Journal of Shanghai Jiaotong University (Science), Vol. E-10, No. 2, 2005, 203~207

Article ID: 1007-1172(2005)02-0203-05

On Restricted Connectivity and Extra Connectivity of Hypercubes and Folded Hypercubes

*XU Jun-ming** (徐俊明), *ZHU Qiang* (朱　强), *HOU Xin-min* (侯新民), *ZHOU Tao* (周　涛)

(Dept. of Mathematics, Univ. of Science and Technology of China, Hefei 230026, China)

Abstract: Given a graph G and a non-negative integer h, the h-restricted connectivity $\kappa^{h}(G)$ of G is the minimum cardinality of a set of vertices of G, in which at least h neighbors of any vertex is not included, if any, whose deletion disconnects G and every remaining component has the minimum degree of vertex at least h; and the h-extra connectivity $\kappa_h(G)$ of G is the minimum cardinality of a set of vertices of G, if any, whose deletion disconnects G and every remaining component has order more than h. This paper shows that for the hypercube Q_n and the folded hypercube FQ_n, $\kappa_1(Q_n)=\kappa^{(1)}(Q_n)=2n-2$ for $n\geqslant 3$, $\kappa_2(Q_n)=3n-5$ for $n\geqslant 4$, $\kappa_1(FQ_n)=\kappa^{(1)}(FQ_n)=2n$ for $n\geqslant 4$ and $\kappa^{(2)}(FQ_n)=4n-4$ for $n\geqslant 8$.

Key words: connectivity; conditional connectivity; restricted connectivity; extra connectivity; hypercube; folded hypercube

Document code: A

Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a graph G, the classical connectivity $\kappa(G)$ of G, defined as the minimum cardinality $|S|$ of a vertex-cut S, has been used as a deterministic measure of reliability and fault-tolerance of the network. In this paper, we consider other two kinds of connectivies. Given a graph G and a non-negative integer h, the h-extra connectivity $\kappa_h(G)$ of G is the minimum cardinality of a set of vertices of G, if any, whose deletion disconnects G and every remaining component has order more than h[1]; and the h-restricted connectivity $\kappa^{(h)}(G)$ of G is the minimum cardinality of a set of vertices of G, in which at least h neighbors of any vertex is not included, if any, whose deletion disconnects G and every remaining component has the minimum degree of vertex at least h[2~4]. The two concepts are generalizations of the classical connectivity and can provide more accurate measures for fault tolerance of a large-scale parallel processing system.

Fàbrega and Fiol[1] showed $\kappa_h(G)\leqslant(h+1)n-2h$ for an n-regular graph G if $\kappa_h(G)$ exists. Latifi *et al*[4] determined $\kappa^{(h)}(Q_n)=(n-h)2^h$ for $0\leqslant h\leqslant\lfloor\frac{n}{2}\rfloor$, where Q_n is the n-dimensional hypercube. Wu and Guo[5] generalized Latifi *et al*'s result to the m-ary n-dimensional generalized hypercube. However, for any n-regular graph G and any integer $h\leqslant n$, we have not yet known whether $\kappa_h(G)$ or $\kappa^{(h)}(G)$ exists or not.

We are, in this paper, interested in the hypercube Q_n and the folded hypercube FQ_n, which have been widely used in design and analysis of interconnection networks[6]. It is known that $\kappa(Q_n)=n$ and $\kappa(FQ_n)=n+1$. We determine

$$\begin{aligned}
&\kappa_1(Q_n)=\kappa^{(1)}(Q_n)=2n-2 && n\geqslant 3\\
&\kappa_2(Q_n)=3n-5 && n\geqslant 4\\
&\kappa_1(FQ_n)=\kappa^{(1)}(FQ_n)=2n && n\geqslant 4\\
&\kappa^{(2)}(FQ_n)=4n-4 && n\geqslant 8
\end{aligned}$$

The proofs of our results are given in Section 2 and Section 3, respectively.

