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On Edge Addition of Altered Graphs*

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Abstract: This paper proves that, for any integers $t \geqslant 6$ and $d \geqslant 2$, the upper bound of minimum diameter of a connected graph obtained from a single path of length d by adding t extra edges is $\left\lceil \frac{d-2}{t+1} \right\rceil + 2$ for $d \in I'(t,k) = \{2k(t+1)+1, 2k(t+1)+2, 2k(t+1)-t+1\} \cup \{2k(t+1)-t+h : h = 6, 7, \cdots, t\}$ and $\left\lceil \frac{d-2}{t+1} \right\rceil + 1$ otherwise for any integer $k \geqslant 1$, which improves the known results. The bound $\left\lceil \frac{d-2}{t+1} \right\rceil + 1$ is best possible.

Key words: diameter; altered graph; edge addition

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AMS Subject Classification(2000): 05C12

0 Introduction

We follow ref. [1] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a simple undirected graph, where $V = V(G)$ and $E = E(G)$ are the vertex-set and the edge-set of G, respectively. Let t_d denote the maximum number of edges that can be added to a path of length d, and $P(t,d)$ the minimum diameter of a graph obtained by adding t extra edges to a path of length d. Clearly, the symbol $P(t,d)$ means $t \leqslant t_d$. Determining $P(t,d)$ for given d and t, proposed by ref. [2], is of important interest in designing and analyzing interconnection networks.

For some small t's and special d's, the values of $P(t,d)$ have been determined. It is easy to verify that $P(1,d) = \left\lfloor \frac{d+1}{2} \right\rfloor$ for $d \geqslant 2$; Schoone *et al.*[3] determined $P(2,d) = \left\lceil \frac{d+1}{3} \right\rceil$ for $d \geqslant 3$ and $P(3,d) = \left\lceil \frac{d+2}{4} \right\rceil$ for $d \geqslant 5$; Deng and Xu[4] determined

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$P(t,(2k-1)(t+1)+1)=2k$ for any positive integer k, $\left\lceil\frac{d}{t+1}\right\rceil \leqslant P(t,d) \leqslant \left\lceil\frac{d}{t+1}\right\rceil+1$ for $t=4,5$ and $d \geqslant 4$, and, in general, $\left\lceil\frac{d}{t+1}\right\rceil \leqslant P(t,d) \leqslant \left\lfloor\frac{d-2}{t+1}\right\rfloor+3$. For any integer $k \geqslant 1$, let

$$I'(t,k)=\{2k(t+1)+1,\ 2k(t+1)+2,\ 2k(t+1)-t+1\} \cup \{2k(t+1)-t+h : h=6,7,\cdots,t\}.$$

In this paper, we improve the upper bound of $P(t,d)$ to

$$P(t,d) \leqslant \begin{cases} \left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{if } d \in I'(t,k) \text{ for some } t \text{ and } k, \\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{otherwise} \end{cases}$$

for the integers $t \geqslant 6$, $d \geqslant 2$ and $k \geqslant 1$. This upper bound is tight for some t and d.

1 Several lemmas

Lemma 1.1[3] For any integers k and h with $2 \leqslant h \leqslant t+1$,

$$P(t,(t+1)(2k-1)+h) \leqslant 2k+1, \text{ when } t=4,5.$$

The following lemma is simple, but useful.

Lemma 1.2 Let t_d be the maximum number of edges added to a path of length $d>1$, then

$$t_d=\frac{d(d-1)}{2}.$$

Lemma 1.3① $P(t,d) \leqslant 2k$, where $d \geqslant 2$, $d-2 \leqslant t < t_d$ for $k=1$, and $\left\lceil\frac{d-1}{2k-1}\right\rceil-1 \leqslant t \leqslant \left\lceil\frac{d}{2k-2}\right\rceil-2$ for any integer $k \geqslant 2$.

Proof To prove the lemma, we construct an altered graph G from a single path $P=x_1x_2\cdots x_{d+1}$ by adding t extra edges such that the diameter of G is at most $2k$.

When $k=1$, choose $r=1$ or 2. We add t extra edges x_rx_j to P, where $j=r+2, r+3,\cdots,d+1$. Since every vertex can reach the vertex x_r within one step, then the distance between any two vertices is at most 2. Thus $P(t,d) \leqslant 2$ for $t=d-2$. Since $P(t,d) \geqslant P(t',d)$ if $t \leqslant t' < t_d$, so $P(t,d) \leqslant 2$ when $d-2 \leqslant t < t_d$.

