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ON RESTRICTED ARC-CONNECTIVITY OF REGULAR DIGRAPHS

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Abstract. The restricted arc-connectivity λ' of a strongly connected digraph G is the minimum cardinality of an arc cut F in G such that every strongly connected component of $G-F$ contains at least two vertices. This paper shows that for a d-regular strongly connected digraph with order n and diameter $k \geq 4$, if λ' exists, then

$$\lambda'(G) \geq \min\left\{ \frac{(n-d^{k-1})(d-1)}{d^{k-1}+d-2},\ 2d-2 \right\}$$

As consequences, the restricted arc-connectivity of the de Bruijn and Kautz digraph and the generalized de Bruijn and Kautz digraph are determined.

1. INTRODUCTION

It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph or strongly connected digraph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity of G is an important measurement for fault-tolerance of the network. We are, in this paper, interested in the edge-failures instead of vertex-failures, that is, we consider the edge-connectivity $\lambda(G)$ as a measurement for fault-tolerance of G.

Suppose that all vertices are perfectly reliable and that all edges fail independently with the same probability p. The parameter

$$R(G,p) = 1 - \sum_{i=\lambda}^{\varepsilon} c_i p^i (1-p)^{\varepsilon-i} \tag{1}$$

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is an important measurement of global reliability of G, where $\varepsilon = |E(G)|$ and $\lambda = \lambda(G)$, and c_i is the number of edge-cuts of cardinality i in G. It has been proved by Ball [1] that the computation of $R(G,p)$ is NP-hard for a graph G in general. To minimize c_λ in (1), Bauer et al. [2] suggested to investigate a class of super edge-connected graphs. A (strongly) connected (di)graph G is said to be super edge-connected, if every minimum edge-cut isolates a vertex of G. Since then one has found that many well-known graphs are super edge-connected. In particular, Soneoka [10] showed that the de Bruijn digraph $B(d,n)$ is super edge-connected for any $d \geq 2$ and $n \geq 1$; Fàbrega and Fiol [6] proved that the Kautz digraph $K(d,n)$ is super edge-connected for any $d \geq 3$ and $n \geq 2$.

A very natural question is how many edges must be removed to disconnect the graph such that every connected component of the resulting graph contains no isolated vertex. To measure this type of edge-connectivity, Esfahanian and Hakimi [4, 5] introduced the concept of the restricted edge-connectivity of a graph. The restricted edge-connectivity of a graph G, $\lambda'(G)$, is defined to be the minimum number of edges whose deletion results in a disconnected graph such that each connected component has at least two vertices. They showed that if G is neither $K_{1,n}$ nor K_3, then $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G)$ is the minimum edge-degree of G. Clearly, if G is super edge-connected, then $\lambda(G) < \lambda'(G)$. Since then one has payed much attention to the concept and determined the restricted edge-connectivity for many well-known graphs (see, for example, [8, 9, 11-13]).

The concept of the restricted edge-connectivity is also valid for digraphs, in which we replace edges by arcs. Up to now, however, all known results deal with only undirected graphs. This paper shows that for a d-regular digraph with order n and diameter $k \geq 4$, if λ' exists, then

$$\lambda'(G) \geq \min\left\{ \frac{(n-d^{k-1})(d-1)}{d^{k-1}+d-2},\ 2d-2\right\}$$

As consequences, the restricted arc-connectivity of the de Bruijn and Kautz digraph and the generalized de Bruijn and Kautz digraph are determined.

2. SOME LEMMAS

We follow [3] or [14] for graph-theoretical terminology and notation not defined here. Let $G = (V, A)$ be a strongly connected digraph, where V is the set of vertices and A is the set of arcs, in which there are no two arcs with the same end-vertices have the same orientation, but loops are allowed here.

The restricted arc-connectivity of a digraph G, $\lambda'(G)$, is defined to be the minimum number of arcs whose deletion results in a disconnected (i.e., not strongly connected) digraph such that each strongly connected component has at least two vertices. An arc cut F of G is called an R-arc-cut if $|F| = \lambda'(G)$.

