *G is super-λ if G is super-κ and $\delta(G) \geqslant 4$.*

Proof. We first prove the assertion (a). Let F be an edge-cut of G with $|F| = \lambda(G)$. The vertex set $V(G)$ can be partitioned into two nonempty subsets X and Y such that $G - F$ contains no edges between X and Y and every edge in F has one end-vertex in X and the other end-vertex in Y. Let X_0 and Y_0 be the set of the end-vertices of the edges in F in X and Y, respectively. Clearly, $|X_0| \leqslant |F|$ and $|Y_0| \leqslant |F|$.

Without loss of generality, assume $|X_0| \leqslant |Y_0|$. Thus, we only need to prove that $|X_0| = |F|$ if F is not an incidence edge-set of some vertex of G since, in this case, every vertex in X_0 is matched by an edge in F with a unique vertex in Y_0. In fact, if $X - X_0 \neq \emptyset$, then X_0 is a vertex-cut of G, so $|F| \geqslant |X_0| \geqslant \kappa(G) = \lambda(G) = |F|$.

We assume $X - X_0 = \emptyset$. It is clear that $|X_0| \geqslant 2$ if F is not an incidence edge-set of some vertex of G. Let $|X_0| = t$ and $E(x) = \{xy \in E(G) \mid y \in Y\}$ for $x \in X$. Since $2 \leqslant t \leqslant \lambda(G) \leqslant \delta(G)$, we have

$$\begin{aligned}\delta(G) \geqslant \lambda(G) = |F| \\ &= \sum_{x \in X_0} \bigl|E(x)\bigr| = \sum_{x \in X_0} d(x) - 2\bigl|E\bigl(G[X_0]\bigr)\bigr| \\ &\geqslant \delta(G)t - t(t-1) = -t^2 + \bigl(\delta(G) + 1\bigr)t \\ &\geqslant \delta(G).\end{aligned}$$

The last inequality holds because the function $f(t) = -t^2 + (\delta(G) + 1)t$ is convex in the integer interval $[2, \delta(G)]$ and reaches the minimum value at the right end-point of the interval, that is, $f(t) \geqslant f(\delta(G)) = \delta(G)$. The equality is true if and only if $t = |X_0| = \delta(G) = |F|$.

We now prove the assertion (b).

Since G is super-κ, $\kappa(G) = \delta(G)$. We have $\kappa(G) = \lambda(G) = \delta(G)$ immediately from $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$. Suppose to the contrary that G is not super-λ. Then there exists a minimum edge-cut F with $|F| = \lambda(G) \geqslant 4$, which is not an incidence edge-set of some vertex of G. By the assertion (a), F is an independent edge-set. Let X, Y, X_0 and Y_0 be defined as before. Let $\lambda = |F|$ and

$$X_0 = \{x_1, x_2, \ldots, x_\lambda\} \quad \text{and} \quad Y_0 = \{y_1, y_2, \ldots, y_\lambda\},$$

where x_i is matched with y_i by an edge in F for $i = 1, 2, \ldots, \lambda$. Consider the following set of vertices in G:

$$S = \{x_1, x_2, y_3, y_4, \ldots, y_\lambda\}.$$

It is clear that S is a minimum vertex-cut of G. Since G is super-κ, S must be a neighbor-set of some vertex, say u, of G. If $u \in X$, $y_1u, y_2u \in F$, contradicting to the assumption that F is an independent edge-set. If $u \in Y$, $x_1u, x_2u \in F$, a contradiction too. Thus, G is super-λ, so the assertion (b) follows. □

Remark. In Theorem 1, the condition $\delta(G) \geqslant 4$ in the assertion (b) is necessary. For example, the graph obtained by joining two triangles by three edges such that the graph is 3-regular. It is easy to see that the graph is super-κ but not super-λ.

Corollary 1. *Let G be a connected graph with $\delta(G) \geqslant 3$. If the line graph $L = L(G)$ is super-κ, then L is super-λ.*

Proof. Since L is super-κ and $\delta(L) = \xi(G) \geqslant 2\delta(G) - 2 \geqslant 4$ for $\delta(G) \geqslant 3$, the corollary follows from Theorem 1. □

3. Super connectivity of line graphs

Many properties of line graphs can be found in [11], two of which are the following lemmas.

Lemma 1. *Let G be a graph with at least two edges. Then G is connected if and only if the line graph $L(G)$ is connected.*

Lemma 2. *Let G be a graph with at least two edges. Then*

$$\kappa(G) \leqslant \lambda(G) \leqslant \kappa\big(L(G)\big) \leqslant \lambda\big(L(G)\big).$$

Lemma 3 [4]. *If G is a λ'-graph, then*

$$\lambda(G) \leqslant \lambda'(G) \leqslant \xi(G).$$

For a subset $E' \subseteq E(G)$, we use $G[E']$ to denote the edge-induced subgraph of G by E'. Let L_1 be a subgraph of $L(G)$ and $E_1 = V(L_1)$. Define $G_1 = G[E_1]$.

