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Edge-Pancyclicity of Crossed Cubes*

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Abstract: The crossed cube CQ_n is a variant of the hypercube Q_n and has better properties than Q_n with the same number of links and processors. It has been shown that CQ_n contains a cycle of every length from 4 to 2^n. The result is improved by showing that every edge of CQ_n lies on a cycle of every length from 4 to 2^n inclusive.

Key words: cycle; crossed cubes; hypercubes; pancyclicity; edge-pancyclicity

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0 Introduction

The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and is easy to implement. As a variant of Q_n, the crossed cube CQ_n, first proposed by Efe[1,2], has many properties superior to Q_n.[1~9]. In particular, CHANG *et al*[4] proved that CQ_n contains a cycle of length from 4 to 2^n. In this note, we improve this result by showing the following theorem.

Theorem Every edge of CQ_n lies on a cycle of every length from 4 to 2^n inclusive for $n \geqslant 2$.

The proof of the theorem is in Section 2. In Section 1, the definition and basic properties of CQ_n are given.

1 Crossed Cubes

The architecture of an interconnection work is usually represented by a connected simple graph $G=(V,E)$, where the vertex-set V is the set of processors and the edge-set E is the set of communication links in the network.

The binary strings x of length n will be written as $x_{n-1}x_{n-2}\cdots x_1x_0$.

The complement of x_i will be denoted by $\bar{x}_i$ (i. e., $\bar{0}=1$ and $\bar{1}=0$). Two binary strings $x=x_1x_0$ and $y=y_1y_0$ of length two are pair-related, denoted as $x\sim y$, if and only

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if $(x,y) \in \{(00,00), (10,10), (01,11), (11,01)\}$. The n-dimensional crossed cube CQ_n has 2^n vertices labelled by binary strings of length n and can be recursively defined as follows.

CQ_1 is a complete graph with two vertices labelled by 0 and 1. For $n \geqslant 2$, CQ_n is obtained by taking two copies of CQ_{n-1}, denoted by CQ_{n-1}^0 with vertex-set

$$V(CQ_{n-1}^0) = \{0x_{n-2}\cdots x_1x_0 : x_i = 0 \text{ or } 1\}$$

and CQ_{n-1}^1 with vertex-set

$$V(CQ_{n-1}^1) = \{1y_{n-2}\cdots y_1y_0 : y_i = 0 \text{ or } 1\},$$

respectively, and adding an edge joining $0x_{n-2}\cdots x_1x_0 \in V(CQ_{n-1}^0)$ and $1y_{n-2}\cdots y_1y_0 \in V(CQ_{n-1}^1)$ if and only if

(1) $x_{n-2} = y_{n-2}$ if n is even, and

(2) $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i < \left\lfloor \frac{n-1}{2} \right\rfloor$.

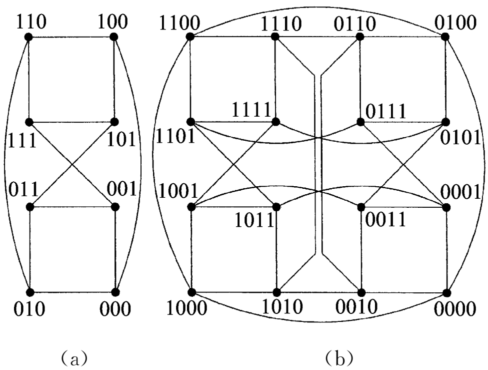

Fig. 1 (a) CQ_3 (b) CQ_4.

The graphs shown in Fig. 1 (a) and (b) are CQ_3 and CQ_4, respectively.

According to the above definition, we can denote $CQ_n = L \oplus R$, where $L \cong CQ_{n-1}^0$ and $R \cong CQ_{n-1}^1$, and call edges between L and R cross edges. Moreover, we write a cross edge as (u_L, u_R), where $u_L \in L$ and $u_R \in R$.