Let X and Y be two disjoint subsets of $V(G)$. We use (X, Y) to denote the set of arcs in G from X to Y and let $A_G^+(X) = (X, \overline{X})$ and $A_G^-(X) = (\overline{X}, X)$, where $\overline{X} = V(G) \setminus X$. A digraph G is said to be d-regular if the out-degree and in-degree of every vertex in G are equal to d. The following properties on a regular digraph are simple and useful, and the detail proofs can be found in Example 1.4.1 in [14].

Lemma 1. *A regular digraph is strongly connected if and only if it is connected. Moreover, if G is a regular digraph, then $|A_G^+(X)| = |A_G^-(X)|$ for any nonempty subset $X \subset V(G)$.*

The generalized de Bruijn digraphs $B_G(n, d)$ and the generalized Kautz digraphs $K_G(n, d)$ are two important classes of regular digraphs. They are widely used in design and analysis of interconnection networks. We first recall the definitions and basic properties of $B_G(n, d)$ and $K_G(n, d)$. Their vertex-sets are both $V = \{0, 1, \cdots, n-1\}$ and their arc-sets are, respectively,

$$A(B_G(n,d)) = \{(i,j): \ j \equiv id + r (\bmod n), r = 0, 1, \cdots, d-1\}, \ d \geq 2$$

$$A(K_G(n,d)) = \{(i,j): \ j \equiv -id - r (\bmod n), r = 1, 2, \cdots, d\}, \ d \geq 2.$$

From the definitions, $B_G(n, d)$ and $K_G(n, d)$ are both d-regular. It has been shown that the diameter of $B_G(n, d)$ is $\lceil \log_d n \rceil$; while the diameter of $K_G(n, d)$ is $\lceil \log_d n \rceil - 1$ if $n = d^p + d^{p-q}$ (where p is an integer and q is an odd integer less than or equal to p), and $\lceil \log_d n \rceil$ otherwise. These imply that if the diameters of $B_G(n, d)$ and $K_G(n, d)$ are k then $n > d^{k-1}$. The proofs of these results and the following lemma can be found in Section 3.2 and Section 3.3 in [12].

Lemma 2. *If their diameters are not less than four, then the connectivity of $B_G(n, d)$ is $(d-1)$ and the connectivity of $K_G(n, d)$ is at least $(d-1)$. Moreover, if its diameter is not less than five, then the connectivity of $K_G(n, d)$ is d if and only if g.c.d $(n, d) \geq 2$ and n is divisible by $(d+1)$.*

In particular, for any positive integer k and $d \geq 2$, $B_G(d^k, d)$ and $K_G(d^k + d^{k-1}, d)$ are the well-known de Bruijn digraph $B(d, k)$ and the Kautz digraph $K(d, k)$, respectively. Their diameters are k. The connectivity of $B(d, k)$ is $d-1$, while the connectivity of $K(d, k)$ is d.

Lemma 3. *Every $B_G(n, d)$ or $K_G(n, d)$ contains a pair of symmetric arcs.*

Proof. From the definition of $K_G(n, d)$, it is clear that there is a pair of symmetric arcs between two vertices 0 and $n-1$ in any $K_G(n, d)$.

From the definition of $B_G(n, d)$, it contains a pair of symmetric arcs between two vertices i and j if and only if i and j satisfy the congruence equations

$$\begin{cases} j \equiv id + r_1 (\bmod n) \\ i \equiv jd + r_2 (\bmod n), \end{cases} \qquad r_1, r_2 \in \{0, 1, \cdots, d-1\},$$

that is, i must satisfy the congruence equation

$$i \equiv id^2 + r_1 d + r_2 (\mathrm{mod}\, n), \quad r_1, r_2 \in \{0, 1, \cdots, d-1\}. \tag{2}$$

Since $n = \lfloor n/(d^2-1) \rfloor (d^2-1) + u$ for some u with $0 \le u < d^2-1$, there exist $a, b \in \{0, 1, \cdots, d-1\}$ so that $u = ad + b$ and, thus, $i = \lfloor n/(d^2-1) \rfloor$ is a solution of the equation (2). Therefore, $B_G(n,d)$ contains a pair of symmetric arcs between two vertices i and $j = id + a$. ∎

2. MAIN RESULTS

We use the symbol $G(n,d,k)$ to denote a d-regular connected digraph G with n vertices, diameter k, and no loops at the end-vertices of any pair of symmetric arcs. If $n \le 3$, then $\lambda'(G)$ does not exist clearly. If $d = 1$, then G is a directed cycle C_n, so $\lambda'(G)$ does not exist. If $k = 1$, then G is a complete digraph K_{d+1}, so, $\lambda'(G)$ does not exist for $d \le 2$; $\lambda'(G) = 2d-2$ for $d \ge 3$. In the following discussion, we assume $n \ge 4$, $d \ge 2$ and $k \ge 2$.