We follow Ref. [7] for graph-theoretical ter-

Received date: 2004-06-23

Foundation item: National Natural Science Foundation of China (No. 10271114)

* **E-mail**: xujm@ustc.edu.cn

minology and notation not defined here. For a graph $G=(V,E)$ and $S\subset V(G)$ or $S\subset G$, let $N_G(S)=\{y\in V(G-S): xy\in E(G)$ for some $x\in S\}$ and $A_G(S)=N_G(S)\cup S$. A vertex-cut S of G is called an R_h-cut if every component of $G-S$ is not a tree of order at most h. If there exists an R_h-cut in G, then $\kappa_h(G)=\min\{|S| : S$ is an R_h-cut of $G\}$. A subset $S\subset V(G)$ is called an $R^{(h)}$-set, if at least h neighbors of any vertex is not included in S; and an $R^{(h)}$-set S is called an $R^{(h)}$-cut if $G-S$ is disconnected and every remaining component has the minimum degree of vertex at least h. If there exists an $R^{(h)}$-cut S in G, then $\kappa^{(h)}(G)=\min\{|S| : S$ is an $R^{(h)}$-cut of $G\}$.

1 Results on Hypercube

An n-dimensional hypercube Q_n, also called an n-cube, is an undirected graph with vertices 2^n, each labeled with a distinct binary sequences $x_1 x_2 \cdots x_n$. Two vertices are linked by an edge if and only if their label sequences differ in exactly one coordinate. The hypercube Q_n has been widely used in network design since it possesses many attractive properties, such as n-regular n-connected and vertex-transitive (see a new book by Xu[8] for details).

Following Esfahanian[2], we express Q_n as $Q_n=L\odot R$, where L and R are the two $(n-1)$-subcubes of Q_n induced by the vertices with the leftmost coordinate 0 and 1, respectively. We call edges between L and R cross edges, and use u_l and u_r to denote two vertices in L and R, respectively, linked by the cross edge u_lu_r in Q_n, where $u_l=0\,u_2\,u_3\cdots u_n$ and $u_r=1\,u_2\,u_3\cdots u_n$.

Theorem 1 $\kappa_1(Q_n)=\kappa^{(1)}(Q_n)=2n-2, n\geqslant 3$.

Proof Let uv be an arbitrary edge in Q_n and $S=N_{Q_n}(uv)$. Then $|S|=2n-2$ since Q_n contains no triangle, and Q_n-S is disconnected since

$$|V(Q_n-A_{Q_n}(S))| = 2^n-2n>2 \text{ for } n\geqslant 3$$

Since every vertex in S is adjacent with u or v and any two distinct vertices have common neighbors at most two, S is an $R^{(1)}$-cut of Q_n and so

$$\kappa_1(Q_n)\leqslant \kappa^{(1)}(Q_n)\leqslant |S| = 2n-2 \text{ for } n\geqslant 3$$

In order to complete the proof of the theorem, we need to prove $\kappa_1(Q_n)\geqslant |S|=2n-2$. To the end, we only need to show that for any $F\subset V(Q_n)$ with $|F|\leqslant 2n-3$, if Q_n-F has no isolated vertex, then Q_n-F is connected.

Let $Q_n=L\odot R$, $F_l=F\cap L$, and $F_r=F\cap R$. Obviously, $F_l\cap F_r=\varnothing$. Thus, either $|F_l|\leqslant n-2$ or $|F_r|\leqslant n-2$. We can, without loss of generality, suppose that $|F_r|\leqslant n-2$. Then $R-F_r$ is connected since $\kappa(R)=\kappa(Q_{n-1})=n-1$. We show that any vertex u_l in $L-F_l$ can be connected to the connected graph $R-F_r$. Let u_lu_r be the cross edge in $Q_n=L\odot R$. If $u_r\notin F_r$, then we are done. So we assume that $u_r\in F_r$. Since there is no isolated vertex in Q_n-F, there exist a vertex v_l adjacent to u_l in $L-F_l$. Let v_lv_r be the cross edge in $Q_n=L\odot R$. If $v_r\notin F_r$, then we are done. So we suppose that $v_r\in F_r$, and let $X=N_L(u_lv_l)$ and $F'=F-\{u_r,v_r\}$. Then $|X|=2n-4$ and $|F'|\leqslant 2n-5$. Thus, there is a vertex $x_l\in X$, such that x_l and x_r are not in F. This implies that u_l in $L-F_l$ can be connected to $R-F_r$ via a path passing through $\{u_l, v_l, x_l, x_r\}$. The theorem follows. □