Now, assume $k>1$. Put $m=\left\lceil\frac{d-1}{2k-1}\right\rceil-1$. We add m edges $x_{k+1}x_{d-(k-2)-i(2k-1)}$ for $i=0,1,\cdots,m-1$. The end-vertices of these edges divide P into $m+2$ segments $L_1, L_2, \cdots, L_{m+1}, L_{m+2}$, where

① Clearly, the lemma is invalid for some d and k when $\left\lceil\frac{d-1}{2k-1}\right\rceil-1>\left\lceil\frac{d}{2k-2}\right\rceil-2$ i. e. $2k+1 \leqslant P(t,d)$ or $P(t,d) \leqslant 2k-1$ for any $t \geqslant 1$.

$$L_1 = P(x_1, x_{k+1}),$$
$$L_2 = P(x_{k+1}, x_{d-(k-2)-(m-1)(2k-1)}),$$
$$L_{m+1-i} = P(x_{d-(k-2)-(i+1)(2k-1)}, x_{d-(k-2)-i(2k-1)}),\ i = 0, 1, \cdots, m-2,$$
$$L_{m+2} = P(x_{d-(k-2)}, x_{d+1}).$$

See Fig. 1 for an example. Since $m = \left\lceil \frac{d-1}{2k-1} \right\rceil - 1 \geqslant \frac{d-1}{2k-1} - 1$, we have

$$d-(k-2)-(m-1)(2k-1)-(k+1) \leqslant$$
$$d-(k-2)-\left(\frac{d-1}{2k-1}-2\right)(2k-1)-(k+1) = 2k.$$

Indeed,

$$d(L_1) = k,$$
$$d(L_2) \leqslant 2k,$$
$$d(L_{m+1-i}) = 2k-1,\ i = 0,\ 1,\ \cdots,\ m-2, \text{if } m \geqslant 2,$$
$$d(L_{m+2}) = k-1.$$

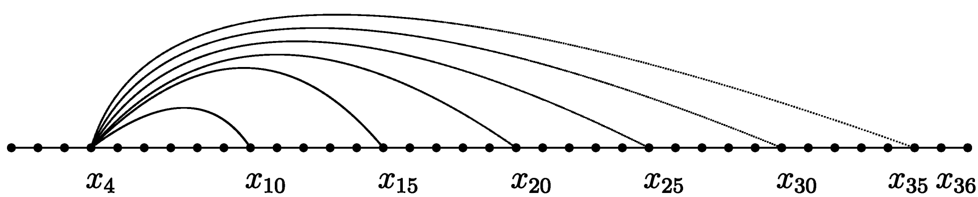

Fig. 1 Construction of Lemma 1.3 for $k = 3, d = 36$ and $m = t = 6$

We need to prove that the diameter of G is at most $2k$. Since any vertex can reach the vertex x_{k+1} within k steps, the distance of any two vertices in G is at most $2k$, which means $P(m, d) \leqslant 2k$. Since $t \geqslant m$ and $P(t, d) \leqslant P(t', d)$ if $t \geqslant t'$, so $P(t, d) \leqslant 2k$ when $d \geqslant 2$ and $\left\lceil \frac{d-1}{2k-1} \right\rceil - 1 \leqslant t \leqslant \left\lceil \frac{d}{2k-2} \right\rceil - 2$ for any integer $k \geqslant 2$. □

Lemma 1.4 Lemma 1.3 is equivalent to the statement that $P(t, d) \leqslant 2k$, where $\frac{1+\sqrt{9+8t}}{2} \leqslant d \leqslant t+2$ for $k = 1$, and $d = 2k(t+1) - t - h, h = 0, 1, \cdots, t+1$ for any integer $k \geqslant 2$.

Proof Clearly, the lemma satisfies when $k = 1$, since $d - 2 \leqslant t < t_d$ from lemma 1.3. So we only need to prove the lemma for $k \geqslant 2$. Let $t = \left\lceil \frac{d-1}{2k-1} \right\rceil - 1$, then $t \geqslant \frac{d-1-(2k-1)}{2k-1}$, that is, $d \leqslant 2k(t+1) - t$. Now let $t \leqslant \left\lceil \frac{d}{2k-1} \right\rceil - 2$, we prove $d \geqslant 2k(t+1) - 2t - 1$ by contradiction. Suppose $d \leqslant 2k(t+1) - 2t - 2$. Then $d \leqslant (2k-2) \cdot (t+1)$, which implies $t \geqslant \left\lceil \frac{d}{2k-2} \right\rceil - 1$, a contradiction. Thus $d \geqslant 2k(t+1) - 2t - 1$. □

The following lemma extends Lemma 1.1 for any $t \geqslant 6$.

