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The super connectivity of shuffle-cubes ☆

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Abstract

The shuffle-cube SQ_n, where $n \equiv 2 \pmod 4$, a new variation of hypercubes proposed by Li et al. [T.-K. Li, J.J.M. Tan, L.-H. Hsu, T.-Y. Sung, The shuffle-cubes and their generalization, Inform. Process. Lett. 77 (2001) 35–41], is an n-regular n-connected graph. This paper determines that the super connectivity of SQ_n is $2n-4$ and the super edge-connectivity is $2n-2$ for $n \geqslant 6$.

Keywords: Combinatorial problems; Shuffle-cubes; Super connectivity; Super edge-connectivity; Hypercubes

1. Introduction

The shuffle-cube, denoted by SQ_n, where $n \equiv 2 \pmod 4$, as an interconnection network topology proposed by Li et al. [4], is a new variation of hypercubes Q_n obtained by changing some links. For an n-bit binary string $u = u_{n-1}u_{n-2}\ldots u_1u_0 \in V(SQ_n)$, let $p_j(u) = u_{n-1}u_{n-2}\ldots u_{n-j}$ and $s_i(u) = u_{i-1}u_{i-2}\ldots u_1u_0$. The *$n$-dimensional shuffle-cube* SQ_n, $n \equiv 2 \pmod 4$, is recursively defined as follows: SQ_2 is Q_2. For $n \geqslant 3$, SQ_n consists of 16 subcube $SQ_{n-4}^{i_1i_2i_3i_4}$'s, where $i_j \in \{0,1\}$ for $1 \leqslant j \leqslant 4$ and $p_4(u) = i_1i_2i_3i_4$ for all vertices u in $SQ_{n-4}^{i_1i_2i_3i_4}$. The vertices $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ in different $(n-4)$-dimensional subcubes are linked by an edge in SQ_n if and only if $s_{n-4}(u) = s_{n-4}(v)$ and $p_4(u) \oplus p_4(v) \in V_{s_2(u)}$, where the symbol $\oplus$ denotes the addition with modulo 2 and

$$V_{00} = \{1111, 0001, 0010, 0011\},$$

$$V_{01} = \{0100, 0101, 0110, 0111\},$$

$$V_{10} = \{1000, 1001, 1010, 1011\},$$

$$V_{11} = \{1100, 1101, 1110, 1111\}.$$

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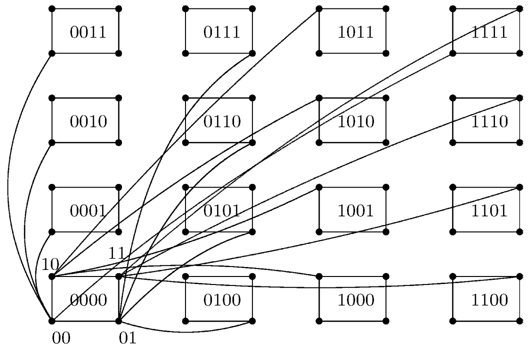

Fig. 1. A shuffle-cube SQ_6.

We illustrate SQ_6 in Fig. 1 showing only edges incident at vertices in SQ_2^{0000} and omitting others.

It is convenient to let $n = 4k + 2$ and $u = u_{n-1}u_{n-2} \ldots u_1u_0 = u_4^k u_4^{k-1} \ldots u_4^1 u_4^0$, where $u_4^0 = u_1u_0$ and $u_4^j = u_{4j+1}u_{4j}u_{4j-1}u_{4j-2}$ for $1 \leqslant j \leqslant k$. Then two vertices u and v in SQ_n are linked by an edge if and only if one of the following conditions holds:

(1) $u_4^{j^*} \oplus v_4^{j^*} \in V_{u_4^0}$ for exactly one j^* satisfying $1 \leqslant j^* \leqslant k$ and $u_4^j = v_4^j$ for all $0 \leqslant j \neq j^* \leqslant k$.
(2) $u_4^0 \oplus v_4^0 \in \{01, 10\}$ and $u_4^j = v_4^j$ for all $1 \leqslant j \leqslant k$.