Lemma 1 (Esfahanian[2]) Let x be an arbitrary vertex in Q_n. Then $Q_n-A_{Q_n}(x)$ is connected.

Lemma 2 Any path P of length two in Q_n has exactly $3n-5$ neighbors in Q_n-P, which form an R_2-cut of Q_n for $n\geqslant 4$.

Proof Let $P=(x, y, z)$ be a path of length two, and, without loss of generality, suppose that $x=x_1\,x_2\,x_3\cdots x_n$, $y=\bar{x}_1\,x_2\,x_3\cdots x_n$ and $z=\bar{x}_1\,\bar{x}_2\,x_3\cdots x_n$, where $\bar{x}_i=\{0, 1\}\setminus\{x_i\}$. Since Q_n contains no triangle and x and z have exactly two neighbors y and $u=x_1\,\bar{x}_2\,x_3\cdots x_n$ in common, the number of neighbors of $\{x, y, z\}$ in Q_n-P is equal to $3n-5$.

Let $S=A_{Q_n}(P)$ and $H=Q_n-S$. Then

$$|V(H)| = 2^n-(3n-2)>2 \text{ for } n\geqslant 4$$

To prove the latter assertion, it is sufficient to prove that H is connected. In fact, let $Q_n=L\odot R$, and suppose, without loss of generality, that R contains one vertex x in $\{x,y,z\}$. Then by Lemma 1, $R-A_R(x)$ is connected. Since $A_R(x)\subset S$ and every vertex in $L\cap V(H)$ can be connected to $R\cap V(H)$, H is connected. □

Latifi et al[4] have determined

$$\kappa^2(Q_n) = 4n - 8 \text{ for } n \geqslant 4$$

The following theorem determines $\kappa_2(Q_n)$.

Theorem 2 $\kappa_2(Q_n)=3n-5, n\geqslant 4$.

Proof Clearly, $\kappa_2(Q_n)\leqslant 3n-5$ by Lemma 2. We need to prove that $\kappa_2(Q_n)\geqslant 3n-5$. We can easily verify that $\kappa_2(Q_4)\geqslant 7$. Suppose $n\geqslant 5$ below. It is sufficient to show that for any $F\subset V(Q_n)$ with $|F|\leqslant 3n-6$, if Q_n-F contains neither isolated vertex nor isolated edge, then Q_n-F is connected.

Let $Q_n = L\odot R$, $F_l = F\cap V(L)$ and $F_r = F\cap V(R)$. Without loss of generality, we may assume that $|F_l|\leqslant |F_r|$. Then, $|F_l|\leqslant \frac{1}{2}(3n-6)< 2n-5$ for $n\geqslant 5$. We prove that Q_n-F is connected by considering two cases respectively.

Case 1 $L-F_l$ contains no isolated vertex.

By Theorem 1, we have

$$\kappa_1(L) = \kappa_1(Q_{n-1}) = 2(n-1) - 2 = 2n - 4 > 2n - 5$$

Thus, $L-F_l$ is connected. Let u_r be any vertex in $R-F_r$. In order to prove that Q_n-F is connected, we only need to prove that u_r can be connected to $L-F_l$.

Since Q_n-F contains neither isolated vertex nor isolated edge, there exists a path P of length two containing u_r. If $V(P)\cap V(L)\neq\varnothing$, then u_r can be connected to $L-F_l$ via P. Suppose P is in R below and let $N=N_{Q_n}(P)$. Since $|N|\geqslant 3n-5>3n-6$, there exists at least one vertex v_r in N such that the cross edge v_lv_r avoids F. Thus, u_r can be connected to $L-F_l$ via P and v_lv_r.