We use CQ_{n-2}^{ij} to denote the $n-2$-dimensional crossed cube which is a subgraph of CQ_n induced by the vertices labelled $ijx_{n-3}\cdots x_0$. We say an edge of CQ_n is critical if it is an edge in CQ_{n-1}^i with one endpoint in CQ_{n-2}^{i0} and the other in CQ_{n-2}^{i1} for $i \in \{0,1\}$.

Lemma 1 Let $CQ_n = L \oplus R$ with $n \geqslant 2$. If $(u_L, v_L) \in E(L)$ is a critical edge of CQ_n, then $(u_R, v_R) \in E(R)$ is also a critical edge of CQ_n, where u_R and v_R are the neighbors of u_L and v_L in R, respectively.

Proof Suppose that $(u_L, v_L) \in E(L)$ is a critical edge of CQ_n and, without loss of generality, assume that $u_L = 00x_{n-3}x_{n-4}\cdots x_1x_0$. We show that $(u_R, v_R) \in E(R)$ is also a critical edge of CQ_n by considering the two cases according to the parity of n, respectively. For $n = 2,3$, the result holds clearly, and assume $n \geqslant 4$ below.

Case 1. n is even.

In this case, $v_L = 01y_{n-3}y_{n-4}\cdots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i \leqslant (n-4)/2$. Hence $u_R = 10y_{n-3}y_{n-4}\cdots y_1y_0$ and $v_R = 11x_{n-3}x_{n-4}\cdots x_1x_0$. By definition of CQ_n, u_R and v_R are adjacent, and so (u_R, v_R) is a critical edge in R.

Case 2. n is odd.

In this case, we have either $x_{n-3} = 1$ or 0. If $x_{n-3} = 1$, then $u_L = 001x_{n-4}\cdots x_1x_0$ and $v_L = 011y_{n-4}\cdots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i \leqslant (n-5)/2$. Hence $u_R = 111y_{n-4}\cdots y_1y_0$ and $v_R = 101x_{n-4}\cdots x_1x_0$. By definition of CQ_n, u_R and v_R are adjacent, and so (u_R, v_R) is a critical edge in R. Similarly, we can show that the statement is also true for $x_{n-3} = 0$.

Note that if $CQ_n = L \oplus R$ then, for any two adjacent u_L and v_L in L, their neighbors u_R and v_R in R are not always adjacent in R, and vice versa. However, it is clear from Lemma 1 that if (u_L, v_L) is a critical edge, then their neighbors u_R and v_R in R must be adjacent in R, and vice versa. Thus, the vertices u_L, v_L, v_R, u_R, u_L form a 4-cycle. Critical edges play an important role in the proof of our theorem. A cycle in CQ_n is called a 2-critical if it contains at least two critical edges. It is easy to see that every vertex in CQ_n is incident with a critical edge and every cross edge lies on a 2-critical cycle of length four.

Lemma 2 If the length of a cycle is greater than 2^{n-2} in the subgraph CQ_{n-1}^0 of CQ_n for $n \geqslant 4$, then it must be a 2-critical cycle.

Proof Note that the $(n-2)$-dimensional crossed cube $CQ_{n-2}{}^{0j}$ for $j \in \{0,1\}$ has only 2^{n-2} vertices. Since any cycle in $CQ_{n-2}{}^{00}$ or in $CQ_{n-2}{}^{01}$ has length at most 2^{n-2}, any cycle of length greater than 2^{n-2} in $CQ_{n-1}{}^{0}$ must contain vertices in both $CQ_{n-2}{}^{00}$ and $CQ_{n-2}{}^{01}$ and so contain at least two critical edges between $CQ_{n-2}{}^{00}$ and $CQ_{n-2}{}^{01}$.

2 Proof of Theorem

In this section, we give the proof of Theorem stated in Introduction.

Proof We prove the theorem by induction on $n \geqslant 2$. The theorem is true for $n = 2$.

For $n = 3$, we only need to prove that every edge of CQ_3 lies on a cycle of every length from 4 to 8 inclusive. We exhaust these cycles as follows.

