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Wide Diameter of Möbius Cubes*

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Abstract

The Möbius cube is a variant of the hypercube and has better performance than the hypercube with the same number of links and processors. This paper shows that the wide diameter of the Möbius cube is at most $\lceil \frac{n+2}{2} \rceil + 2$, about one half of the same dimensional hypercube's.

Keywords Möbius cubes, Hypercubes, Diameter, Wide diameter, Connectivity

AMS Subject Classification: 05C12 90B10

1 Introduction

To have processors work on a problem cooperatively, parallel computer systems need a mechanism for exchanging data. Interconnection networks are one way to meet this need. In an interconnection network, each processor has its own memory and resources and is connected to a number of neighboring processors by communication paths. The processors can work cooperatively by passing messages via the communication paths. The topology of a network determines how well the processors can interact with each other to solve a given problem. The small wider diameter is an important performance parameter in choosing a particular interconnection network for a parallel computer.

The hypercube network has proved to be one of the most popular interconnection networks. Möbius cubes, proposed first by Cull and Larson[3], form a class of hypercube variants that give better performance with the same number of edges and vertices. Like hypercubes, Möbius cubes are expansible, have a simple routing algorithm, and have a high fault tolerance. The Möbius cubes are superior to the hypercube in having about half of the diameter of the hypercube, about two-thirds of the average distance of hypercube. The variously desired properties of Möbius cubes have been extensively investigated in the literature, see, for example, [3, 4, 5, 8, 9]. In this paper, we shows that the wide diameter of a Möbius cube is about one half of the same dimensional hypercube's.

The rest of this paper is organized as follows. Section 2 recalls definition of Möbius cubes and introduces notation used in this paper. In section 3, we give some lemmas to prepare for the section 4, in which we prove the wide diameter of the n-Möbius cubes is at most $\lceil \frac{n+2}{2} \rceil + 2$.

2 Definitions and Notation

The architecture of an interconnection network is usually represented by a graph. Throughout this paper, we use network and graph, processor and vertex, and link and edge, interchangeably.

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Our fundamental graph terminology is referred to [2] when using an undirected graph to model an interconnection network. This paper considers finite, simple, and loopless graph $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G, respectively. The edge connecting two vertices x and y is denoted by (x, y).

Suppose that G is a w-connected graph. By Menger's theorem there exist w internally disjoint paths between any two distinct vertices in G, the smallest number ℓ for which there are w internally disjoint paths of length at most ℓ between any two vertices in G is called the wide diameter of G, denoted by $d_w(G)$, proposed first by Hsu [7]. In a real-time processing system, the wide diameter is an important parameter to measure performance of the network.

An n-dimensional Möbius cube, denoted by M_n, has 2^n vertices. Each vertex has a unique n-component binary vector on $\{0, 1\}$ for an address, also called an n-bit string. For instance, a vertex X has address $x_1x_2\cdots x_n$, The vertex X connects to n neighbors $Y_1, Y_2, \cdots, Y_n$, where each Y_i satisfies one of the following equations:

$$Y_i = x_1 \cdots x_{i-1}\bar{x}_i x_{i+1} \cdots x_n \quad \text{if } x_{i-1} = 0 \tag{1}$$

$$Y_i = x_1 \cdots x_{i-1}\bar{x}_i \bar{x}_{i+1} \cdots \bar{x}_n \quad \text{if } x_{i-1} = 1 \tag{2}$$

where $\bar{x}_i$ is the complement of the bit x_i in $\{0, 1\}$.

More informally, a vertex X connects to a neighbor that differs in a bit x_i if $x_{i-1} = 0$, and to a neighbor that differs in bits x_i through x_n if $x_{i-1} = 1$. The connection between X and Y along dimension 1 has x_0 undefined, so we can assume x_0 is either equal to 0 or equal to 1, which gives us slightly different network topologies. If we assume $x_0 = 0$, we call the network a "0-Möbius cube", denoted by M_n^0; and if we assume $x_0 = 1$, we call the network a "1-Möbius cube", denoted by M_n^1. Figure 1 and Figure 2 show the 0-Möbius cube M_4^0 and the 1-Möbius cube M_4^1.

According to the above definition, it is not difficult to see that M_n^0 (respectively, M_n^1) can be recursively constructed from M_{n-1}^0 and M_{n-1}^1 by adding 2^{n-1} edges. For any vertex $X = x_1x_2\cdots x_{n-1}$ in M_{n-1}^0 or M_{n-1}^1, we construct a new vertex $X' = x_1'x_2'\cdots x_n'$, where $x_2' = x_1, x_3' = x_2, \cdots, x_n' = x_{n-1}$, then assigning $x_1' = 0$ if X is in M_{n-1}^0, or $x_1' = 1$ if X is in M_{n-1}^1. M_n^0 is constructed by connecting all pairs of vertices that differ only in the first bit, and M_n^1 is constructed by connecting all pairs of vertices that differ in the first through the nth bits. For short, we denote $M_n^0 = M_{n-1}^0 \oplus M_{n-1}^1$ and $M_n^1 = M_{n-1}^0 \otimes M_{n-1}^1$.

We define $\widetilde{M_n^0}$ ($\widetilde{M_n^1}$, respectively) obtained from two disjoint copies of M_{n-1}^0 (M_{n-1}^1, respectively) by adding 2^{n-1} edges between copies to link two vertices that have not any differences, that is, $\widetilde{M_n^0} = M_{n-1}^0 \times K_2$ ($\widetilde{M_n^1} = M_{n-1}^1 \times K_2$, respectively), where "$G \times K_2$" denotes the cartesian product of the graph G and the complete graph K_2 of order 2.

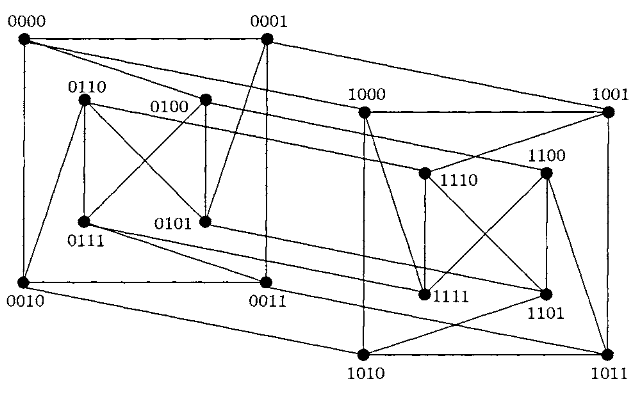

Figure 1: 0-Möbius cube M_4^0

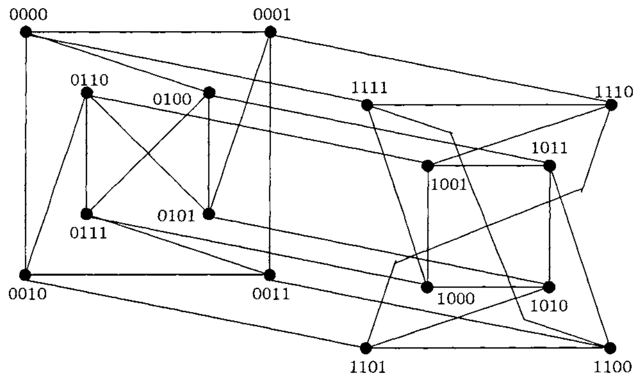

Figure 2: 1-Möbius cube M_4^1

Let $X = x_1 x_2 \cdots x_n$ be a vertex of M_n. For a positive integer $\ell \leq n$, the ℓ-prefix of X, $p_\ell(X)$, is defined as an ℓ-bit string $x_1 x_2 \cdots x_\ell$.