Lemma 4. *Using the above notations, we have that if L_1 is a connected subgraph of L with at least two vertices, then the subgraph $G_1 \subseteq G$ is connected and $|V(G_1)| \geqslant 3$.*

Proof. Assume that x and y are any two vertices of G_1. There is an edge e of G_1 such that x is incident with the edge e. Without loss of generality, we denote the edge e by xz. If $z = y$, x can reach y by the edge e. If $z \neq y$, y is incident with another edge e'. Without loss of generality, we can assume the edge $e' = wy$. So, $e = xz$ and $e' = wy$ are two vertices in L_1. Since L_1 is connected, there is a path in L_1 connecting $e = xz$ to $e' = wy$: $(xz, zz_1, z_1z_2, \ldots, z_kw, wy)$. The corresponding edges $xz, zz_1, z_1z_2, \ldots, z_kw, wy$ form a walk in G_1 between x and y: $(x, z, z_1, z_2, \ldots, z_k, w, y)$. Therefore, x and y are connected, which implies G_1 is connected. Since $|E(G_1)| = |V(L_1)| \geqslant 2$ and G is a simple graph, $|V(G_1)| \geqslant 3$. □

The complete bipartite graph $K_{1,3}$ is usually called a *claw*, and any graph that does not contain an induced claw is called claw-free. It is easy to see that every line graph is *claw-free*.

Lemma 5. *Let G be a connected claw-free graph. Then $G - S$ contains exactly two components for any minimum super vertex-cut S of G.*

Proof. Let S be a minimum super vertex-cut of G and $G_1, G_2, \ldots, G_t$ be connected components of $G - S$. Then G_i contains at least two vertices for $i = 1, 2, \ldots, t$, $t \geqslant 2$, and there are no edges between G_i and G_j for any $i \neq j$. Since S is a vertex-cut, there is a vertex x adjacent to some component. Furthermore, the vertex $x \in S$ is adjacent to every component, otherwise $S' = S \setminus \{x\}$ is also a super vertex-cut, contradicting the minimality of S. If $t \geqslant 3$, the vertex x is adjacent to at least three components, which is impossible for G is a claw-free graph. Therefore, $t = 2$ and the lemma follows. □

Theorem 2. *Let G be a λ'-graph. Then $\kappa'(L(G))$ exists and $\kappa'(L(G)) = \lambda'(G)$ if and only if G is not super-λ'.*

Proof. We first note that $\lambda'(G)$ exists for a λ'-graph G. Suppose G is not super-λ', so there exists a minimum super edge-cut F such that each of the two components of $G - F$, G_1 and G_2, has order at least three. By Lemma 1, $L(G_1), L(G_2)$ are both connected in $L = L(G)$. And $|V(L(G_i))| = |E(G_i)| \geqslant 2$ for $i = 1, 2$. There are no edges between $L(G_1)$ and $L(G_2)$ in $L - F$ and, hence, F is a super vertex-cut of L. It follows that $\kappa'(L)$ exists and $\kappa'(L) \leqslant |F| = \lambda'(G)$.

We now show that $\kappa'(L) \geqslant \lambda'(G)$. To the end, let S be a minimum super vertex-cut of L. Since every line graph is claw-free, by Lemma 5 $L - S$ contains exactly two components, L_1 and L_2 with $|V(L_i)| \geqslant 2$ for $i = 1, 2$. By Lemma 4, for each $i = 1, 2$, $G_i = G[V(L_i)]$, the subgraph of G induced by $V(L_i)$, is connected and $|V(G_i)| \geqslant 3$. Thus S is a super edge-cut of G. It follows that $\lambda'(G) \leqslant |S| = \kappa'(L)$. Therefore, $\kappa'(L) = \lambda'(G)$.

Conversely, suppose $\kappa'(L)$ exists and $\kappa'(L) = \lambda'(G)$. We prove that G is not super-λ' by contradiction. Assume that G is super-λ', which means every minimum super edge-cut isolates one edge. Let S be a super vertex-cut of L with $|S| = \kappa'(L) = \lambda'(G)$. By Lemma 5, $L - S$ is partitioned into two components, denoted by L_1 and L_2, respectively. Applying Lemma 4 to G_1 and G_2, the subgraph of G induced by $V(L_1)$ and $V(L_2)$, respectively, are connected and $|V(G_1)| \geqslant 3$, $|V(G_2)| \geqslant 3$. There are no edges between G_1 and G_2, so S is a super edge-cut of G with $|S| = \kappa'(L) = \lambda'(G)$, which implies that S is a minimum super edge-cut of G. But $G - S$ contains no isolated edges. We get a contradiction, so G is not super-λ'. □

Corollary 2. *Let G be a connected graph with $\xi(G) > \delta(G)$. Then $\kappa'(L(G))$ exists and $\kappa'(L(G)) = \lambda(G)$ if and only if G is not super-λ.*

Proof. Assume G is not super-λ. Then $\lambda'(G)$ exists and $\lambda'(G) = \lambda(G) \leqslant \delta(G)$. Since $\xi(G) > \delta(G)$, G is not super-λ'. By Theorem 2, $\kappa'(L)$ exists and $\kappa'(L) = \lambda'(G) = \lambda(G)$.