The union of the following four cycles of length four covers all edges of CQ_3.

$$\langle 000,001,011,010,000\rangle,\quad \langle 100,101,111,110,100\rangle,$$
$$\langle 000,100,110,010,000\rangle,\quad \langle 101,011,001,111,101\rangle.$$

The union of the following five cycles of length five covers all edges of CQ_3.

$$\langle 000,001,111,110,010,000\rangle,\quad \langle 001,111,110,010,011,001\rangle,$$
$$\langle 111,110,010,011,101,111\rangle,\quad \langle 110,010,011,101,100,110\rangle,$$
$$\langle 100,000,010,011,101,100\rangle.$$

The union of the following three cycles of length six covers all edges of CQ_3.

$$\langle 000,001,111,101,011,010,000\rangle,\langle 000,100,101,111,110,010,000\rangle,$$
$$\langle 100,000,001,011,010,110,100\rangle.$$

The union of the following three cycles of length seven covers all edges of CQ_3.

$$\langle 000,100,110,111,101,011,010,000\rangle,\langle 100,101,111,001,011,010,110,100\rangle,$$
$$\langle 000,001,011,101,111,110,010,000\rangle.$$

The union of the following two cycles of length eight covers all edges of CQ_3.

$$\langle 000,100,101,011,001,111,110,010,000\rangle,$$
$$\langle 100,000,001,111,101,011,010,110,100\rangle.$$

Thus the theorem is true for $n = 3$.

Assume now that the theorem is true for all $3 \leqslant k < n$. Let e be any edge of CQ_n and let l be any integer with $4\leqslant l\leqslant 2^n$, where $n\geqslant 4$. To complete the proof of the theorem, we need to show that e is contained in a cycle of length l by considering two cases according to whether e is a cross edge or not.

Case 1. Edge e is not a cross edge. Then edge e is in L or R. Without loss of generality, we may assume e is in L.

If $4\leqslant l\leqslant 2^{n-1}$, by the induction hypothesis, there exists a cycle of length l in $L\subset CQ_n$ that contains e.

Suppose that $2^{n-1}+1\leqslant l\leqslant 2^{n-1}+3$. By the induction hypothesis, there exists a cycle C of length $2^{n-1}-3$ in L containing e. For $n\geqslant 4$, we have $2^{n-1}-3>2^{n-2}$, and so C is a 2 − critical cycle by Lemma 2. Thus, we can choose a critical edge (u_L, v_L) in C different from e. Then the neighbors of u_L, v_L are u_R, v_R in R with $(u_R, v_R)\in E(R)$ by Lemma 1. By the induction hypothesis, there exists a cycle C' of length $4\leqslant l'\leqslant 6$ in R containing (u_R, v_R). Thus $P'=C'-(u_R, v_R)$ is a path between v_R and u_R in R. Let $P=C-(u_L, v_L)$. Then P contains e and $P+(v_L, v_R)+P'+(u_R, u_L)$ is a cycle of length l in CQ_n containing e.

Suppose that $2^{n-1}+4\leqslant l\leqslant 2^n$. Let $l'=l-2^{n-1}$. Then $4\leqslant l'\leqslant 2^{n-1}$. By the induction hypothesis and Lemma 2, there exists a 2-critical cycle C of length 2^{n-1} in L containing e. We can choose a critical edge (u_L, v_L) different from e. Without loss of generality, let u_R and v_R be the neighbors of u_L and v_L, respectively. Then u_R and v_R are adjacent in R. Let $P=C-(u_L, v_L)$. Obviously e lies on P. By the induction hypothesis there exists a cycle C' of length l' in R that contains (u_R, v_R). Let $P'=C'-(v_R, u_R)$. Then $P+(v_L, v_R)+P'+(u_R, u_L)$ is a cycle of length l in CQ_n and contains e.

Case 2. Edge e is a cross edge between L and R. We may assume $e=(u_L, u_R)$.

In order to find cycles of lengths 4 and 5 containing e, we consider two cases according to the parity of n, respectively.

Note that if $x_1x_0\sim y_1y_0$, then $\bar{x}_1x_0\sim\bar{y}_1y_0$ and $\bar{x}_1\bar{x}_0\sim y_1\bar{y}_0$.