For an ℓ-bit string S with $\ell \leq n$, we use $M_n(S)$ to denote the subgraph of M_n induced by the set of vertices with the prefix S. Let S_1 and S_2 be two distinct ℓ-bit strings with $\ell \leq n$. If $M_n(S_1)$ and $M_n(S_2)$ can be joined by an edge in M_n, then $M_n(S_1)$ and $M_n(S_2)$ are called two adjacent subgraphs of M_n. $M_n(S_1) \cup M_n(S_2)$ denotes the subgraph of M_n induced by $V(M_n(S_1)) \cup V(M_n(S_2))$.

Let X be a vertex in M_n, ℓ a positive integer with $\ell < n$ and $S_1 = p_\ell(X)$. Assume S_2 is an ℓ-bit string with $S_1 \neq S_2$. If $(S_1, S_2) \in E(M_\ell)$, then, clearly, a vertex Y satisfying $p_\ell(Y) = S_2$ and $(X, Y) \in E(M_n)$ can be uniquely determined by S_1, S_2 and X. We can thus denote such a vertex Y by $f(X; S_1, S_2)$. For example, given $(S_1, S_2) = (00, 11) \in E(M_2^1)$ and $X = 0011$, the neighbor of X in M_4^1 with prefix 11 can be uniquely identified and is given by 1100, that is, $f(X; S_1, S_2) = 1100$.

Let $P = (X_0, X_1, \cdots, X_m)$ be a path in M_ℓ. The length of P, the number of edges in P, is denoted by $|P|$. Two terminal vertices X_0 and X_m of P can be denoted by $t(X_0; P) = X_m$ and $t(X_m, P) = X_0$, X_i is called the immediate predecessor of X_{i+1}. For a given vertex X in M_n with $p_\ell(X) = X_0$, we use $P(X; P)$ to denote the path $(W_0, W_1, \cdots, W_m)$ in M_n associated with P, where $W_0 = X$ and $W_i = f(W_{i-1}; X_{i-1}, X_i)$ for all $1 \leq i \leq m$(here, we must require M_n and M_ℓ are both "0-Möbius cube" or both "1-Möbius cube", otherwise, W_i and W_{i+1} possibly haven't edge to connect in M_n for $i = 0, \cdots, m-1$.). Then $|P(X; P)| = |P|$. For example, let $P = (00, 10, 11, 01)$ be a path of M_2^0 and $X = 0011$. Then $P(X; P) = (0011, 1011, 1100, 0100)$ is a path in M_4^0.

For a given path $P = (X_0, X_1, \cdots, X_m)$ in M_ℓ, let $I = \{M_n(X_i) \cup M_n(X_{i+1}) : M_n(X_i) \cup M_n(X_{i+1})$ is isomorphic to the $(n-\ell+1)$-dimensional $M_{n-\ell+1}^1$, $1 \leq i \leq m-2\}$, and let $\gamma(P) = |I|$. From definition, we can note that $M_n(X_0) \cup M_n(X_1)$ and $M_n(X_{m-1}) \cup M_n(X_m)$ don't belong the set I whether they are isomorphic to $MQ_{n-\ell+1}^1$ or not. For example, let $P = (00, 10, 11, 01)$ be a path of M_2^0, then $\gamma(P) = 1$.

3 Some Lemmas

Lemma 1 (Cull and Larson [3]) The Möbius cube M_n is n-regular n-connected, the diameter of M_n^0 is equal to $\lceil \frac{n+2}{2} \rceil$ for $n \geq 4$, and the diameter of M_n^1 is equal to $\lceil \frac{n+1}{2} \rceil$ for $n \geq 1$.

Lemma 2 Let $S = x_1 x_2 \cdots x_\ell$ be an ℓ-bit string with $l \leq n-1$. Then $M_n(S)$ is isomorphic to $M_{n-\ell}^0$ if $x_\ell = 0$, and isomorphic to $M_{n-\ell}^1$ if $x_\ell = 1$.

Proof Assume $S = x_1x_2\cdots x_\ell$ be an ℓ-bit string with $l \le n-1$. If $x_\ell = 0$, then the mapping $\theta: V(M_n(S)) \to V(M^0_{n-\ell})$ defined by $\theta(x_1x_2\cdots x_\ell x_{\ell+1}\cdots x_n) = x_{\ell+1}\cdots x_n$ is bijective, and for any $X, Y \in V(M_n(S))$, $(X,Y) \in E(M_n(S)) \Leftrightarrow (\theta(X), \theta(Y)) \in E(M^0_{n-\ell})$ as $x_\ell = 0$. Thus, θ is an isomorphism from $M_n(S)$ to $M^0_{n-\ell}$. In a similar argument, we can prove the other part of the lemma. ∎

In the M^0_4 shown in Figure 1, we can find that $M^0_4(01) \cup M^0_4(11)$ (resp. $M^0_4(10) \cup M^0_4(00)$) is isomorphic to the $\widetilde{M^1_3}$ (resp. $\widetilde{M^0_3}$), $M^0_4(01) \cup M^0_4(00)$ (resp. $M^0_4(10) \cup M^0_4(11)$) is isomorphic to the M^0_3 (resp. M^1_3). In general, we have the following result.

Lemma 3 For any $X = x_1x_2\cdots x_n$, $Y = y_1y_2\cdots y_n \in V(M_n)$, let $S_1 = p_\ell(X)$, $S_2 = p_\ell(Y)$ with $\ell \le n-1$. If $S_1 \ne S_2$, then $M_n(S_1) \cup M_n(S_2)$ is isomorphic to

(1) $M^0_{n-\ell+1}$ if $x_\ell \ne y_\ell, x_{\ell-1} = y_{\ell-1} = 0 (\ell \ge 1)$, and $x_1\cdots x_{\ell-2} = y_1\cdots y_{\ell-2} (\ell \ge 2)$;

(2) $M^1_{n-\ell+1}$ if $x_\ell \ne y_\ell$, and there exists an i with $0 \le i \le \ell-1$ such that $x_i = y_i = 1, x_1\cdots x_{i-1} = y_1\cdots y_{i-1} (i \ge 1)$ and $\bar{x}_{i+1}\cdots\bar{x}_\ell = y_{i+1}\cdots y_\ell$;

(3) $\widetilde{M^0}_{n-\ell+1}$ if $x_\ell = y_\ell = 0$, and there exists an i with $1 \le i \le \ell-1$ such that $x_{i-1} = y_{i-1} = 0$, $x_i \ne y_i$ and $x_j = y_j$ for each $j = 1, 2, \cdots, i-2, i+1, \cdots, \ell$;

(4) $\widetilde{M^1}_{n-\ell+1}$ if $x_\ell = y_\ell = 1$, and there exists an i with $1 \le i \le \ell-1$ such that $x_{i-1} = y_{i-1} = 0, x_i \ne y_i$ and $x_j = y_j$ for each $j = 1, 2, \cdots, i-2, i+1, \cdots, \ell$.

Proof Let $X = x_1x_2\cdots x_n$ and $Y = y_1y_2\cdots y_n$ be two distinct vertices in $V(M_n)$, and let $S_1 = p_\ell(X)$ and $S_2 = p_\ell(Y)$ with $S_1 \ne S_2$. We only prove the assertions (1) and (3) since the assertions (2) and (4) can be proved similarly.