Lemma 1.5 $P(t, 2k(t+1) - t + 6 - h') \leqslant 2k+1$ for the integers $t \geqslant 6, k \geqslant 1$, and

$1 \leqslant h' \leqslant 5$.

Proof Like the proof of Lemma 1.3, we construct an altered graph G from a single path $P = x_1 x_2 \cdots x_{d+1}$ by adding t extra edges such that the diameter of G is at most $2k+1$.

Let $d = 2k(t+1) - t + 5$. We add t edges

$$e_{2i-1} = x_{2k(3-r)+(1-r)} x_{d-2(i-1)(2k-1)+1}, \quad \text{for } i = 1, 2, \cdots, (t-r)/2 - 2$$
$$e_{2i} = x_{2k(2+r)+r} x_{d-(2i-1)(2k-1)-1}, \quad \text{for } i = 1, 2, \cdots, (t+r)/2 - 3$$
$$e_{t-4} = x_{4k} x_{12k+1},\ e_{t-3} = x_{6k+1} x_{10k+1},$$
$$e_{t-2} = x_{4k} x_{8k+1},\ e_{t-1} = x_{2k} x_{6k+1},\ e_t = x_1 x_{4k},$$

where $r = \begin{cases} 0 & \text{if } t \text{ is even}, \\ 1 & \text{if } t \text{ is odd}. \end{cases}$

Now the end-vertices of these edges divide P into $t+1$ segments, since

$$d - (t-5)(2k-1) + 1 = 2k(t+1) - t + 5 - (t-5)(2k-1) + 1 = 12k + 1,$$

we have

$$L_i = P(x_{d-i(2k-1)+1}, x_{d-(i-1)(2k-1)+1}), \quad \text{for } i = 1, 2, \cdots, t-5$$
$$L_{t-4} = P(x_{10k+1}, x_{12k+1}), \quad L_{t-3} = P(x_{8k+1}, x_{10k+1}),$$
$$L_{t-2} = P(x_{6k+1}, x_{8k+1}), \quad L_{t-1} = P(x_{4k}, x_{6k+1}),$$
$$L_t = P(x_{2k}, x_{4k}), \quad L_{t+1} = P(x_1, x_{2k}),$$

See Fig. 2 for an example.

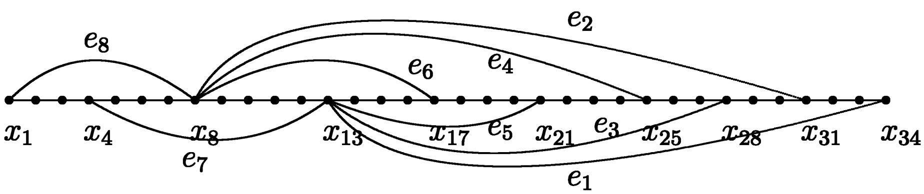

Fig. 2 Construction of Lemma 1.5 for $k = 2, t = 8$ and $d = 33$.

Let $X = \{x_1, x_2, \cdots, x_{12k+1}\}$ and $X' = \{x_{2k+1}, x_{2k+2}, \cdots, x_d\}$. From the proof of Lemma 1.1 we have the distance between any two vertices $x, y \in X$ is less than or equal to $2k+1$. So we only need to prove the lemma when $x, y \in X'$ or $x \in X$ and $y \in X'$ is less than or equal to $2k+1$. For $j = 1, 2, \cdots, t-5$ we define $\left(\dfrac{t(t+1)}{2} - 15\right)$ cycles $C_j^1, C_j^2, \cdots, C_j^{t-j+1}$ as

$$C_j^1 = L_j \cup + L_{j+1} + e_j + e_{j+2},$$
$$C_j^i = L_j \cup + L_{i+j} + e_j + e_{j+1} + e_{i+j} + e_{i+j+1}, \text{for } i = 2, 3, \cdots, t-j-3,$$
$$C_j^{t-j-2} = L_j \cup + L_{t-2} + e_j + e_{j+1} + e_{t-2},$$
$$C_j^{t-j-1} = L_j \cup + L_{t-1} + e_j + e_{j+1},$$
$$C_j^{t-j} = L_j \cup + L_t + e_j + e_{j+1} + e_{t-1},$$
$$C_j^{t-j+1} = L_j \cup + L_{t+1} + e_j + e_{j+1} + e_{t-1} + e_t.$$