Theorem 1. *For a connected digraph $G = G(n,d,k)$, if $\lambda'(G)$ exists, then*

$$\lambda'(G) \ge \begin{cases} \min\left\{ \dfrac{(n-d^{k-1})(d-1)}{d^{k-1}+d-2},\ 2d-2 \right\} & \text{for } k \ne 3; \\[2ex] \min\left\{ \dfrac{n}{2d+2},\ 2d-2 \right\} & \text{for } k = 3. \end{cases}$$

Proof. Since G is d-regular and connected, by Lemma 1, G is strongly connected. To prove the theorem, it is sufficient to show that if $\lambda' = \lambda'(G) < 2d-2$ then

$$n \le \begin{cases} \lambda' \, \dfrac{d^{k-1}+d-2}{d-1} + d^{k-1} & \text{for } k \ne 3; \\[2ex] 2\,\lambda'\,(d+1) & \text{for } k = 3. \end{cases} \tag{3}$$

To the end, let F be an R-arc-cut of G such that $|F| = \lambda'$. Then, $V(G)$ can be partitioned into two disjoint nonempty sets X and Y such that $F = (X, Y)$. Let X_0 and Y_0 be the sets of the initial and terminal vertices of the arcs of F, respectively. Let

$$d_G(x, X_0) = \min\{d_G(x,u) :\ u \in X_0\}, \qquad m = \max\{d_G(x, X_0) :\ x \in X\};$$
$$d_G(Y_0, y) = \min\{d_G(v,y) :\ v \in Y_0\}, \qquad m' = \max\{d_G(Y_0, y) :\ y \in Y\},$$

where $d_G(u,v)$ denotes the distance from u to v in G. For any $x_0 \in X_0$ and $y_0 \in Y_0$, let

$$X_\ell^-(x_0) = \{x \in X : \; d_G(x,x_0) = \ell\}, \quad 0 \le \ell \le m;$$
$$Y_{\ell'}^+(y_0) = \{y \in Y : \; d_G(y_0,y) = \ell'\}, \quad 0 \le \ell' \le m'.$$

Noting that $|X_0| \le |F|$ and $|Y_0| \le |F|$, we have that

$$\begin{aligned} |X| &\le \sum_{x_0 \in X_0} \sum_{\ell=0}^{m} |X_\ell^-(x_0)| &\le |F|(1+d+d^2+\cdots+d^m); \\ |Y| &\le \sum_{y_0 \in Y_0} \sum_{\ell'=0}^{m'} |Y_{\ell'}^+(y_0)| &\le |F|(1+d+d^2+\cdots+d^{m'}). \end{aligned} \tag{4}$$

We now consider the relationship between m and m'. Choose $x \in X$ and $y \in Y$ such that $d_G(x,X_0) = m$ and $d_G(Y_0,y) = m'$. Since any (x,y)-path in G must go through F, there exists an arc $e = x_0y_0 \in F$, $x_0 \in X_0, y_0 \in Y_0$, such that

$$d_G(x,x_0) + 1 + d_G(y_0,y) = d_G(x,y) \le k.$$

Because of the choices of x and y, we have $d_G(x,x_0) \ge m$ and $d_G(y_0,y) \ge m'$. Thus,

$$m' \le d_G(y_0,y) \le k - d_G(x,x_0) - 1 \le k - m - 1.$$

It follows from (4) that

$$n = |X| + |Y| \le |F| \, \frac{d^{m+1} + d^{k-m} - 2}{d-1}, \tag{5}$$

where $0 \le m \le k-1$.

Since G is d-regular, $|(X,Y)| = |(Y,X)|$. Without loss of generality, we can suppose $m \le m'$ in the following discussion. There are two cases.