(1) Assume $x_\ell \ne y_\ell$, $x_{\ell-1} = y_{\ell-1} = 0$, and $x_1\cdots x_{\ell-2} = y_1\cdots y_{\ell-2}$. We can, without loss of generality, assume $x_\ell = 0, y_\ell = 1$. By Lemma 2, $M_n(S_1)$ is isomorphic to $M^0_{n-\ell}$ and $M_n(S_2)$ isomorphic to $M^1_{n-\ell}$. Thus, to prove that $M_n(S_1) \cup M_n(S_2)$ is isomorphic to $M^0_{n-\ell+1}$, it is sufficient to prove $M_n(S_1) \cup M_n(S_2)$ is isomorphic to $M^0_{n-\ell} \oplus M^1_{n-\ell}$. To the end, we can define a mapping

$$\theta:\ V(M_n(S_1) \cup M_n(S_2)) \to V(M^0_{n-\ell} \oplus M^1_{n-\ell})$$

subject to, for $Z = z_1\cdots z_\ell z_{\ell+1}\cdots z_n$,

$$\theta(Z) = \begin{cases} 0z_{\ell+1}\cdots z_n & \text{if } Z \in V(M_n(S_1)); \\ 1z_{\ell+1}\cdots z_n & \text{if } Z \in V(M_n(S_2)). \end{cases}$$

By the recursively constructed definition of M^0_n, it is not difficult to see that θ is an isomorphism from $M_n(S_1) \cup M_n(S_2)$ to $M^0_{n-\ell} \oplus M^1_{n-\ell}$.

(3) If $x_\ell = y_\ell = 0$, then by Lemma 2, both $M^0_n(S_1)$ and $M^0_n(S_1)$ are isomorphic to $M^0_{n-\ell}$. Since there is an i $(1 \le i \le \ell-1)$ such that $x_{i-1} = y_{i-1} = 0, x_i \ne y_i, x_j = y_j$ $(1 \le j \le \ell, j \ne i, i-1)$, $(X, x_1\cdots x_{i-1}\overline{x_i}x_{i+1}\cdots x_\ell x_{\ell+1}\cdots x_n) \in E(M_n(S_1) \cup M_n(S_2))$, e.g., $(S_1x_{\ell+1}\cdots x_n, S_2x_{\ell+1}\cdots x_n) \in E(M_n(S_1) \cup M_n(S_2))$, so there exist a bijection such that $M_n(S_1) \cup M_n(S_2)$ is isomorphic to $\widetilde{M^0}_{n-\ell+1}$. ∎

Lemma 4 Let $X = x_1x_2\cdots x_n$ and $Y = y_1y_2\cdots y_n$ be two vertices in M_n. If there are $(n-2)$ internally disjoint paths $P^*_1, P^*_2, \cdots, P^*_{n-2}$ from $x_1x_2\cdots x_{n-2}$ to $y_1y_2\cdots y_{n-2}$ in M_{n-2} (Here, M_n and M_{n-2} are both either "0-Möbius cube" or "1-Möbius cube".) with $|P^*_1| \le \lceil \frac{n}{2} \rceil$ and $|P^*_i| \le \lceil \frac{n}{2} \rceil + 2$ for each $i = 2, \cdots, n-2$, and $\gamma(P^*_i) \ge 1$ for each $i = 1, 2, \cdots, n-2$, then there are n internally disjoint paths $P_1, P_2, \cdots, P_n$ joining X and Y such that $|P_1| \le \lceil \frac{n+2}{2} \rceil$, $|P_i| \le \lceil \frac{n+2}{2} \rceil + 2$ for each $i = 2, \cdots, n$, and $\gamma(P_i) \ge 1$ for each $i = 1, 2, \cdots, n$.

Proof Our aim is to construct n paths $P_1, P_2, \cdots, P_n$ satisfying the requirement in Lemma. Since M_n and M_{n-2} are both either "0-Möbius cube" or "1-Möbius cube", we first construct $(n-2)$ paths in M_n as follows :

$$P'_i = P(X; P^*_i) = (X, X_{i,1}, \cdots, X_{i,|P^*_i|} = W_i) \text{ for } i = 1, 2, \cdots, n-2.$$

We can observe that all of P_i' are disjoint and $|P_i'| = |P_i^*|$ for each $i = 1, 2 \cdots, n-2$. Furthermore, since $P_{n-2}(W_i) = p_{n-2}(Y)$, it follows that W_i and Y are both in a subgraph which is isomorphic to either M_2^0 or M_2^1. For given i $(1 \le i \le n-2$), if $W_i = Y$, we simply let $P_i = P_i'$; if $W_i \ne Y$ we make the following modifications of P_i' to obtain a path P_i.

Since $\gamma(P_i^*) \ge 1$, we can assume $M_n(p_{n-2}(X_{i,k})) \cup M_n(p_{n-2}(X_{i,k+1}))$ is isomorphic to M_{n-1}^1 $(1 \le k < |P_i'| - 1)$. Because $M_n(p_{n-2}(X_{i,|P_i'|-1}))$ and $M_n(p_{n-2}(W_i))$ have edges to connect, thus $M_n(p_{n-2}(X_{i,|P_i'|-1})) \cup M_n(p_{n-2}(W_i))$ is isomorphic to one of M_{n-1}^0, M_{n-1}^1, $\widetilde{M_{n-1}^0}$ and $\widetilde{M_{n-1}^1}$. Let $X'_{i,|P_i^*|-1}$ be a neighbor predecessor of Y in $MQ_n(p_{n-2}(X_{i,|P_i^*|-1}))$, $X'_{i,j}$ be a neighbor predecessor of $X'_{i,j+1}$ in $M_n(p_{n-2}(X_{i,j}))$ for each $j = |P_i^*| - 2, \cdots, k$. We will prove that there exists ℓ with $k \le \ell \le |P_i^*| - 1$ such that $X'_{i,\ell}$ is a neighbor of $X_{i,\ell}$ in $M_n(p_{n-2}(X_{i,\ell}))$. Without loss of generality, we can assume $M_n(p_{n-2}(X_{i,|P_i'|-1})) \cup M_n$ $(p_{n-2}(W_i))$ is isomorphic to M_{n-1}^0, other situations can be similarly dealt with.

For each $i = 1, 2, \cdots, n-2$, assume $W_i = w_1^i \cdots w_n^i$. Without loss of generality, let $w_{n-2}^i = y_{n-2} = 0$. By Lemma 2, W_i and Y belong to $M_n(p_{n-2}(W_i))$ which is isomorphic to M_2^0. If $w_{n-1}^i w_n^i = 00$, $y_{n-1}y_n = 01$ or $w_{n-1}^i w_n^i = 10$, $y_{n-1}y_n = 11$, then we can let $\ell = |P_i'| - 1$. If W_i and Y do not have an edge to connect, such as $w_{n-1}^i w_n^i = 00$, $y_{n-1}y_n = 11$ or $w_{n-1}^i w_n^i = 10$, $y_{n-1}y_n = 01$, since $M_n(p_{n-2}(X_{i,|P_i'|-1}))$ is isomorphic to M_2^1, thus $X_{i,|P_i'|-1}$ is a neighbor of $X'_{i,|P_i'|-1}$. Thus we can also let $\ell = |P_i'| - 1$. If $w_{n-1}^i w_n^i = 00$, $y_{n-1}y_n = 10$ or $w_{n-1}^i w_n^i = 01$, $y_{n-1}y_n = 11$, then $X_{i,|P_i'|-1}$ is not a neighbor of $X'_{i,|P_i'|-1}$. But if there is j $(k < j < |P_i'| - 1)$ such that $M_n(p_{n-2}(X_{i,j})) \cup M_n(p_{n-2}(X_{i,j+1}))$ is isomorphic to M_{n-1}^0, then we can let $\ell = j$. If there is not such j, then we can let $\ell = k$.