It has been shown that SQ_n is n-regular n-connected in [4]. In this paper, we further discuss its super connectivity, a more refined parameter than the connectivity for measuring the reliability and the fault tolerance of a network [2,3].

Let $G = (V, E)$ be a graph. A subset $S \subset V$ (respectively $F \subset E$) is called a *super vertex-cut* (respectively *super edge-cut*) if $G - S$ (respectively $G - F$) is not connected and every component contains at least two vertices. The *super connectivity* $\kappa'(G)$ (respectively *super edge-connectivity* $\lambda'(G)$) is the minimum cardinality over all super vertex-cuts (respectively super edge-cuts) in G if they exist.

In [2], Esfahanian proved that $\kappa'(Q_n) = \lambda'(Q_n) = 2n - 2$ for $n \geqslant 3$. In this paper, we prove that $\kappa'(SQ_n) = 2n - 4$ and $\lambda'(SQ_n) = 2n - 2$, where $n \equiv 2 \pmod 4$ and $n \geqslant 6$.

2. Some lemmas

We follow [1] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a simple connected graph. For $x \in V(G)$, let $N_G(x)$ be the set of neighbors of x and $d_G(x) = |N_G(x)|$, the degree of x. For $xy \in E(G)$, let $N_G(xy) = N_G(x) \cup N_G(y) \setminus \{x, y\}$, and let

$$\zeta_G(xy) = \left|N_G(xy)\right|,$$
$$\zeta(G) = \min\left\{\zeta_G(xy)\colon\ xy \in E(G)\right\};$$
$$\xi_G(xy) = d_G(x) + d_G(y) - 2,$$
$$\xi(G) = \min\left\{\xi_G(xy)\colon\ xy \in E(G)\right\}.$$

Lemma 1 [3]. *$\lambda'(G) \leqslant \xi(G)$ for any graph G with order at least four and not a star.*

Lemma 2. *$\zeta(SQ_n) = 2n - 4$, and the edge uv which attains this value is only if $u = u_4^k \ldots u_4^1 u_4^0$ and $v = u_4^k \ldots u_4^{i+1}(u_4^i \oplus e)u_4^{i-1} \ldots u_4^0$, where $e \in \{0001, 0010, 0011\} \subseteq V_{00}$ and $u_4^0 = v_4^0 = 00$.*

Proof. Let uv be an edge in SQ_n with $u = u_4^k u_4^{k-1} \ldots u_4^1 u_4^0$ and $v = v_4^k v_4^{k-1} \ldots v_4^1 v_4^0$, where $u_4^0 = u_1u_0$. Then

$$v = \begin{cases} u_4^k \ldots u_4^{i+1}(u_4^i \oplus e_1)u_4^{i-1} \ldots u_4^1 u_4^0 \\ \quad \text{for } i \neq 0,\ e_1 \in V_{u_4^0}, \text{ or} \\ u_4^k u_4^{k-1} \ldots u_4^1 (u_4^0 \oplus e_2) \\ \quad \text{for } i = 0, e_2 \in \{01, 10\}. \end{cases}$$

For convenience, we denote $v = u_4^k \ldots u_4^{i+1}(u_4^i \oplus e_3)u_4^{i-1} \ldots u_4^1 u_4^0$ for the possible two cases.

If u and v have no neighbors in common, then $\zeta_{SQ_n}(uv) = 2n - 2 > 2n - 4$.