Case 2 $L-F_l$ contains an isolated vertex.

Suppose that u_l is an isolated vertex in $L-F_l$. We first show that $L-(F_l\cup\{u_l\})$ is connected. In fact, since any pair of two nonadjacent vertices in L can share at most two common neighbors and $|F_l|\leqslant 2n-5$, thus, u_l is the only isolated vertex in L, that is, $L-F_l\cup\{u_l\}$ contains no isolated vertex, which implies that if $L-F_l\cup\{u_l\}$ is disconnected, then $F_l\cup\{u_l\}$ is an R_1-cut of L. By Theorem 1, we can obtain a contradiction as follows:

$$2n-4 = \kappa_1(Q_{n-1}) = \kappa_1(L) \leqslant |F_l\cup\{u_l\}| \leqslant 2n-5$$

Now, in the same argument as Case 1, we can prove that any vertex u_r in $R-F_r$ can be connected to $L-F_l$.

Combining Case 1 with Case 2, we prove that $|F|\geqslant 3n-5$ for any R_2-cut F of Q_n, which implies

$$\kappa_2(Q_n) \geqslant 3n - 5 \text{ for } n \geqslant 4$$

2 Results on Folded Hypercube

The folded n-cube FQ_n, proposed by El-Amawy and Latifi[6], is the graph obtained from the hypercube Q_n by adding an edge between any two complementary vertices $x=(x_1,x_2,\cdots,x_n)$ and $\bar{x}=(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)$ (such edges called complementary edges).

Like Q_n, the folded hypercube FQ_n can be expressed as $FQ_n=L\odot R$, where L and R are the two $(n-1)$-subcubes of Q_n, induced by the vertices with the leftmost coordinate is 0 and 1, respectively. Between L and R, apart from the cross edges, there exists a complementary edge joining u_l and $\bar{u}_l\in R$ for any $u_l\in L$.

Lemma 3 FQ_n contains no triangle for $n\geqslant 3$, and any two nonadjacent vertices in FQ_n have common neighbors at most two for $n\geqslant 4$.

Proof Suppose that (x, y, z) is a triangle in FQ_n. If $x=x_1x_2\cdots x_n$, then $y=x_1\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_n$ and $z=x_1\cdots x_{j-1}\bar{x}_jx_{j+1}\cdots x_n$ with $i\neq j$ or $y=x_1\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_n$ and $z=\bar{x}_1\cdots\bar{x}_{h-1}x_h\bar{x}_{h+1}\cdots\bar{x}_n$. If y and z are defined by the former, then since $yz\in E(FQ_n)$ and $i\neq j$, we have $n=2$ and yz is a complementary edge; if y and z are defined by the latter, then since $xz\in E(FQ_n)$, we also have $n=2$ and and xz is a cross edge, a contradiction.

Suppose that two nonadjacent vertices x and y in FQ_n have a common neighbor z. Then the distance between x and y is two. Suppose $x=x_1x_2\cdots x_n$. Then $y=x_1\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_{j-1}\bar{x}_jx_{j+1}\cdots x_n$ or $y=\bar{x}_1\cdots\bar{x}_{h-1}x_h\bar{x}_{h+1}\cdots\bar{x}_n$. For $n\geqslant 4$, the former allows that x and y have exactly two common neighbors $z=x_1\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_n$ and $u=x_1\cdots x_{j-1}\bar{x}_jx_{j+1}\cdots x_n$, the latter allows that x and y have only one common neighbor $v=\bar{x}_1\bar{x}_2\cdots\bar{x}_n$. □

Theorem 3 $\kappa_1(FQ_n)=\kappa^{(1)}(FQ_n)=2n$, $n\geqslant 4$.

Proof Arbitrarily choose an edge uv in FQ_n, and let $S=N_{FQ_n}(uv)$. Clearly, FQ_n-S is disconnected and contains no isolated vertex for $n\geqslant 4$ by Lemma 3. Thus, $\kappa_1(FQ_n)\leqslant\kappa^{(1)}(FQ_n)\leqslant|S|=2n$.