So there exists ℓ with $k \le \ell \le |P_i^*| - 1$ such that $X'_{i,\ell}$ is a neighbor of $X_{i,\ell}$ in $M_n(p_{n-2}(X_{i,\ell}))$. Let $P_i = (P_i' - (X_{i,\ell}, \cdots, W_i)) \cup (X_{i,\ell}, X'_{i,\ell}, \cdots, X'_{i,|P_i'|-1}, Y)$. Obviously, $|P_i| \le |P_i'| + 1$ and $\gamma(P_i) \ge 1$.

Obviously, these $(n-2)$ paths $P_1, P_2, \cdots, P_{n-2}$ are internally disjoint. Now we construct other two paths using P_1^* as follows. For $i = n-1$ and n, let X^i be the two distinct neighbors of X in $M_n(p_{n-2}(X))$. Let $P'_{n-1} = P(X^{n-1}; P_1^*)$, $P'_n = P(X^n; P_1^*)$, $W_{n-1} = t(X^{n-1}; P'_{n-1})$, $W_n = t(X^n; P'_n)$. Note that $|P'_{n-1}| = |P'_n| = |P'_1|$. Furthermore, $W_{n-1} \ne W_n$, W_{n-1} and W_n are in a subgraph isomorphic to M_2^0 or M_2^1.

Without loss of generality, we consider that W_{n-1} and W_n are different from Y. Let $d_{M_n(p_{n-2}(Y))}(W_{n-1}, Y) = 1$, $d_{M_n(p_{n-2}(Y))}(W_n, Y) = 2$, the path from W_n to Y in $M_n(p_{n-2}(Y))$ be (W_n, Z, Y), then we can construct

$$P_{n-1} = (X, X^{n-1}, P'_{n-1}, W_{n-1}, Y) \quad \text{and} \quad P_n = (X, X^n, P'_n, W_n, Z, Y).$$

Obviously, $|P_{n-1}| \le \lceil \frac{n}{2} \rceil + 2$, $|P_n| \le \lceil \frac{n+2}{2} \rceil + 2$ and $\gamma(P_{n-1}) \ge 1, \gamma(P_n) \ge 1$. These n paths $P_1, P_2, \cdots, P_n$ are internally disjoint, $|P_1| \le \lceil \frac{n+2}{2} \rceil$ and $|P_i| \le \lceil \frac{n+2}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n$, as desired. The lemma follows. ∎

4 Wide diameter of Möbius Cubes

Theorem 1 Let X and Y be two distinct vertices of M_n for $n \ge 1$. Then, there are n internally disjoint paths $P_1, P_2, \cdots, P_n$ joining X to Y such that $|P_1| \le \lceil \frac{n+2}{2} \rceil$ and $|P_i| \le \lceil \frac{n+2}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n$.

Proof We prove this theorem by induction on n. It is easy to verify that the theorem is true for $n = 1, 2, 3$. Assume the theorem is true for M_ℓ for all ℓ with $3 < \ell < n$. We proceed to the induction step according to as M_n is M_n^1 or M_n^0.

Case 1 We first consider M_n is M_n^1. Without loss of generality, assume that $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_n$ are in M_n^1 and assume $x_1x_2 = 01$ since other situations can be considered in a similar argument. We partition M_n^1 into four $(n-2)$-Möbius cubes shown in Figure 3. We distinguish the following cases according to different locations of Y.

Subcase 1.1 $y_1y_2 = 01$, that is, both X and Y are in $M_n(01)$.

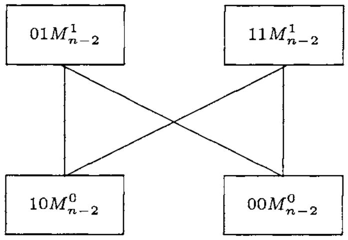

Figure 3: The partition of M^1_n into four $(n-2)$-Möbius cubes

By Lemma 2, $M^1_n(01)$ which is isomorphic to M^1_{n-2}. By the induction hypothesis, there are $(n-2)$ internally disjoint paths $P_1,\cdots,P_{n-2}$ joining X to Y in the $M^1_n(01)$ such that $|P_1|\le\lceil\frac{n}{2}\rceil$ and $|P_i|\le\lceil\frac{n}{2}\rceil+2$ for each $i=2,3,\cdots,n-2$. Let $P_{n-1}=(X,10\bar{x}_3\cdots\bar{x}_n,P(10\bar{x}_3\cdots\bar{x}_n\ ,10\bar{y}_3\cdots\bar{y}_n),Y)$, where $P(10\bar{x}_3\cdots\bar{x}_n,10\bar{y}_3\cdots\bar{y}_n)$ is a shortest path from $10\bar{x}_3\cdots\bar{x}_n$ to $10\bar{y}_3\cdots\bar{y}_n$ in $M^1_n(10)$. Since the diameter of $M^1_n(01)$ is $\lceil\frac{n-1}{2}\rceil$, thus $|P_{n-1}|\le\lceil\frac{n-1}{2}\rceil+2$. Let $P_n=(X,00x_3\cdots x_n,P(00x_3\cdots x_n,00y_3\cdots y_n),Y)$, where $P(00x_3\cdots x_n,00y_3\cdots y_n)$ is a shortest path from $00x_3\cdots x_n$ to $00y_3\cdots y_n$ in $M^1(00)$, Since the diameter of $M^1_n(00)$ is $\lceil\frac{n}{2}\rceil$, thus $|P_n|\le\lceil\frac{n}{2}\rceil+2$. Obviously, $P_1,\cdots,P_n$ are required paths joining X to Y.

Subcase 1.2 $y_1y_2=00$, that is, Y is in $M^1_n(00)$.
By Lemma 2, $M^1_n(01)\cup M^1_n(00)$ is isomorphic to M^0_{n-1}. By the induction hypothesis, there are $(n-1)$ internally disjoint paths $P_1,\cdots,P_{n-1}$ joining X to Y in $M^1_n(01)\cup M^1_n(00)$ such that $|P_1|\le\lceil\frac{n+1}{2}\rceil$ and $|P_i|\le\lceil\frac{n+1}{2}\rceil+2$ for each $i=2,3,\cdots,n-1$. Let $P_n=(X,10\bar{x}_3\cdots\bar{x}_n,P(10\bar{x}_3\cdots\bar{x}_n,11\bar{y}_3\cdots\bar{y}_n),Y)$, where $P(10\bar{x}_3\cdots\bar{x}_n,11\bar{y}_3\cdots\bar{y}_n)$ is a shortest path from $10\bar{x}_3\cdots\bar{x}_n$ to $11\bar{y}_3\cdots\bar{y}_n$ in $M^1_n(10)\cup M^1_n(11)$. By Lemma 2, $M^1_n(10)\cup M^1_n(11)$ is isomorphic to the M^1_{n-1}. Since the diameter of M^1_{n-1} is $\lceil\frac{n}{2}\rceil$, thus $|P_n|\le\lceil\frac{n}{2}\rceil+2$. Obviously, these n paths are required.

Subcase 1.3 $y_1y_2=10$. The proof is similar to that of Subcase 1.2.

Subcase 1.4 $y_1y_2=11$, that is, Y is in $M^0_n(11)$.
This is a difficult case for us. We partition M^1_n into eight $(n-3)$-Möbius cubes shown in Figure 4. Without loss of generality, we can assume $x_1x_2x_3=010$ since a similar argument can deal with the case of $x_1x_2x_3=011$. We distinguish the following cases according to different locations of Y.