Suppose now that u and v have a neighbor w in common. Since w is a neighbor of u, then

$$w = \begin{cases} u_4^k \ldots u_4^{j+1}(u_4^j \oplus e_1')u_4^{j-1} \ldots u_4^1 u_4^0 \\ \quad \text{for } j \neq 0, e_1' \in V_{u_4^0}, \text{ or} \\ u_4^k u_4^{k-1} \ldots u_4^1 (u_4^0 \oplus e_2') \\ \quad \text{for } j = 0, e_2' \in \{01, 10\}. \end{cases}$$

For convenience, we denote $w = u_4^k \ldots u_4^{j+1}(u_4^j \oplus e')u_4^{j-1} \ldots u_4^1 u_4^0$ for the possible two cases. Since w is a neighbor of v, then

$$w = \begin{cases} v_4^k \ldots v_4^{s+1}(v_4^s \oplus e_1'')v_4^{s-1} \ldots v_4^1 v_4^0 \\ \quad \text{for } s \neq 0, e_1'' \in V_{v_4^0}, \text{ or} \\ v_4^k v_4^{k-1} \ldots v_4^1 (v_4^0 \oplus e_2'') \\ \quad \text{for } s = 0, e_2'' \in \{01, 10\}. \end{cases}$$

We denote $w = v_4^k \dots v_4^{s+1}(v_4^s \oplus e'')v_4^{s-1} \dots v_4^1 v_4^0$ for the possible two cases. Then $i = j = s$, $u_4^i \oplus e' = (u_4^i \oplus e_3) \oplus e'' = u_4^i \oplus (e_3 \oplus e'')$. If $i = 0$, then $e' = e_3 \oplus e''$ does not holds for $\{e_3, e', e''\} \subseteq \{01, 10\}$. If $i \neq 0$, then $e' = e_3 \oplus e''$ holds only for $u_4^0 = 00$ and $\{e_3, e', e''\} = \{0001, 0010, 0011\}$. The two vertices $u_4^k \dots u_4^{i+1}(u_4^i \oplus e')u_4^{i-1} \dots u_4^1 u_4^0$ and $u_4^k \dots u_4^{i+1}(u_4^i \oplus e'')u_4^{i-1} \dots u_4^1 u_4^0$ are all neighbors of both u and v. So $\zeta_{SQ_n}(uv) = 2n - 4$ and the lemma follows by the arbitrary choice of the edge uv. □

Lemma 3. $\kappa'(SQ_n) \leqslant 2n - 4$ *for* $n \geqslant 6$.

Proof. By Lemma 2, let uv be an edge of SQ_n such that $\zeta_{SQ_n}(uv) = 2n - 4$, where $u = u_4^k \dots u_4^1 u_4^0$ and $v = u_4^k \dots u_4^{i+1}(u_4^i \oplus e)u_4^{i-1} \dots u_4^0$, $e \in V_{00}$ and $u_4^0 = v_4^0 = 00$. We prove that $N_{SQ_n}(uv)$ is a super vertex-cut, which means $\kappa'(SQ_n) \leqslant \zeta_{SQ_n}(uv) = 2n - 4$. To the end, we need to prove that $SQ_n - (N_{SQ_n}(uv) \cup \{u, v\})$ has no isolated vertices.

Suppose that $w = w_4^k \dots w_4^1 w_4^0$ is a vertex in $SQ_n - (N_{SQ_n}(uv) \cup \{u, v\})$. We now prove $N_{SQ_n}(w) \nsubseteq N_{SQ_n}(uv)$, where $w_4^0 \in \{00, 01, 10, 11\}$.

If $w_4^0 = 00$, since $w \notin N_{SQ_n}(uv) \cup \{u, v\}$, then the vertex $w' = w_4^k \dots w_4^1(w_4^0 \oplus 01)$ is a neighbor of w and $w' \notin N_{SQ_n}(uv)$.

If $w_4^0 \in \{01, 10\}$, the vertex $w' = w_4^k \dots w_4^1 11$ is a neighbor of w and $w' \notin N_{SQ_n}(uv)$.

If $w_4^0 = 11$, the vertex $w' = w_4^k \dots w_4^{i+1}(w_4^i \oplus e')w_4^{i-1} \dots w_4^1 w_4^0$ is a neighbor of w and $w' \notin N_{SQ_n}(uv)$ where $e' \in V_{11}$.

Owning to the above discussion, we have $N_{SQ_n}(w) \nsubseteq N_{SQ_n}(uv)$ for $n \geqslant 6$. □

Lemma 4. *Let* $n = 4k + 2$, $k \geqslant 1$. *Then the number of non-adjacent edges between any two distinct* $(4k - 2)$*-subcubes is at least* 2^{4k-4}.