To prove the theorem, it is sufficient to show $\kappa_1(FQ_n)\geqslant 2n$. To the end, we prove that for any $F\subset V(FQ_n)$ with $|F|\leqslant 2n-1$, if FQ_n-F contains no isolated vertex, then FQ_n-F is connected.

Let $FQ_n=L\otimes R$, and let $F_l=F\cap L$, and $F_r=F\cap R$. It is obvious that either $|F_l|\leqslant n-1$ or $|F_r|\leqslant n-1$, without loss of generality we can assume that $|F_r|\leqslant n-1$. We consider two cases.

Case 1 $R-F_r$ is connected. We can show that any vertex u_l in $L-F_l$ can be connected to $R-F_r$. In fact, if u_l is an isolated vertex in $L-F_l$, since FQ_n-F contains no isolated vertex, then u_l has at least one neighbor u_r or $\bar{u}_l$ in $R-F_r$, the result follows. If not, there exists a neighbor v_l of u_l in $L-F_l$. If one of $\{u_r,\bar{u}_l,v_r,\bar{v}_l\}$ is not in F_r, then we are done. Otherwise, $n\geqslant 5$ and let $N=N_L(u_lv_l)$. Then $|N|=2n-4$ and by Lemma 3 there must be a vertex $w_l\in N$ such that w_r or $\bar{w}_l$ is not in F_r, that is, u_l can be connected to the connected graph $R-F_r$ via w_l.

Case 2 $R-F_r$ is disconnected. In this case, $|F_r|=n-1$ since $\kappa(R)=\kappa(Q_{n-1})=n-1$. Then by Theorem 1, $R-F_r$ contains a unique isolated vertex v_r. By Lemma 1, $R-(F_r\cup\{v_r\})$ is connected. In the same argument as Case 1, we can prove FQ_n-F is connected. □

Theorem 4 $\kappa^{(2)}(FQ_n)=4n-4$ for $n\geqslant 8$.

Proof Let G_{00}, G_{01}, G_{10} and G_{11} be the four subgraphs of FQ_n induced by the vertices with the leftmost two coordinates 00, 01, 10 and 11, respectively. Then, it is clear that G_{00}, G_{01}, G_{10} and G_{11} all are isomorphic to Q_{n-2}, and the union of any two distinct subgraphs is isomorphic to Q_{n-1}.

We first show that $\kappa^{(2)}(FQ_n)\leqslant 4(n-1)$. Let $S=\{x_{00}, x_{01}, x_{10}, x_{11}\}$, where $x_{ij}=(iju_3u_4\cdots u_n)\in V(G_{ij})$ for $i,j\in\{0,1\}$. Clearly, the subgraph induced by S is a cycle of length four. Let $F=N_{FQ_n}(S)$ and $A_{FQ_n}(S)=S\cup F$. Consider a neighbor y_{00} of x_{00} in G_{00}. Since FQ_n contains no triangle for $n\geqslant 3$ by Lemma 3, y_{00} is not a neighbor of neither x_{01} nor x_{10}. Since two nonadjacent vertices in FQ_n have at most two common neighbors for $n\geqslant 4$ by Lemma 3, y_{00} is not a neighbor of x_{11}. These facts imply that F does not contain the neighbor-set of any vertex in FQ_n and $|F|=4(n-1)$. Hence FQ_n-F is disconnected since $FQ_n-A_{FQ_n}(S)$ has vertices $2^n-4n>2$ for $n\geqslant 7$.

To prove that F is an $R^{(2)}$-set of FQ_n, we only need to show that every vertex of $H=FQ_n-A_{FQ_n}(S)$ has degree at least two. Arbitrarily choose a vertex x in H. Because of the symmetry, without loss of generality, suppose that x is in G_{00}. Note that exactly three neighbors of x are in G_{01}, G_{10} and G_{11}, respectively, and x and x_{00} have common neighbors at most two. In other words, neighbors of x are in F at most five. Thus

$$|N_H(x)|\geqslant(n+1)-5\geqslant 2 \text{ for } n\geqslant 6$$

And so, F is an $R^{(2)}$-cut of FQ_n.