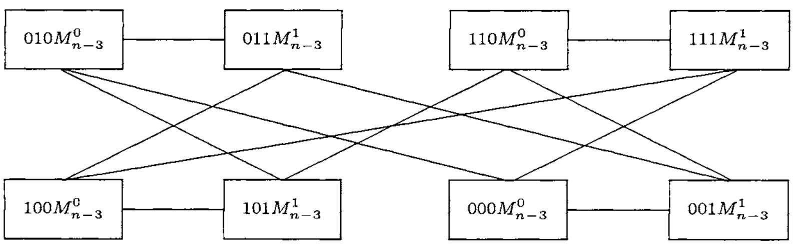

Figure 4: The partition of M^1_n into eight $(n-3)$-Möbius cubes

Subcase 1.4.1 $y_1y_2y_3=111$.
Since $000x_4\cdots x_n$ and $111y_4\cdots y_n$ belong to the $M^1_n(111)\cup M^1_n(000)$ which is isomorphic to the M^1_{n-2} by Lemma 2, there are $n-2$ internally disjoint paths joining $000x_4\cdots x_n$ to $111y_4\cdots y_n$ in the $M^1_n(111)\cup M^1_n(000)$. Let

$$N_{M^1_n(000)}(000x_4\cdots x_n)=\{000z_{i4}\cdots z_{in}:1\le i\le n-3\}$$

be neighbors of $000x_4\cdots x_n$ in $M_n^1(000)$. Assume these $(n-2)$ paths have the following forms.

$$\begin{array}{ll} P_i^* & = 000x_4\cdots x_n \to 000z_{i4}\cdots z_{in} \to \cdots \to Y \text{ for } i=1,2\cdots,n-3; \\ P_{n-2}^* & = 000x_4\cdots x_n \to 111\bar{x}_4\cdots\bar{x}_n \to \cdots \to Y. \end{array}$$

By the induction hypothesis, $|P_i^*| \le \lceil\frac{n}{2}\rceil + 2$ and $|P_1^*| \le \lceil\frac{n}{2}\rceil$. By Lemma 3, $M_n^1(010) \cup M_n^1(000)$ is isomorphic to the $\widetilde{M_{n-2}^0}$, we can construct n paths as follows.

$$\begin{array}{ll} P_i & = X \to 010z_{i4}\cdots z_{in} \to 000z_{i4}\cdots z_{in} \to P_i^* - (000x_4\cdots x_n, 000z_{i4}\cdots z_{in}); \\ P_{n-2} & = X \to 000x_4\cdots x_n \to P_{n-2}^*; \\ P_{n-1} & = X \to 101\bar{x}_4\cdots\bar{x}_n \to 110x_4\cdots x_n \to \cdots \to 110\bar{y}_4\cdots\bar{y}_n \to Y; \\ P_n & = X \to 011\bar{x}_4\cdots\bar{x}_n \to 100x_4\cdots x_n \to \cdots \to 100\bar{y}_4\cdots\bar{y}_n \to Y, \end{array}$$

where $1 \le i \le n-3$, $110x_4\cdots x_n \to \cdots \to 110\bar{y}_4\cdots\bar{y}_n$ and $100x_4\cdots x_n \to \cdots \to 100\bar{y}_4\cdots\bar{y}_n$ are shortest paths in $M_n^1(110)$ and $M_n^1(100)$, respectively. Obviously, these n paths are internally disjoint and $|P_1| \le \lceil\frac{n+2}{2}\rceil$ and $|P_i| \le \lceil\frac{n+2}{2}\rceil + 2$ for each $i = 2,\cdots,n$.

Subcase 1.4.2 $y_1y_2y_3 = 110$.

First assume $x_4\cdots x_n = y_4\cdots y_n$. Let

$$N_{M_n^1(010)}(X) = \{010x_{14}\cdots x_{1n},\cdots,010x_{(n-3)4}\cdots x_{(n-3)n}\}$$

be neighbors of X in $M_n^1(010)$. Obviously,

$$N_{M_n^1(110)}(Y) = \{110x_{14}\cdots x_{1n},\cdots,110x_{(n-3)4}\cdots x_{(n-3)n}\}$$

is the neighbors of Y in $M_n^1(110)$. We can construct n paths joining X to Y in M_n as follows.

$$\begin{array}{ll} P_1 & = X \to 101\bar{x}_4\cdots\bar{x}_n \to 110x_4\cdots x_n = Y; \\ P_{i+1} & = X \to 010x_{i4}\cdots x_{in} \to 101\bar{x}_{i4}\cdots\bar{x}_{in} \to 110x_{i4}\cdots x_{in} \to Y,\ 1 \le i \le n-3; \\ P_{n-1} & = X \to 000x_4\cdots x_n \to 111\bar{x}_4\cdots\bar{x}_n \to Y; \\ P_n & = X \to 011\bar{x}_4\cdots\bar{x}_n \to 001\bar{x}_4\cdots\bar{x}_n \to Y. \end{array}$$

By a simple observation, these n paths are internally disjoint and $|P_i| = 4$ for $i = 2,3,\cdots,n-2$, $|P_1| = 2$ and $|P_{n-1}| = |P_n| = 3$.

We now assume $x_4\cdots x_n \ne y_4\cdots y_n$. We consider two subcases according to the parity of n.

Subcase 1.4.2a n is odd.

Obviously, both $M_n^1(010) \cup M_n^1(101)$ and $M_n^1(101) \cup M_n^1(110)$ are isomorphic to M_{n-2}^1. By the induction hypothesis, there are $(n-3)$ internally disjoint paths joining $110x_3\cdots x_n$ to Y in $M_n^1(110)$. Let

$$N_{M_n^1(110)}(110x_4\cdots x_n) = \{110z_{i4}\cdots z_{in} : 1 \le i \le n-3\}.$$

be neighbors of $110x_4\cdots x_n$ in $M_n^1(110)$. Then, these paths have the following forms.

$$P_i^* = 110x_4\cdots x_n \to 110z_{i4}\cdots z_{in} \to \cdots \to Y,$$

and $1 \le |P_1^*| \le \lceil\frac{n-1}{2}\rceil$, $2 \le |P_i^*| \le \lceil\frac{n-1}{2}\rceil + 2$ for each $i = 2,3,\cdots,n-3$. We construct n paths internally disjoint from X to Y in M_n as follows.

$$\begin{array}{ll} P_1 & = X \to 010z_{14}\cdots z_{1n} \to \cdots \to 010y_4\cdots y_n \to 101\bar{y}_4\cdots\bar{y}_n \to Y, \\ P_i & = X \to 010z_{i4}\cdots z_{in} \to 101\bar{z}_{i4}\cdots\bar{z}_{in} \to 110z_{i4}\cdots z_{in} \to \cdots \to Y, \\ P_{n-2} & = X \to 101\bar{x}_4\cdots\bar{x}_n \to 110x_4\cdots x_n \to P_1^*, \\ P_{n-1} & = X \to 000x_4\cdots x_n \to \cdots \to 000y_4\cdots y_n \to 111\bar{y}_4\cdots\bar{y}_n \to Y, \\ P_n & = X \to 011\bar{x}_4\cdots\bar{x}_n \to \cdots \to 011\bar{y}_4\cdots\bar{y}_n \to 001\bar{y}_4\cdots\bar{y}_n \to Y, \end{array}$$

where $i = 2,3,\cdots,n-3$; the path from X to $010y_4\cdots y_n$ in P_1 is along the responding path P_1^* in the $M_n^1(010)$; the path from $110z_{i4}\cdots z_{in}$ to Y in P_i is along the path P_i^*, $2 \le i \le n-3$; the path from