Proof. Let $SQ_{4k-2}^{i_1 i_2 i_3 i_4}$ and $SQ_{4k-2}^{j_1 j_2 j_3 j_4}$ be two distinct $(4k - 2)$-subcubes in SQ_{4k+2}. Obviously, the edges between $SQ_{4k-2}^{i_1 i_2 i_3 i_4}$ and $SQ_{4k-2}^{j_1 j_2 j_3 j_4}$ are non-adjacent since $i_1 i_2 i_3 i_4 \neq j_1 j_2 j_3 j_4$. By the definition of SQ_{4k+2}, every edge uv between $SQ_{4k-2}^{i_1 i_2 i_3 i_4}$ and $SQ_{4k-2}^{j_1 j_2 j_3 j_4}$ satisfies $s_{4k-2}(u) = s_{4k-2}(v)$, and $p_4(u) \oplus p_4(v) = i_1 i_2 i_3 i_4 \oplus j_1 j_2 j_3 j_4 \in V_{s_2(u)}$. Therefore, $s_2(u)$ is determined. Then the number of non-adjacent edges between $SQ_{4k-2}^{i_1 i_2 i_3 i_4}$ and $SQ_{4k-2}^{j_1 j_2 j_3 j_4}$ is at least $2^{4k+2-4-2} = 2^{4k-4}$. □

3. Main results

Theorem 1. $\kappa'(SQ_n) = 2n - 4$, *where* $n = 4k + 2$ *and* $k \geqslant 1$.

Proof. By Lemma 3, we only need to prove $\kappa'(SQ_n) \geqslant 2n - 4$. To the end, let F be an arbitrary set of vertices in SQ_n such that $|F| \leqslant 2n - 5$ and $SQ_n - F$ has no isolated vertices. We prove that $SQ_n - F$ is connected. By definition, SQ_n consists of 16 subcube SQ_{n-4}'s. We partition 16 subcube SQ_{n-4}'s of SQ_n into two subsets S_1 and S_2, where $S_1 = \{SQ_{n-4} \mid SQ_{n-4}$ contain at least $n - 4$ vertices in $F\}$, $S_2 = \{SQ_{n-4} \mid SQ_{n-4}$ contain at most $n - 5$ vertices in $F\}$. Then S_1 consists of at most three subcube SQ_{n-4}'s since $4(n - 4) > 2n - 5$ for $n \geqslant 6$, and so $S_2 \neq \emptyset$.

We prove that $SQ_n - F$ is connected from the following claims.

Claim 1. $S_2 - F$ *is connected.*

Proof. Let $n = 4k + 2$, $k \geqslant 1$. Since every $(4k - 2)$-subcube in S_2 is $(4k - 2)$-connected and contains at most $n - 5$ $(= 4k - 3)$ vertices in F, it is connected in $S_2 - F$.

If $k = 1$, then $n = 6$, $|F| \leqslant 7$ and every subcube SQ_2 in S_2 contains at most one vertex in F. We decompose S_2 into two subgraphs H_1 and H_2, where $H_1 = \{SQ_2 \mid V(SQ_2) \cap F \neq \emptyset\}$ and $H_2 = \{SQ_2 \mid V(SQ_2) \cap F = \emptyset\}$. Then H_2 contains at least 9 subcube SQ_2's. It is easy to observe that H_2 is connected. If $H_1 = \emptyset$, then the claim follows. Assume $H_1 \neq \emptyset$ below. Let $G' = SQ_2$ be a subcube in H_1. Since G' contains at most one vertex in F, $G' - F$ is connected and G' has at least 11 neighbors in other subcube SQ_2's, at least one of them is in H_2 since there are at most 7 subcubes in $H_1 \cup S_1$. Since H_2 is connected, each subcube in H_1 connects with H_2 in $SQ_6 - F$, and so $S_2 - F$ is connected.