Conversely, suppose $\kappa'(L)$ exists and $\kappa'(L) = \lambda(G)$. We assume G is super-λ, which means that every minimum edge-cut isolates one vertex. Let S be a super vertex-cut of L with $|S| = \kappa'(L) = \lambda(G)$. By Lemma 5, $L - S$ is partitioned into exactly two components L_1 and L_2 with at least two vertices. By Lemma 4, G_1 and G_2, the induced subgraph of G by $V(L_1)$ and $V(L_2)$, respectively, are connected and $|V(G_1)| \geqslant 3$, $|V(G_2)| \geqslant 3$. There are no edges between G_1 and G_2 in $G - S$. So S is an edge-cut of G with $|S| = \kappa'(L) = \lambda(G)$, which implies that S is a minimum edge-cut of G. But $G - S$ contains no isolated vertices, a contradiction, so G is not super-λ. □

Corollary 3. *Let G be a λ'-graph. Then $L(G)$ is super-κ if and only if G is super-λ'.*

Proof. Suppose that $L = L(G)$ is super-κ. Then $\kappa(L) = \delta(L)$ and $\kappa'(L) > \kappa(L)$. Assume G is not super-λ'. By Theorem 2, we have $\kappa'(L) = \lambda'(G)$. By Lemma 3,

$$\xi(G) = \delta(L) = \kappa(L) < \kappa'(L) = \lambda'(G) \leqslant \xi(G),$$

a contradiction, so G is super-λ'.

Conversely, suppose that G is super-λ'. Then $\lambda'(G) = \xi(G)$ and every minimum super edge-cut of G isolates one edge. Suppose to the contrary that L is not super-κ. Then $\kappa'(L) = \kappa(L)$. Let S be a super vertex-cut of L with $|S| = \kappa'(L) = \kappa(L)$. In the same way as above, we have that S is a super edge-cut of G and that the two components of $G - S$ both contains at least three vertices. Hence,

$$\xi(G) = \lambda'(G) \leqslant |S| = \kappa(L) \leqslant \delta(L) = \xi(G),$$

which implies that $|S| = \lambda'(G)$. Thus, S is a minimum super edge-cut of G but S does not isolate one edge, a contradiction. Thus, L is super-κ. □

Using Theorem 2, we obtain the main result in [5].

Corollary 4. *$\kappa(L(G)) = \lambda'(G)$ for any λ'-graph G.*

Proof. If G is not super-λ' then $\kappa'(L(G)) = \lambda'(G)$ by Theorem 2. Thus, by Corollary 3, $L(G)$ is not super-κ, which means $\kappa(L(G)) = \kappa'(L(G))$. Therefore, $\kappa(L(G)) = \lambda'(G)$.

If G is a super-λ' graph, $\lambda'(G) = \xi(G)$. By Corollary 3, $L(G)$ is super-κ, so $\kappa(L(G)) = \delta(L(G)) = \xi(G) = \lambda'(G)$. □

Corollaries 1 and 3 lead to the following corollary immediately.

Corollary 5. *Let G be a λ'- graph with $\delta(G) \geqslant 3$. If G is super-λ', $L(G)$ is super-λ.*

References

[1] D. Bauer, F. Boesch, C. Suffel, R. Tindell, Connectivity extremal problems and the design of reliable probabilistic networks, in: The Theory and Application of Graphs, Wiley, New York, 1981, pp. 45–54.

[2] F.T. Boesch, Synthesis of reliable networks. A survey, IEEE Trans. Reliability 35 (1986) 240–246.

[3] A.H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Trans. Comput. 38 (11) (1989) 1586–1591.

[4] A.H. Esfahanian, S.L. Hakimi, On computing a conditional edge-connectivity of a graph, Inform. Process. Lett. 27 (1988) 195–199.

[5] A. Hellwig, D. Rautenbach, L. Volkmann, Note on the connectivity of line graphs, Inform. Process. Lett. 91 (1) (2004) 7–10.

[6] A. Hellwig, L. Volkmann, Sufficient conditions for λ'-optimality in graphs of diameter 2, Discrete Math. 283 (2004) 113–120.

[7] A. Hellwig, L. Volkmann, Sufficient conditions for graphs to be λ'-optimal, super-edge-connected and maximally edge-connected, J. Graph Theory, submitted for publication.

[8] J.-X. Meng, Y.-H. Ji, On a kind of restricted edge connectivity of graphs, Discrete Appl. Math. 117 (2002) 183–193.

[9] M. Wang, Q. Li, Conditional edge connectivity properties, reliability comparisons and transitivity of graphs, Discrete Math. 258 (2002) 205–214.

[10] Y.Q. Wang, Super restricted edge-connectivity of vertex-transitive graphs, Discrete Math. 289 (2004) 199–205.

[11] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht, 2001.

[12] J.-M. Xu, Theory and Application of Graphs, Kluwer Academic Publishers, Dordrecht, 2003.

[13] J.-M. Xu, K.-L. Xu, On restricted edge-connectivity of graphs, Discrete Math. 243 (1–3) (2002) 291–298.