If n is even, we may assume $u_L=0x_{n-2}x_{n-3}\cdots x_1x_0$, then $u_R=1x_{n-2}y_{n-3}\cdots y_1y_0$, where $x_{2i+1}x_{2i}\sim y_{2i+1}y_{2i}$ for $0\leqslant i\leqslant(n-4)/2$. Let $v_L=0\bar{x}_{n-2}y_{n-3}\cdots y_1y_0$, and $v_R=1\bar{x}_{n-2}x_{n-3}\cdots x_1x_0$, then (u_L, v_L) and (u_R, v_R) are critical edges and $\langle u_L, v_L, v_R, u_R, u_L\rangle$ is a cycle of length four in CQ_n containing e. And

$$\langle 0x_{n-2}x_{n-3}\cdots x_1x_0,\ 0x_{n-2}x_{n-3}\cdots\bar{x}_1x_0,\ 0x_{n-2}x_{n-3}\cdots\bar{x}_1\bar{x}_0,$$
$$1x_{n-2}y_{n-3}y_{n-4}\cdots y_1\bar{y}_0,\ 1x_{n-2}y_{n-3}\cdots y_1y_0,\ 0x_{n-2}x_{n-3}\cdots x_1x_0\rangle.$$

is a cycle of length five in CQ_n containing e.

If n is odd, we may assume $u_L=0x_{n-2}x_{n-3}\cdots x_1x_0$, then $u_R=1y_{n-2}y_{n-3}\cdots y_1y_0$, where $x_{2i+1}x_{2i}\sim y_{2i+1}y_{2i}$ for $0\leqslant i\leqslant(n-3)/2$. Let $v_L=0\bar{x}_{n-2}x_{n-3}y_{n-4}\cdots y_1y_0$ and $v_R=1\bar{y}_{n-2}y_{n-3}x_{n-4}\cdots x_1x_0$, then (u_L, v_L) and (u_R, v_R) are critical edges and $\langle u_L, v_L, v_R, u_R, u_L\rangle$ is a cycle of length four in CQ_n containing e. And

$$\langle 0x_{n-2}x_{n-3}\cdots x_1x_0,\ 0x_{n-2}x_{n-3}\cdots\bar{x}_1x_0,\ 0x_{n-2}x_{n-3}\cdots\bar{x}_1\bar{x}_0,$$
$$1y_{n-2}y_{n-3}y_{n-4}\cdots y_1\bar{y}_0,\ 1y_{n-2}y_{n-3}\cdots y_1y_0,\ 0x_{n-2}x_{n-3}\cdots x_1x_0\rangle.$$

is a cycle of length five in CQ_n containing e.

For $l \geqslant 6$, it does not matter whether n is even or not. We can write $l = l_1 + l_2$ where $l_1 = 2, l_2 \geqslant 4$ or $l_1 \geqslant 4$, $l_2 \geqslant 4$. Consider the cycle $\langle u_L, v_L, v_R, u_R, u_L \rangle$ of length four in CQ_n containing e. By the induction hypothesis, there exists a cycle C of length l_1 in L containing (u_L, v_L) if $l_1 \geqslant 4$ and a cycle C' of length l_2 in R containing (u_R, v_R). Let $P = (u_L, v_L)$ if $l_1 = 2$ or $P = C - (u_L, v_L)$ if $l_1 \geqslant 4$; $P' = C' - (v_R, u_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length l in CQ_n and contains e.

By the induction principle, the theorem follows.

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交叉超立方体网络的边泛圈性

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摘要: 作为超立方体 Q_n 的变型,在点数和边数都相同的情况下,交叉超立方体 CQ_n 有比超立方体更好的性质. 在已获证明的 CQ_n 包含所有长度(从 4 到 2^n)的圈的基础上,进一步改进了这一结果,证明了 CQ_n 中每条边落在所有长度(从 4 到 2^n)的圈中.

关键词: 圈;交叉超立方体;超立方体;泛圈;边一泛圈性