If $k \geqslant 2$, let G_1 and G_2 be two arbitrary distinct $(4k - 2)$-subcubes in S_2. We only need to prove that G_1 connects with G_2 in $S_2 - F$. Let E_{12} be the set of non-adjacent edges between G_1 and G_2. Then $|E_{12}| \geqslant 2^{4k-4}$ by Lemma 4. Since $2^{4k-4} > 2(4k - 3) = 2(n -$

5) for $k \geqslant 2$, G_1 connects with G_2 in $S_2 - F$ for $k \geqslant 2$. □

If $S_1 = \emptyset$, then there is nothing to do by Claim 1. Assume $S_1 \neq \emptyset$ below.

Claim 2. *Let G_1 be a subcube SQ_{n-4} in S_1 and T a connected component with t vertices in $G_1 - F$. Then T connects with $S_2 - F$.*

Proof. If $t = 1$, since $SQ_n - F$ has no isolated vertices, the vertex u_1 in T connects with a vertex u_2 in $(SQ_n - G_1) - F$.

If $t \geqslant 2$, for any edge xy in T, we have $|V(T) \cap N_{G_1}(xy)| \leqslant t - 2$, and $N_{G_1}(xy) \geqslant 2(n-4) - 4$ by Lemma 2. Since every vertex in G_1 has the same kth 4-bit, different vertices in G_1 have different neighbors in $SQ_n - G_1$. So T has $4t$ neighbors in $SQ_n - G_1$. Thus, T has at least $2(n-4) - 4 - (t-2) + 4t$ neighbors in $SQ_n - T$. Since $2(n-4) - 4 - (t-2) + 4t = 2n + 3t - 10 > 2n - 5 \geqslant |F|$ and T is a component of $G_1 - F$, there exist a vertex u_1 in T and a vertex u_2 in $(SQ_n - G_1) - F$ such that $u_1u_2 \in E(SQ_n - F)$.

Let G_2 be a subcube SQ_{n-4} that contains u_2. If $G_2 \in S_2$, then the claim follows. Assume $G_2 \in S_1$ below. Since each of G_1 and G_2 contains at least $n - 4$ vertices in F and $(2n-5) - 2(n-4) = 3$, F contains at most three vertices of $SQ_n - G_1 - G_2$. By Lemma 2, u_1 and u_2 have at most two neighbors in common. Note that each vertex in a subcube SQ_{n-4} has four neighbors in other subcubes. Thus, $\{u_1, u_2\}$ has at least four neighbors in $SQ_n - G_1 - G_2$, at least one of them, say u_3, is not in F and $|V(G_3) \cap F| \leqslant 1$, where G_3 is the $(n-4)$-subcube containing u_3. Since $|V(G_3) \cap F| \leqslant 1 \leqslant n - 5$, $G_3 \in S_2$, and so the claim follows.

By the above discussion, we prove that $SQ_n - F$ is connected, which means $\kappa'(SQ_n) \geqslant 2n - 4$ for $n \geqslant 6$. The theorem follows. □

Theorem 2. $\lambda'(SQ_n) = 2n - 2$, *where $n = 4k + 2$ and $k \geqslant 1$.*

Proof. By Lemma 1, we only need to prove $\lambda'(SQ_n) \geqslant 2n - 2$ for $n \geqslant 6$.

Let F be an arbitrary set of edges in SQ_n such that $|F| \leqslant 2n - 3$ and $SQ_n - F$ has no isolated vertices. We prove that $SQ_n - F$ is connected.

We partition 16 subcube SQ_{n-4}'s of SQ_n into two subsets S_1 and S_2, where $S_1 = \{SQ_{n-4} \mid SQ_{n-4}$ contain at least $n - 4$ edges in $F\}$, $S_2 = \{SQ_{n-4} \mid SQ_{n-4}$ contain at most $n - 5$ edges in $F\}$. Then S_1 consists of at most four subcube SQ_{n-4}'s since $5(n-4) > 2n - 3$ for $n \geqslant 6$, and so $S_2 \neq \emptyset$. We complete the proof by the following claims.