$000x_4\cdots x_n$ to $000y_4\cdots y_n$ in P_{n-1} is a shortest path from $000x_4\cdots x_n$ to $000y_4\cdots y_n$ in $M_n^1(000)$; the path from $011\bar{x}_4\cdots\bar{x}_n$ to $011\bar{y}_4\cdots\bar{y}_n$ in P_n is a shortest path from $011\bar{x}_4\cdots\bar{x}_n$ to $011\bar{y}_4\cdots\bar{y}_n$ in $M_n^1(011)$. Obviously,

$$|P_1| \le \lceil\tfrac{n+3}{2}\rceil = \lceil\tfrac{n+2}{2}\rceil,\ |P_{n-1}| \le \lceil\tfrac{n+1}{2}\rceil + 2,\ |P_n| \le \lceil\tfrac{n+1}{2}\rceil + 2;$$
$$|P_i| \le \lceil\tfrac{n+3}{2}\rceil + 2 = \lceil\tfrac{n+2}{2}\rceil + 2,\ \text{for each } i = 2,3,\cdots,n-2.$$

Subcase 1.4.2b n is even. We partition M_n^1 into sixteen $(n-4)$-Möbius cubes shown as Figure 5. Without loss of generality, we can assume $x_1x_2x_3x_4 = 0100$ since the case of $x_1x_2x_3x_4 = 0101$ can be dealt with similarly.

If $y_1y_2y_3y_4 = 1100$, then, since both $M_n^1(0100) \cup M_n^1(1011)$ and $M_n^1(1011) \cup M_n^1(1100)$ are isomorphic to M_{n-3}^1, we can construct n required paths from X to Y similar to Case 1.4.2a. We now consider the case of $y_1y_2y_3y_4 = 1101$.

If $n = 4$, in M_4^1, we can construct 4 internally disjoint paths joining 0100 and 1101 as follows.

$$\begin{aligned}
P_1^* &= 0100 \to 1011 \to 1010 \to 1101;\\
P_2^* &= 0100 \to 0111 \to 0011 \to 0010 \to 1101;\\
P_3^* &= 0100 \to 0101 \to 0001 \to 1110 \to 1101;\\
P_4^* &= 0100 \to 0000 \to 1111 \to 1100 \to 1101.
\end{aligned}$$

Obviously, $\gamma(P_i^*) \ge 1$ for each $i = 1,2,3,4$ and $|P_1^*| = 3 \le \lceil\frac{n+2}{2}\rceil$ and $|P_i^*| = 4 \le \lceil\frac{n+2}{2}\rceil + 2$ for $2 \le i \le 4$. Thus these 4 paths satisfy the conditions of Lemma 4. We can construct n required paths joining X and Y in M_n^1 by applying Lemma 4 continuously.

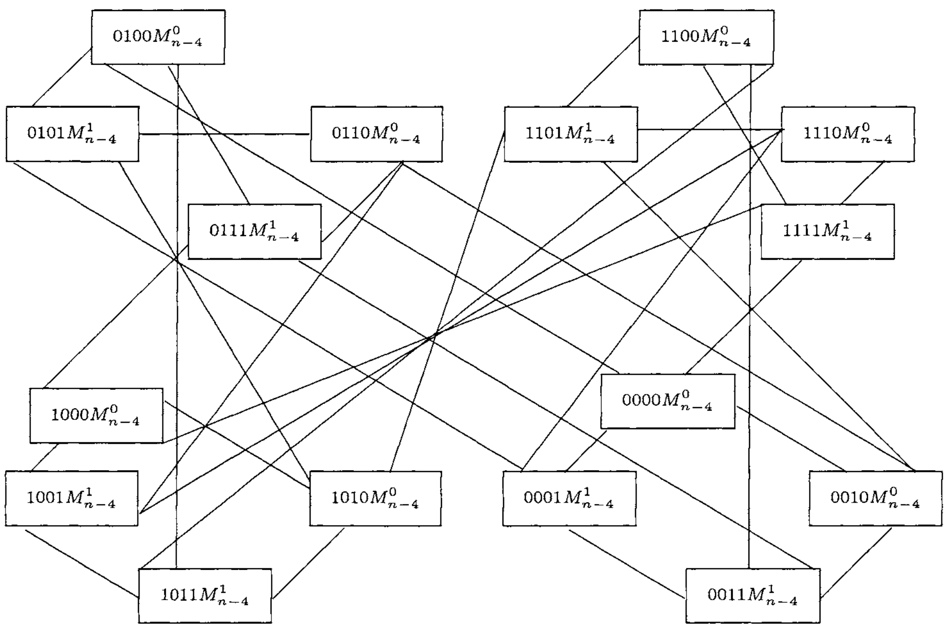

Figure 5: The partition of M_n^1 into sixteen $(n-4)$-Möbius cubes

Case 2 We now consider M_n to be M_n^0. The partition of M_n^0 into four $(n-2)$-Möbius cubes are shown in Figure 6. There are four cases.

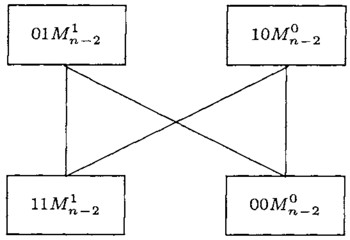

Figure 6: The partition of M^0_n into four $(n-2)$-Möbius cubes

In this case, X and Y belong to a subgraph $M^0_n(01)$ which is isomorphic to M^1_{n-2}. By the induction hypothesis, there are $(n-2)$ disjoint paths $P_1, \cdots, P_{n-2}$ joining X to Y in the $M^0_n(01)$ such that $|P_1| \le \lceil \frac{n}{2} \rceil$ and $|P_i| \le \lceil \frac{n}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n-2$. Let

$$\begin{aligned} P_{n-1} &= X \to 00x_3 \cdots x_n \to \cdots \to 00y_3 \cdots y_n \to 01y_3 \cdots y_n, \\ P_n &= X \to 11x_3 \cdots x_n \to \cdots \to 11y_3 \cdots y_n \to Y, \end{aligned}$$

where the path from $00x_3 \cdots x_n$ to $00y_3 \cdots y_n$ in P_{n-1} is a shortest path from $00x_3 \cdots x_n$ to $00y_3 \cdots y_n$ in the $M^0_n(00)$, and the path from $11x_3 \cdots x_n$ to $11y_3 \cdots y_n$ in P_n is a shortest path from $11x_3 \cdots x_n$ to $11y_3 \cdots y_n$ in $M^0_n(11)$.

Since $d_{M^0_n(00)}(00x_3 \cdots x_n, 00y_3 \cdots y_n) \le \lceil \frac{n}{2} \rceil$ and $d_{M^0_n(11)}(11x_3 \cdots x_n, 11y_3 \cdots y_n) \le \lceil \frac{n-1}{2} \rceil$, thus, $|P_{n-1}| = \lceil \frac{n}{2} \rceil + 2$ and $|P_n| = \lceil \frac{n-1}{2} \rceil + 2$. Obviously, $P_1, \cdots, P_n$ are disjoint paths joining X to Y and satisfy our requirements.

Subcase 2.1 $y_1y_2 = 01$.

Subcase 2.2 $y_1y_2 = 00$.