Their lengths are

$$\varepsilon(C_j^1) = 4k, \quad \text{if } j < t-5;$$

$$\varepsilon(C_{t-5}^{1}) = 4k+1;$$
$$\varepsilon(C_j^i) = 4k+2, \quad \text{for } i = 2,3,\cdots,t-j-5, j < t-6, t > 7;$$
$$\varepsilon(C_j^{t-j-4}) = 4k+3, \quad \text{if } j < t-5, t \neq 6;$$
$$\varepsilon(C_j^{t-j-3}) = 4k+3;$$
$$\varepsilon(C_j^i) = 4k+2, \text{ for } t-j-2 \leqslant i \leqslant t-j+1.$$

It is easy to see that any two vertices $x, y \in X'$ or $x \in X$ and $y \in X'$ are contained in some cycle C_j^i defined above. The fact

$$\max\{d(C_j^i): 1 \leqslant j \leqslant t-5, 1 \leqslant i \leqslant t-j+1\} \leqslant \left\lfloor \frac{4k+3}{2} \right\rfloor = 2k+1$$

means $$P(t, 2k(t+1)-t+5) \leqslant d(G) \leqslant 2k+1.$$

Since $P(t,d) \leqslant P(t,d')$ if $d \leqslant d'$, $P(t, 2k(t+1)-t+6-h') \leqslant 2k+1$ when $1 \leqslant h' \leqslant 5$. □

2 Proof of main result

In this section, we only consider $t \geqslant 6$. For a given t, let $d(k) = 2k(t+1)-2t-1$. Then $d(k+1) = 2k(t+1)+1 = d(k)+2t+2$. We give an upper bound of $P(t,d)$ when d is any integer in the interval $I(t,k) = [2k(t+1)-2t-1,\ 2k(t+1)]$ for any $k \geqslant 1$ and $t \geqslant 6$. To state our theorem, let $I'(t,k) = \{2k(t+1)+1,\ 2k(t+1)+2,\ 2k(t+1)-t+1\} \cup \{2k(t+1)-t+h : h = 6,7,\cdots, t\}$.

Theorem 2.1 For any $t \geqslant 6$ and $k \geqslant 1$, if $d \in I(t,k)$ then

$$P(t,d) \leqslant \begin{cases} \left\lceil \dfrac{d-2}{t+1} \right\rceil + 2 & \text{if } d \in I'(t,k) \text{ for some } t \text{ and } k; \\ \left\lceil \dfrac{d-2}{t+1} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

Proof First, we consider $k=1$. In this case, $\dfrac{1+\sqrt{9+8t}}{2} \leqslant d \leqslant t+2$ and $d \geqslant 3$ since $t \geqslant 1$. For $d \leqslant t+2$ we have $d = t+2-i$ for some $i = 0,1,\cdots, t-1$ and, thus,

$$\frac{d-2}{t+1} = \frac{(t+2-i)-2}{t+1} = \frac{t-i}{t+1},$$

which means $\left\lceil \dfrac{d-2}{t+1} \right\rceil = 1$. From Lemma 1.4 we have

$$P(t,d) \leqslant 2 = \left\lceil \frac{d-2}{t+1} \right\rceil + 1.$$

Secondly, we consider $k \geqslant 2$ and $d = 2k(t+1)-t-h$, $h = 0,1,\cdots,t+1$. In this case, we have

$$\frac{d-2}{t+1} = \frac{2k(t+1)-t-h-2}{t+1} = 2k-1-\frac{h+1}{t+1},$$

which implies that

$$2k = \frac{d-2}{t+1} + 1 + \frac{h+1}{t+1}. \tag{1}$$

For $h=0,1,\cdots,t$, we have $\frac{h+1}{t+1}\leqslant 1$. When $h=t$ the equality holds and, hence,

$$2k-2=\frac{d-2}{t+1}=\left\lceil\frac{d-2}{t+1}\right\rceil. \tag{2}$$

Also when $h=t+1, d=2k(t+1)-2t-1$. From (1) we have

$$2k-2=\frac{d-2}{t+1}+\frac{1}{t+1}=\left\lceil\frac{d-2}{t+1}\right\rceil. \tag{3}$$