We now show that $\kappa^{(2)}(FQ_n)\geqslant 4n-4$. To the end, it is sufficient to prove that $|F|\geqslant 4n-4$ for any minimum vertex-cut F of FQ_n with minimum degree $\delta(FQ_n-F)\geqslant 2$ since $|S|\geqslant|F|$ for any $R^{(2)}$-cut S of FQ_n. Let $F_{ij}=V(G_{ij})\cap F$ for $i,j\in\{0,1\}$, and let H be a component of FQ_n-F. Because of the minimality of F, $N_{FQ_n}(H)=F$ and $|N_{FQ_n}(H)|=|F|$.

Case 1 H is contained in exactly one of $\{G_{ij}-F_{ij}: i,j\in\{0,1\}\}$, say $G_{11}-F_{11}$. From definition of FQ_n, every vertex u in G_{11} has exactly three neighbors $u_{00}\in V(G_{00})$, $u_{01}\in V(G_{01})$ and $u_{10}\in V(G_{10})$.

If $G_{11}-F_{11}$ contains an isolated vertex u, then $F'=(F\backslash N_{G_{11}}(u))\cup\{u_{00}, u_{01}, u_{10}\}$ is a vertex-cut of FQ_n with $\delta(FQ_n-F')\geqslant 2$, but

$$|F'|=|F|-|N_{G_{11}}(u)|+3<|F|$$

since

$$|N_{G_{11}}(u)|=n-2>3 \text{ for } n\geqslant 6$$

which contradicts the minimality of F.

If $G_{11}-F_{11}$ contains a vertex u of degree one, let v be a neighbor of u in G_{11}, then

$$F'=(F\backslash N_{G_{11}}(u))\cup\{v, u_{00}, u_{01}, u_{10}\}$$

is a vertex-cut of FQ_n with $\delta(FQ_n-F')\geqslant 2$, but

$$|F'|=|F|-|N_{G_{11}}(u)|+4<|F|$$

since

$$|N_{G_{11}}(u)| = n - 3 > 4 \text{ for } n \geqslant 8$$

which contradicts the minimality of F.

Thus, F_{11} is an $R^{(2)}$-cut of G_{11} and

$$|F_{11}| \geqslant \kappa^{(2)}(Q_{n-2}) = 4(n-2) - 8 = 4n - 16$$

Note that $|V(H)| \geqslant 4$ and $N_{FQ_n}(H)$ of H should be deleted to separate H from FQ_n. Thus

$$|F| \geqslant |N_{FQ_n}(H)| \geqslant |F_{11}| + 4 \times 3 \geqslant 4n - 4$$

Case 2 H is contained in exactly two of $\{G_{ij} - F_{ij} : i,j \in \{0, 1\}\}$. Without loss of generality, suppose that H is contained in

$$G_{00} \cup G_{10} - (F_{00} \cup F_{10})$$

Using the minimality of F and in the same argument as Case 1, we can show that $F_{00} \cup F_{10}$ is an $R^{(2)}$-cut of $G_{00} \cup G_{10}$, and so

$$|F_{00} \cup F_{10}| \geqslant \kappa^{(2)}(Q_{n-1}) = 4(n-1) - 8 = 4n - 12$$

We claim that H has at least 8 neighbors in $G_{01} \cup G_{11}$. If neighbors of any two distinct vertices of H are different, then we are done since $|V(H)| \geqslant 4$. If two vertices in H have common neighbors in $G_{00} \cup G_{10}$, then the two vertices must be the form $(00u)$ in G_{00} and $(10\bar{u})$ in G_{10}, the distance between which in H is $n-1$. This fact implies $|H| \geqslant n$. Thus, H has at least $n \geqslant 8$ neighbors in $G_{01} \cup G_{11}$ for $n \geqslant 6$. To separate H from FQ_n, $N_{FQ_n}(H)$ of H should be deleted. It follows that

$$|F| \geqslant |N_{FQ_n}(H)| \geqslant |F_{00} \cup F_{10}| + 8 \geqslant 4n - 4$$