In this case X and Y belong to a subgraph $M^0_n(01) \cup M^0_n(00)$ which is isomorphic to M^0_{n-1}. By the induction hypothesis, there are $(n-1)$ disjoint paths $P_1, \cdots, P_{n-1}$ joining X to Y in the $M^0_n(01) \cup M^0_n(00)$ such that $|P_1| \le \lceil \frac{n+1}{2} \rceil$ and $|P_i| \le \lceil \frac{n+1}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n-1$. Let

$$P_n = X \to 11x_3 \cdots x_n \to \cdots \to 10y_3 \cdots y_n \to Y,$$

where the path from $11x_3 \cdots x_n$ to $10y_3 \cdots y_n$ is a shortest path from $11x_3 \cdots x_n$ to $10y_3 \cdots y_n$ in $M^0_n(11) \cup M^0_n(10)$. By Lemma 3, $M^0_n(11) \cup M^0_n(10)$ is isomorphic to M^1_{n-1}, thus the distance between of $11x_3 \cdots x_n$ and $10y_3 \cdots y_n$ in $M^0_n(11) \cup M^0_n(10)$ is at most $\lceil \frac{n}{2} \rceil$, and so $|P_n| \le \lceil \frac{n}{2} \rceil + 2$. Obviously, $P_1, P_2 \cdots, P_n$ are disjoint paths joining X to Y and satisfy our requirements.

Subcase 2.3 $y_1y_2 = 11$.

Assume $x_3 \cdots x_n = y_3 \cdots y_n$. Let

$$N_{M^0_n(01)}(X) = \{01z_{13} \cdots z_{1n}, \cdots, 01z_{(n-2)3} \cdots x_{(n-2)n}\}$$

be neighbors of X in $M^0_n(01)$. Obviously,

$$N_{M^0_n(11)}(Y) = \{11z_{13} \cdots z_{1n}, \cdots, 11z_{(n-2)3} \cdots z_{(n-2)n}\}$$

is the neighbors of Y in $M^0_n(11)$. We can construct n paths joining X to Y in M_n as follows.

$$\begin{aligned} P_1 &= X \to Y; \\ P_{i+1} &= X \to 01z_{i3} \cdots z_{in} \to 11z_{i3} \cdots z_{in} \to Y, \ i = 1, \cdots, n-2; \\ P_n &= X \to 00x_3 \cdots x_n \to \cdots \to 00\bar{x}_3 \cdots \bar{x}_n \to 10\bar{x}_3 \cdots \bar{x}_n \to Y, \end{aligned}$$

where $00x_3 \cdots x_n$ to $00\bar{x}_3 \cdots \bar{x}_n$ is a shortest path from $00x_3 \cdots x_n$ to $00\bar{x}_3 \cdots \bar{x}_n$ in $M^0_n(00)$. Clearly, these n paths are internally disjoint and $|P_1| = 1$, $|P_i| = 3$, $1 \le i \le n-2$, $|P_n| \le \lceil \frac{n+2}{2} \rceil + 2$.

We now assume $x_3 \cdots x_n \neq y_3 \cdots y_n$. By Lemma 3, X and Y are in $M_n^0(01) \cup M_n^0(11)$ which is isomorphic to $\widetilde{M^1}_{n-1}$. Because $11x_3 \cdots x_n$ and Y are in $M_n^0(11)$ which is isomorphic to M_{n-2}^1. There are $(n-2)$ internally disjoint paths $P_1^*, \cdots, P_{n-2}^*$ joining $11x_3 \cdots x_n$ to Y in $M_n^0(11)$ such that $|P_1^*| \leq \lceil \frac{n}{2} \rceil$ and $|P_i^*| \leq \lceil \frac{n}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n-2$. We can assume that

$$P_i^* = 11x_3 \cdots x_n \to 11z_{i3} \cdots z_{in} \to \cdots \to Y, \quad \text{for each } i = 1, 2, \cdots, n-2.$$

We construct n internally disjoint paths joining X to Y in M_n as follows.

$$\begin{array}{ll} P_1 & = X \to 01z_{13} \cdots z_{1n} \to \cdots \to 01y_3 \cdots y_n \to Y; \\ P_i & = X \to 01z_{i3} \cdots z_{in} \to 11z_{i3} \cdots z_{in} \to \cdots \to Y \text{ for each } i = 2, 3, \cdots, n-2; \\ P_{n-1} & = X \to 11x_3 \cdots x_n \to P_1^*; \\ P_n & = X \to 00x_3 \cdots x_n \to 10x_3 \cdots x_n \to \cdots \to 10\bar{y}_3 \cdots \bar{y}_n \to Y, \end{array}$$

where the path from X to $01y_3 \cdots y_n$ in P_1 is along the corresponding path P_1^* in the $M_n^0(01)$; the path from $11z_{i3} \cdots z_{in}$ to Y in P_i is along the P_i^*; the path from $10x_3 \cdots x_n$ to $10\bar{y}_3 \cdots \bar{y}_n$ in P_n is a shortest path from $10x_3 \cdots x_n$ to $10\bar{y}_3 \cdots \bar{y}_n$ in $M_n^0(10)$ which is isomorphic to the M_{n-2}^0. Clearly, $P_1, P_2 \cdots, P_n$ are internally disjoint. By the induction hypothesis, $|P_1| \leq \lceil \frac{n+2}{2} \rceil$ and $|P_i| \leq \lceil \frac{n+2}{2} \rceil + 2$ for each $i = 2, 3, \cdots n$, as desired.

Subcase 2.4 $y_1 y_2 = 10$.

We partition M_n^0 into eight $(n-3)$-Möbius cubes shown in Figure 7. Without loss of generality, we can assume $x_1x_2x_3 = 010$ since a similar argument can deal with the case of $x_1x_2x_3 = 011$. We distinguish the following cases according to different locations of Y.

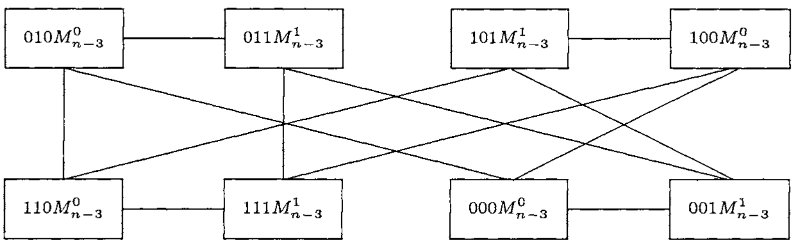

Figure 7: The partition of M_n^0 into eight $(n-3)$-Möbius cubes

Subcase 2.4.1 $x_1x_2x_3 = 010$, $y_1y_2y_3 = 101$.

Since $M_n^0(010) \cup M_n^0(110)$, $M_n^0(110) \cup M_n^0(101)$ are isomorphic to the $\widetilde{M^0}_{n-2}$ and M_{n-2}^1 respectively, the proof is similar to Subcase 1.4.1.

Subcase 2.4.2 $x_1x_2x_3 = 010$, $y_1y_2y_3 = 100$. We consider two subcases according to the parity of n.

Subcase 2.4.2a n is odd.