Claim 1. *$S_2 - F$ is connected.*

Proof. Let $n = 4k + 2$, $k \geqslant 1$. Since SQ_n is n-regular n-connected in [4], we conclude that SQ_n is n-edge-connected. Then every $(4k-2)$-subcube in S_2 is also connected in $S_2 - F$. Let G_1 and G_2 be two arbitrary distinct $(4k-2)$-subcubes in S_2. We only need to prove G_1 connects with G_2 in $S_2 - F$. Let B_{12} be a set of edges between G_1 and G_2. Then $|B_{12}| \geqslant 2^{4k-4}$. If $B_{12} \nsubseteq F$, then there is noting to do. Assume $B_{12} \subseteq F$ below.

Since S_1 consists of at most four $(4k-2)$-subcubes, $|S_2| \geqslant 12$. For each of other 10 subcubes in $S_2 - F$, if at most one of subcube in $\{G_1, G_2\}$ connects with it, then

$$|F| \geqslant (10+1) \cdot |B_{12}| \geqslant 11 \cdot 2^{4k-4} > |F|,$$

a contradiction. Thus, there exists a $(4k-2)$-subcube, say G_3, such that G_3 connects each of G_1 and G_2 in $S_2 - F$. This implies G_1 and G_2 are connected in $S_2 - F$. □

If $S_1 = \emptyset$, then there is nothing to do by Claim 1. Assume $S_1 \neq \emptyset$ below.

Claim 2. *Let G_1 be a subcube SQ_{n-4} in S_1 and T connected component with t vertices in $G_1 - F$. Then T connects with $S_2 - F$.*

Proof. If $t = 1$, since there is no isolated vertex in $SQ_n - F$, the vertex u_1 in T connects with a vertex u_2 in $(SQ_n - G_1) - F$.

If $t = 2$, there exist two vertices u_1 in T and u_2 in $SQ_n - G_1$ such that $u_1u_2 \notin F$ for $\xi(G) = 2n - 2$ and $|F| \leqslant 2n - 3$.

If $t \geqslant 3$, let xy be an edge in T and A the set of edges that are incident with x or y in G_1. Then $|A| = 2(n-4) - 2$ and $|E(T) \cap A| \leqslant t$. Since G_1 is a subcube SQ_{n-4} and every vertex in G_1 has the same kth 4-bit, different vertices in G_1 have different neighbors in $SQ_n - G_1$. So there are $4t$ other edges between T and $SQ_n - G_1$. Thus, there are at least $2(n-4) - 2 - t + 4t$ edges between T and $SQ_n - T$. Since $2(n-4) - 2 - t + 4t = 2n + 3t - 10 > 2n - 3 \geqslant |F|$ and T is a component of $G_1 - F$, there exist two vertices u_1 in T and u_2 in $SQ_n - G_1$ such that $u_1u_2 \notin F$.

Let G_2 be a subcube SQ_{n-4} that contains u_2. If $G_2 \in S_2$, the claim follows. Assume $G_2 \in S_1$ below. Since each of G_1 and G_2 contains at least $n-4$ edges in F and $(2n-3) - 2(n-4) = 5$, F contains at most five edges of $E(SQ_n - G_1 - G_2)$. By Lemma 2, u_1 and u_2 have at most two neighbors in common. Note that each vertex of 16 subcube SQ_{n-4}'s has only four neighbors in other subcubes (see Fig. 1). Thus, $\{u_1, u_2\}$ connects with at least four neighbors in $SQ_n - G_1 - G_2$ by six edges, at least one of them, say u_3, satisfies $u_1u_3 \notin F$ or $u_2u_3 \notin F$ and $|E(G_3) \cap F| \leqslant 1$, where G_3 is the $(n-4)$-subcube containing u_3. Since $|E(G_3) \cap F| \leqslant 1 \leqslant n-5$, $G_3 \in S_2$, and so the claim follows.

By the above discussion, we prove that $SQ_n - F$ is connected, which means $\lambda'(SQ_n) \geqslant 2n - 2$ for $n \geqslant 6$, and so the theorem follows. □

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