Case 1. $m \ge 1$. Then $m' \ge 1$, so $m \le k - m' - 1 \le k-2$ which implies $k \ge 3$. Define a function

$$f(m) = \frac{d^{m+1} + d^{k-m} - 2}{d-1}.$$

It is a convex function in the integer interval $[1, k-2]$ and reaches the maximum value at an end-point of the interval. Since $f(1) = f(k-2)$, it follows from (5) that

$$n \le |F| \, f(m) \le |F| \, f(1) = \lambda' \, \frac{d^{k-1} + d^2 - 2}{d-1}. \tag{6}$$

Case 2. $m = 0$. This case indicates $X = X_0$ and $m' = k - 1$. Let $A(x) = \{(x, v) | v \in Y\}$. If $2 \leq |X| \leq d - 1$, then we can deduce a contradiction as follows (Note that no loops are at the end-vertices of any pair of symmetric arcs.).

$$2d - 3 \geq |F| = \sum_{x \in X} |A(x)| \geq d|X| - |X|(|X| - 1) \geq 2d - 2.$$

Thus, $|X| \geq d$, so there is a vertex $x \in X$ which is adjacent to exactly one vertex in Y_0. Since $d_G(x, y) \leq k$ for any $y \in Y$, the number of the farthest vertices in Y that can be reached from any vertex in Y_0 is at most d^{k-1}, that is,

$$\sum_{y \in Y_0} |Y_{m'}^+(y)| \leq d^{k-1}.$$

It follows that

$$\begin{aligned} n \leq\ & |X| + \sum_{y \in Y_0} \sum_{i=0}^{m'-1} |Y_i^+(y)| + \sum_{y \in Y_0} |Y_{m'}^+(y)| \\ \leq\ & |X| + |Y_0| \sum_{i=0}^{m'-1} d^i + d^{k-1} \\ \leq\ & |F| + |F| \frac{d^{m'} - 1}{d - 1} + d^{k-1} \\ =\ & |F| + |F| \frac{d^{k-1} - 1}{d - 1} + d^{k-1}, \end{aligned}$$

from which we obtain that

$$n \leq \lambda' \, \frac{d^{k-1} + d - 2}{d - 1} + d^{k-1}. \tag{7}$$

Note that (6) is valid for $k \geq 3$ and that (7) is valid for $k \geq 2$. Comparing (6) and (7), we obtain (3) for $k \neq 3$. When $k = 3$, (6) is always valid and (7) is valid only for $|X| \geq d$. Note that the values of the right hand of (6) and (7) are $2\lambda'(d+1)$ and $\lambda'(d + 2) + d^2$, respectively. If (7) is valid, since $\lambda' = |F| \geq |X| \geq d$, then $2\lambda'(d + 1) \geq \lambda'(d + 2) + d^2$, which means $n \leq 2\lambda'(d + 1)$. Thus, we obtain (3) for $k \geq 2$, so the theorem follows. ■

Corollary 1.1. *For a connected digraph $G = G(n, d, k)$, if $\lambda'(G)$ exists and*

$$n \geq \begin{cases} 3d^{k-1} + 2d - 4, & \text{for } k \neq 3; \\ 4(d^2 - 1), & \text{for } k = 3. \end{cases}$$

then $\lambda' \geq 2d - 2$.

Proof. If $\lambda' < 2d-2$, then, when $k \neq 3$, by Theorem 1, we should have that

$$n < 2(d-1)\frac{d^{k-1}+d-2}{d-1} + d^{k-1} = 3d^{k-1} + 2d - 4,$$

which contradicts the hypothesis of n, so $\lambda' \geq 2d-2$. Similarly, when $k = 3$, by Theorem 1, we also have $\lambda' \geq 2d-2$. ■

Corollary 1.2. *For the de Bruijn digraph* $B(d,k)$ *with* $d \geq 4$ *and* $k \geq 2$, $\lambda'(B(d,k)) = 2d-2$.