Since $M_n^0(010) \cup M_n^0(000)$, $M_n^0(000) \cup M_n^0(100)$ are isomorphic to $\widetilde{M^0}_{n-2}$, we can construct $n-2$ internally disjoint paths joining $000x_4 \cdots x_n$ and $100y_4 \cdots y_n$ in the $M_n^0(000) \cup M_n^0(100)$ in a way similar to Subcase 2.3. Let

$$N_{M_n^0(000)}(000x_4 \cdots x_n) = \{000z_{i4} \cdots z_{in} : \ 1 \leq i \leq n-3\}.$$

Then the paths have the forms as follows.

$$\begin{array}{ll} P_i^* & = 000x_4 \cdots x_n \to 000z_{i4} \cdots z_{in} \to \cdots \to Y \text{ for each } i = 1, 2, \cdots, n-3; \\ P_{n-2}^* & = 000x_4 \cdots x_n \to 100x_4 \cdots x_n \to \cdots \to Y. \end{array}$$

And $|P_1^*| \le \lceil \frac{n+1}{2} \rceil$ and $|P_i^*| \le \lceil \frac{n+1}{2} \rceil + 2$ for each $i = 2, 3, \cdots, n-2$. We can construct n internally disjoint paths joining X to Y as follows.

$$\begin{array}{ll} P_i & = X \to 010z_{i4}\cdots z_{in} \to 000z_{i4}\cdots z_{in} \to \cdots \to Y \text{ for each } i = 1, 2, \cdots, n-3; \\ P_{n-2} & = X \to 000x_4\cdots x_n \to P_{n-2}^*; \\ P_{n-1} & = X \to 110x_4\cdots x_n \to 101\bar{x}_4\cdots \bar{x}_n \to \cdots \to 101y_4\cdots y_n \to Y; \\ P_n & = X \to 011x_4\cdots x_n \to 111x_4\cdots x_n \to \cdots \to 111\bar{y}_4\cdots \bar{y}_n \to Y, \end{array}$$

where the path from $000z_{i4}\cdots z_{in}$ to Y in P_i is along the path P_i^*; the path from $101\bar{x}_4\cdots\bar{x}_n$ to $101y_4\cdots y_n$ in P_{n-1} is a shortest path from $101\bar{x}_4\cdots\bar{x}_n$ to $101y_4\cdots y_n$ in $M_n^0(101)$; the path from $111x_4\cdots x_n$ to $111\bar{y}_4\cdots\bar{y}_n$ in P_n is a shortest path from $111x_4\cdots x_n$ to $111\bar{y}_4\cdots\bar{y}_n$ in $M_n^0(111)$. Obviously, $|P_1| \le \lceil \frac{n+3}{2} \rceil = \lceil \frac{n+2}{2} \rceil$, $|P_i| \le \lceil \frac{n+3}{2} \rceil + 2 = \lceil \frac{n+2}{2} \rceil + 2$ for all $2 \le i \le n-2$ and $|P_i| \le \lceil \frac{n+1}{2} \rceil + 2$ for $i = n-1, n$. So these n paths satisfy our requirements.

Subcase 2.4.2b n is even. We partition M_n^0 into sixteen $(n-4)$-Möbius cubes shown as Figure 8.

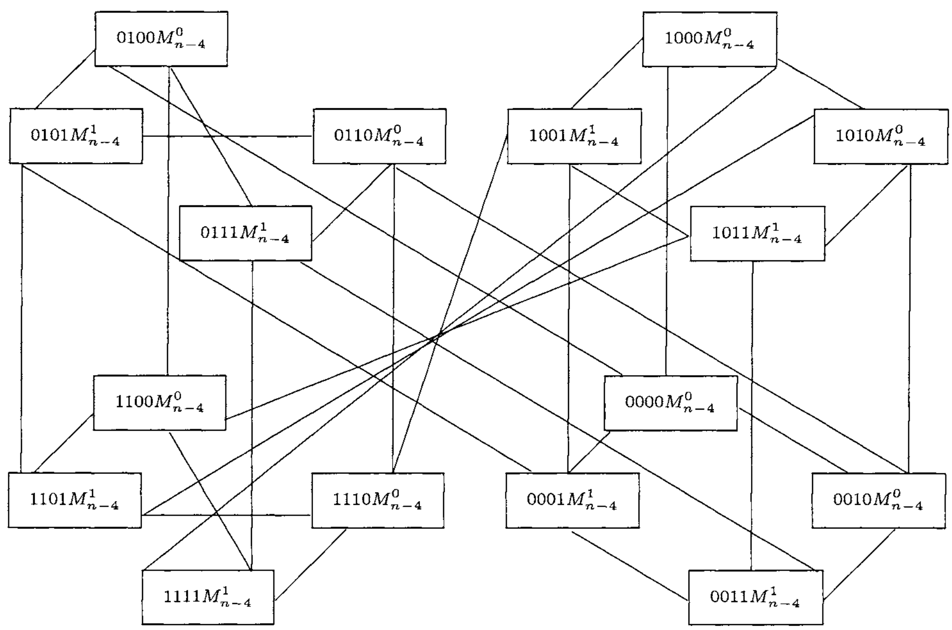

Figure 8: The partition of M_n^0 into sixteen $(n-4)$-Möbius cubes

If $x_1x_2x_3x_4 = 0100$ and $y_1y_2y_3y_4 = 1000$, then, since $M_n^0(0100) \cup M_n^0(0000)$, $M_n^0(0000) \cup M_n^0(1000)$ are isomorphic to the $\widetilde{M^0}_{n-3}$, we can construct n required paths in a way similar to Subcase 2.4.2a.

Now assume $x_1x_2x_3x_4 = 0100$ and $y_1y_2y_3y_4 = 1001$. We first consider $n = 4$, in M_4^0, we can construct 4 internally disjoint paths joining 0100 and 1001 as follows:

$$\begin{array}{ll} P_1^* & = 0100 \to 1100 \to 1011 \to 1001; \\ P_2^* & = 0100 \to 0111 \to 1111 \to 1000 \to 1001; \\ P_3^* & = 0100 \to 0000 \to 0010 \to 0011 \to 0001 \to 1001; \\ P_4^* & = 0100 \to 0101 \to 0110 \to 1110 \to 1001. \end{array}$$

Obviously, $\gamma(P_i^*) = 1$ for each $i = 1, 2, 3, 4$ and $|P_i^*| \le 5 \le \lceil \frac{n+2}{2} \rceil + 2$ for $i = 2, 3, 4$, and $|P_1^*| = 3 \le$

$\lceil \frac{n+2}{2} \rceil$. These 4 paths satisfy the conditions of Lemma 4. Thus, we can construct n required paths by applying Lemma 4 continuously.

The proof of the theorem is complete. ∎

Theorem 2 For any Möbius cube M_n,

$$\begin{array}{rcl} \lceil \frac{n+2}{2} \rceil + 1 \leq & d_n(M_n^0) & \leq \lceil \frac{n+2}{2} \rceil + 2, \text{ for } n \geq 4; \\ \lceil \frac{n+1}{2} \rceil + 1 \leq & d_n(M_n^1) & \leq \lceil \frac{n+2}{2} \rceil + 2, \text{ for } n \geq 1. \end{array}$$

Proof Let G be n-connected and n-regular graph, we can use a basic result that $d_n(G) \geq d(G)+1$ obtained first by Hsu and Luczak [6]. Since M_n is n-connected and n-regular graph, thus $d_n(M_n) \geq d(M_n)+1$. Because $d(M_n^0) = \lceil \frac{n+2}{2} \rceil$, $n \geq 4$, $d(M_n^1) = \lceil \frac{n+1}{2} \rceil$, $n \geq 1$, and by Theorem 1, this theorem follows. ∎

Remarks As we have known, the fault tolerant diameter of n-connected graph G, $D_n(G)$, is not larger than the wide diameter of G. So we have $D_n(M_n) \leq \lceil \frac{n+2}{2} \rceil + 2$ for all $n \geq 1$, immediately.

For the hypercube Q_n, Armstrong and Gray [1], Saad and Schiltz [11] independently showed that $d_n(Q_n) = n+1$, and Krishnamoorthy and Krishanmurthy [10] showed that $D_n(Q_n) = n+1$. From which, we can see that the wide diameter and fault tolerant diameter of Möbius cubes are about one half of those for hypercube respectively.

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