Case 3 H is contained in exactly three of $\{G_{ij} - F_{ij} : i,j \in \{0, 1\}\}$. Without loss of generality, we suppose that H is contained in $G_{00} \cup G_{10} \cup G_{11}$. If each of $H \cap G_{00}$, $H \cap G_{10}$ and $H \cap G_{11}$ contains no isolated vertex, then F_{00}, F_{10} and F_{11} are R'-cuts of $H \cap G_{00}$, $H \cap G_{10}$ and $H \cap G_{11}$, respectively. Thus,

$$|F| \geqslant 3\kappa^{(1)}(Q_{n-2}) = 3(2n-6) = 6n - 18 \geqslant 4n - 4 \text{ for } n \geqslant 7$$

Without loss of generality, suppose that $H \cap G_{00}$ contains an isolated vertex. Then $G_{10} \cup G_{11} - F_{10} \cup F_{11}$ contains no isolated vertex, that is, $F_{10} \cup F_{11}$ is an R'-cut of $G_{10} \cup G_{11}$. Thus

$$|F_{00}| + |F_{10} \cup F_{11}| \geqslant \kappa(Q_{n-2}) + \kappa^{(1)}(Q_{n-1}) = (n-2) + (2n-4) = 3n - 6$$

Since H contains complementary edges, $|V(H)| \geqslant n+2$, and H has at least $n+2$ neighbors in G_{01}. It follows that

$$|F| \geqslant |F_{00}| + |F_{10} \cup F_{11}| + (n+2) \geqslant 4n - 4$$

Case 4 H is contained in all four of $\{G_{ij} - F_{ij} : i,j \in \{0, 1\}\}$. If there is one of $\{H \cap G_{ij} : i,j \in \{0, 1\}\}$, say $H \cap G_{11}$, is of order at least 2, then it has at least two neighbors in G_{10} and at least two neighbors in G_{01}, which implies that both $H \cap G_{10}$ and $H \cap G_{01}$ are of order at least 2, respectively. So, $|F| \geqslant 3 \cdot 2(n-2-1) = 6n - 18 \geqslant 4n - 4$ for $n \geqslant 7$, the result follows. Thus, each of $\{H \cap G_{ij} : i,j \in \{0,1\}\}$ is an isolated vertex, that is $H \cong C_4$ by Lemma 3, so $|F| \geqslant N_{FQ_n}(H) = 4n - 4$. □

References

[1] Fàbrega J, Fiol M A. Extraconnectivity of graphs with large girth[J]. *Discrete Math*, 1994, **127**:163~170.

[2] Esfahanian A H. Generalized measures of fault tolerance with applications to N-cube networks[J]. *IEEE Trans Computers*, 1989, **38**(11): 1586~1591.

[3] Esfahanian A H, Hakimi S L. On computer a conditional edge connectivity of graph [J]. *Information Processing Letters*, 1988, **27**: 195~199.

[4] Latifi S, Hegde M, Naraghi-Pour M. Conditional connectivity measures for large multiprocessor systems[J]. *IEEE Trans Computers*, 1994, **43**(2): 218~221.

[5] Wu J, Guo G. Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets [J]. *IEEE Trans Computers*, 1998, **47**(8): 888~893.

[6] El-Amawy A, Latifi S. Properties and performance of folded hypercubes[J]. *IEEE Trans Paral Distri Systems*, 1991, **2**(3): 31~42.

[7] Xu J-M. Theory and application of graphs[M]. Dordrecht/Boston/London: Kluwer Academic Publishers, 2003.

[8] Xu J-M. Toplogical structure and analysis of interconnection networks [M]. Dordrecht/Boston/London: Kluwer Academic Publishers, 2001.