Proof. Note that $B(d,k)$ contains d pairs of symmetric arcs with no loops at their end-vertices. Choose a pair of symmetric arcs between two vertices, say x and y. Since $B(d,k)$ is $(d-1)$-connected, thus, $B(d,k) - \{x,y\}$ is strongly connected for $d \geq 4$, which implies that $\lambda'(B(d,k))$ exists and that $\lambda'(B(d,k)) \leq |A^+(\{x,y\})| = 2d-2$. On the other hand, since the number of vertices is d^k, which satisfies the conditions of Corollary 1.1 for $d \geq 4$ and $k \geq 2$, $\lambda'(B(d,k)) \geq 2d-2$. Thus, $\lambda'(B(d,k)) = 2d-2$. ■

Corollary 1.3. *For the Kautz digraph* $K(d,k)$ *with* $d \geq 3$ *and* $k \geq 2$, $\lambda'(K(d,k)) = 2d-2$.

Proof. Note that $K(d,k)$ contains no loops and that $K(d,k)$ contains $(d+1)$ pairs of symmetric arcs with no loops at their end-vertices. Choose a pair of symmetric arcs between two vertices, say x and y. Since $K(d,k)$ is d-connected, thus, $K(d,k) - \{x,y\}$ is strongly connected for $d \geq 3$, which implies that $\lambda'(K(d,k))$ exists and that $\lambda'(K(d,k)) \leq 2d-2$. On the other hand, since the number of vertices is $d^k + d^{k-1}$, which satisfies the conditions of Corollary 1.1 for $d \geq 3$ and $k \geq 2$, $\lambda'(K(d,k)) \geq 2d-2$. Thus, $\lambda'(K(d,k)) = 2d-2$. ■

We now consider $\lambda'(B_G(n,d))$ and $\lambda'(K_G(n,d))$. However, they do not always exist in general. For example, $\lambda'(K_G(5,2))$ does not exist. We have the following result.

Theorem 2. *If* $\lambda'(B_G(n,d))$ *and* $\lambda'(K_G(n,d))$ *exist,* $d \geq 3$ *and* $k \geq 4$, *then*

$$\lambda'(B_G(n,d)) \geq 2d-2, \quad \lambda'(K_G(n,d)) \geq 2d-2.$$

Proof. Let G be $B_G(n,d)$ or $K_G(n,d)$ with diameter k. The following notation is useful to prove this theorem. For any vertex x in G, let $J_i^+(x)$ be the set of vertices in G which can be reached from x via a directed walk of length i in G. Imase et al. [7] have shown $|J_i^+(x)| = d^i$ for $i \leq k-1$.

Let F be an R-arc-cut of G with $|F| = \lambda'(G)$. We prove this theorem by refining on the technique in Theorem 1, so the notations X, X_0, Y, Y_0, m, m' are defined as in the proof of Theorem 1.

Since G is d-regular, $|(X,Y)| = |(Y,X)|$. Without loss of generality, we can suppose $m \leq m'$ in the following discussion. We show the theorem by contradiction. If $|F| \leq 2d-3$, then we will deduce a contradiction by considering four cases, respectively.

Case 1. $m = 0$. This case indicates $X = X_0$, so $2 \leq |X| \leq |F| \leq 2d-3$. If $|X| \leq d-1$, we can deduce a contradiction as follows

$$2d-3 \geq |F| = \sum_{x \in X} |A(x)| \geq d|X| - |X|(|X|-1)) \geq 2d-2.$$

We now assume $|X| \geq d$. Noting that $|J_1^+(x) \cap X| = d-1$ for some $x \in X$, and that $|Y_0| \leq |F| \leq 2d-3$, we have that, for $k \geq 4$,

$$\begin{aligned} |J_2^+(x) \cap X| &\geq d(d-1) - |F| \geq d^2 - 3d + 3 \\ 2d-3 \geq |X| \geq |J_3^+(x) \cap X| &\geq d|J_2^+(x) \cap X| - |F| \geq d^3 - 3d^2 + d + 3. \end{aligned}$$

However, this is impossible for $d \geq 3$.

Case 2. $m = 1$. Choose $x \in X$ such that $d_G(x, X_0) = 1$. Then $|J_i^+(x)| = d^i$ for $i \leq k-1$. Noting that $J_1^+(x) \subseteq X$, and that $|J_1^+(x)| = d$, we have that, for $k \geq 4$,

$$\begin{aligned} |J_2^+(x) \cap X| &\geq d^2 - |F| \geq d^2 - 2d + 3 \\ |X| \geq |J_3^+(x) \cap X| &\geq d|J_2^+(x) \cap X| - |F| \\ &\geq d(d^2 - 2d + 3) - 2d + 3 \\ &= d^3 - 2d^2 + d + 3. \end{aligned}$$

However, this is impossible for $d \geq 3$ since $|X| \leq |F| + d|F| \leq 2d^2 - d - 3$ when $m = 1$.