If $0\leqslant h\leqslant t-1$, then from (1) we have

$$2k-1=\frac{d-2}{t+1}+\frac{h+1}{t+1}=\left\lceil\frac{d-2}{t+1}\right\rceil. \tag{4}$$

Thus, from Lemma 1.4 and (1) we have

$$P(t,2k(t+1)-t-h)\leqslant 2k=\frac{d-2}{t+1}+1+\frac{h+1}{t+1}. \tag{5}$$

From (2), (3), (4) and (5) we have

$$P(t,d)\leqslant\begin{cases}\left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{for } d=2k(t+1)-2t-1 \text{ or } 2k(t+1)-2t;\\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{for } 2k(t+1)-2t+1\leqslant d\leqslant 2k(t+1)-t.\end{cases}$$

which implies for $k\geqslant 1$

$$P(t,d)\leqslant\begin{cases}\left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{for } d=2k(t+1)+1 \text{ or } 2k(t+1)+2;\\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{for } 2k(t+1)+3\leqslant d\leqslant 2k(t+1)+t+2.\end{cases}$$

Thirdly, for $k\geqslant 1$ we consider $d=2k(t+1)-t+6-h'$ with $1\leqslant h'\leqslant 5$. In this case, we have

$$\frac{d-2}{t+1}=\frac{2k(t+1)-t+6-h'-2}{t+1}=2k-\frac{t-4+h'}{t+1},$$

that is,

$$2k+1=\frac{d-2}{t+1}+\frac{t-4+h'}{t+1}+1. \tag{6}$$

If $h'=5$ then, from (6), we have

$$2k+1=\frac{d-2}{t+1}+2=\left\lceil\frac{d-2}{t+1}\right\rceil+2. \tag{7}$$

If $1\leqslant h'\leqslant 4$ then, from (6), we have

$$2k+1=\frac{d-2}{t+1}+\frac{t-4+h'}{t+1}+1=\left\lceil\frac{d-2}{t+1}\right\rceil+1. \tag{8}$$

From Lemma 1.5 and the equalities (6), (7) and (8) we have

$$P(t,d)\leqslant\begin{cases}\left\lceil\frac{d-2}{t+1}\right\rceil+2 & \text{for } d=2k(t+1)-t+1;\\ \left\lceil\frac{d-2}{t+1}\right\rceil+1 & \text{for } 2k(t+1)-t+2\leqslant d\leqslant 2k(t+1)-t+5.\end{cases}$$

Finally, since for any $k\geqslant 1$,

$$P(t,2k(t+1)+1) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 2.$$

and $P(t,d') \leqslant P(t,d'')$ when $d' \leqslant d''$, then we have

$$P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 2, \qquad \text{for } 2k(t+1)-t+6 \leqslant d \leqslant 2k(t+1).$$ □

Corollary 2.1[4] $P(t,2k(t+1)-t) = 2k$ for any positive integer k.

Proof Let $d = 2k(t+1)-t$. On the one hand, by $P(t,d) \geqslant \left\lceil \frac{d}{t+1} \right\rceil$, due to ref. [4] and stated in Introduction, we have

$$P(t,d) \geqslant \left\lceil \frac{d}{t+1} \right\rceil = \left\lceil \frac{2k(t+1)-t}{t+1} \right\rceil = 2k.$$

On the other hand, $(t+1) \nmid (d-2)$, by Theorem 2.1, we have

$$P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1 = \left\lceil \frac{2k(t+1)-t-2}{t+1} \right\rceil + 1 = 2k.$$

So, $P(t,2k(t+1)-t) = 2k$ for any positive integer k. □

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变更图的边添加

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摘要:证明了:对任何整数 $t \geqslant 6$ 和 $d \geqslant 2$,从一条长为 d 的简单路通过添加 t 条边后得到的图的最小直径上界为 $\left\lceil \frac{d-2}{t+1} \right\rceil + 2$,如果 $d \in I'(t,k) = \{2k(t+1)+1, 2k(t+1)+2, 2k(t+1)-t+1\} \cup \{2k(t+1)-t+h : h = 6,7,\cdots,t\}$;其他情形为 $\left\lceil \frac{d-2}{t+1} \right\rceil + 1$. 这个证明改进了已知结果,而且 $\left\lceil \frac{d-2}{t+1} \right\rceil + 1$ 是最好的上界.

关键词:直径;变更图;边添加