Case 3. $m = 2$. This case implies $k \geq 5$, since $2 \leq m \leq m'$ and $m + m' \leq k-1$. We can choose a vertex $x \in X$ such that $d_G(x, X_0) = 2$. Then $|J_i^+(x)| = d^i$ for $i \leq k-1$. Noting that $J_2^+(x) \subseteq X$, and $|J_2^+(x)| = d^2$, we have that, for $k \geq 5$,

$$\begin{aligned} |J_3^+(x) \cap X| &\geq d|J_2^+(x) \cap X| - |F| \geq d^3 - 2d + 3 \\ |X| \geq |J_4^+(x) \cap X| &\geq d|J_3^+(x) \cap X| - |F| \end{aligned}$$

$$\begin{aligned} &\geq\ d(d^3-2d+3)-2d+3 \\ &=\ d^4-2d^2+d+3. \end{aligned}$$

However, this is impossible for $d \geq 3$ since $|X| \leq |F|(1+d+d^2) \leq (2d-3)(1+d+d^2) = 2d^3-d^2-d-3$ when $m=2$.

Case 4. $m \geq 3$. In this case, we have $m \leq k-4$ and $k \geq 7$ since $3 \leq m \leq m'$ and $m+m' \leq k-1$. It follows from (5) that

$$\begin{aligned} n &\leq\ |F|\ \frac{d^{m+1}+d^{k-m}-2}{d-1} \\ &\leq\ |F|\ \frac{d^{k-3}+d^4-2}{d-1} \\ &<\ 2\ (d^{k-3}+d^4-2). \end{aligned}$$

However, this is impossible since $d^{k-1} < n$ and $2\ (d^{k-2}+d^3-2) < d^{k-1}$ for $k \geq 7$ and $d \geq 3$. The theorem follows. ■

Corollary 2.1. *For any $K_G(n,d)$ with diameter $k \geq 4$, if either $d \geq 4$ or $d \geq 3, k \geq 5$, g.c.d $(n,d) \geq 2$ and n is divisible by $(d+1)$ then $\lambda'(K_G(n,d)) = 2d-2$.*

Proof. By Lemma 3, choose a pair of symmetric arcs with end-vertices x and y in $K_G(n,d)$. Note that $|J_i^+(x)| = |J_i^+(y)| = d^i$ for $i \leq k-1$ and that the vertices x and y have no loops since $k \geq 4$. By Lemma 2, $K_G(n,d)$ is 3-connected either if $d \geq 4$ and $k \geq 4$ or if $d \geq 3$, $k \geq 5$, g.c.d $(n,d) \geq 2$ and n is divisible by $(d+1)$. Thus, $K_G(n,d) - \{x,y\}$ is strongly connected, which implies that $A^+(\{x,y\})$ is an R-arc-cut of $K_G(n,d)$. Thus, $\lambda'(K_G(n,d)) \leq 2d-2$. By Theorem 2, we have $\lambda'(K_G(n,d)) = 2d-2$. ■

Corollary 2.2. *For any $B_G(n,d)$ with diameter k, if $d \geq 4$ and $k \geq 4$, then $\lambda'(B_G(n,d)) = 2d-2$.*

Proof. By Lemma 3, choose a pair of symmetric arcs with end-vertices x and y in $B_G(n,d)$. Note that $|J_i^+(x)| = |J_i^+(y)| = d^i$ for $i \leq k-1$ and that the vertices x and y have no loops since $k \geq 4$. By Lemma 2, $B_G(n,d)$ is $(d-1)$-connected. Thus, $B_G(n,d) - \{x,y\}$ is strongly connected for $d \geq 4$, which implies that $A^+(\{x,y\})$ is an R-arc-cut of $B_G(n,d)$. Thus, $\lambda'(B_G(n,d)) \leq 2d-2$. By Theorem 2, we have $\lambda'(B_G(n,d)) = 2d-2$. ■

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