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On Edge-Forwarding Index of Graphs with Degree Restriction*

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Abstract: For a given graph G of order n, a routing R is a set of $n(n-1)$ elementary paths such that every ordered pair of vertices in G are connected by a path in the set. The edge-forwarding index of G with respect to a given routing R, denoted by $\pi(G,R)$, is the maximum number of paths in R passing through any edge e of G. The edge-forwarding index $\pi(G)$ of G is the minimum of $\pi(G,R)$ taken over all possible routings R of G. The parameter $\pi_{\Delta,n}$ is the minimum of $\pi(G)$ taken over all graphs G of order n with maximum degree at most Δ. We determine all values of $\pi_{n-2p,n}$ for $n \geqslant 4p+1$ but $n \notin \left[4p+\left\lceil \frac{1}{3}(2p-1)\right\rceil -1, 6p\right]$ for any $p \geqslant 1$.

Key words: forwarding index; vertex-forwarding index; edge-forwarding index; routing

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0 Introduction

Let $G=(V,E)$ be a connected graph of order n. A routing R in G is a set of $n(n-1)$ fixed paths for all ordered pairs (x,y) of vertices of G. The path $R(x,y)$ specified by R carries the data transmitted from the source x to the destination y. It is possible that the fixed paths specified by a given routing R passing through some vertex or edge are too many, which means that the routing R loads the vertex or the edge too much. Load of any vertex or edge is limited by the capacity of the vertex or edge, for otherwise it would affect efficiency of transmission, even resulting in malfunctions of the network. In order to

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measure the load of a vertex or an edge, ref. [1] proposed the notion of the forwarding index.

Let G be a graph with a given routing R, x a vertex of G and e an edge of G. The load of x with respect to R, denoted by $\xi(G,R,x)$ [resp. the load of e with respect to R, denoted by $\pi(G,R,e)$], are defined as the number of paths specified by R passing through x [resp. e]. The vertex-forwarding index and the edge-forwarding index of G with respect to R are, respectively, defined as

$$\xi(G,R) = \max\{\xi(G,R,x) : x \in V(G)\}$$

and

$$\pi(G,R) = \max\{\pi(G,R,e) : e \in E(G)\}.$$

The vertex-forwarding index and the edge-forwarding index of G are, respectively, defined as

$$\xi(G) = \min\{\xi(G,R) : R \text{ is a routing of } G\}$$

and

$$\pi(G) = \min\{\pi(G,R) : R \text{ is a routing of } G\}.$$

It is not difficult to see that maximizing network capacity clearly reduces to minimizing vertex-[resp. edge-] forwarding index of a routing. Thus, it becomes very significant to decide the vertex and edge-forwarding indices of a given graph.

Many authors are interested in the forwarding indices of a graph [1~5]. However, Saad[5] has showed that for any graph determining the forwarding index problem is NP-complete. One can still determine the exact value of the forwarding index with some graph-theoretical parameters. For example, ref. [1] proposed the following problem: Given Δ and n, determine $\xi_{\Delta,n}$, the minimum of $\xi(G)$ taken over all graphs of order n with maximum degree at most Δ. This problem has been solved asymptotically in ref. [1] and determined $\xi_{\Delta,n}$ for $n \leqslant 15$ and $\frac{1}{3}(n+4) \leqslant \Delta \leqslant n-1$ in ref. [4].

Ref. [3] considers the same problem for edge-forwarding index: Given Δ and n, determine $\pi_{\Delta,n}$, the minimum of $\pi(G)$ taken over all graphs of order n with maximum degree at most Δ. Ref. [2] determins $\pi_{\Delta,n}$ for some values of n and Δ. For example, $\pi_{2,n} = \left\lfloor \frac{1}{4} n^2 \right\rfloor$ for any $n \geqslant 3$; $\pi_{n-1,n} = 2$ for any $n \geqslant 2$; $\pi_{n-2,n} = 3$ for any $n \geqslant 6, n \neq 7$ and $\pi_{n-2,n} = 4$ for any $n = 4,5,7$. In particular, they have obtained that for any $p \geqslant 1$,

$$\pi_{n-2p-1,n} = \begin{cases} 3, & \text{if } n \geqslant 10p+1; \\ 4, & \text{if } 6p+1 \leqslant n < 10p+1; \\ 6, & \text{if } 4p+1 \leqslant n \leqslant 4p + \left\lceil \frac{2}{3} p \right\rceil. \end{cases} \tag{1}$$

In this paper, we obtain the following result.

Theorem 1 For any $p \geqslant 1$, we have

$$\pi_{n-2p,n} = \begin{cases} 3, & \text{if } n \geqslant 10p-2 \text{ or } n = 10p-4; \\ 4, & \text{if } 6p+1 \leqslant n < 10p-4 \text{ or } n = 10p-3; \\ 6, & \text{if } 4p+1 \leqslant n \leqslant 4p + \left\lceil \frac{1}{3}(2p-1) \right\rceil - 2. \end{cases}$$

For the definitions and notations not given here, the reader should refer to ref. [1].

1 Lemmas

In this section, we give some lemmas to be used in the proof of our theorem.

Lemma 1[2] (ⅰ) $\Delta\pi_{\Delta,n} \geqslant 2\xi_{\Delta,n} + 2(n-1)$ for any n and Δ; (ⅱ) $\pi_{\Delta,n} \leqslant \pi_{\Delta',n}$ for any n, $\Delta' \leqslant n-1$ and $\Delta' < \Delta$.

Lemma 2[4] $\xi_{\Delta,n} \geqslant n-\Delta-1$ for any n and Δ, and $\xi_{\Delta,n} \geqslant n-\Delta$ if n and Δ are odd.

Lemma 3 For any $p \geqslant 1$,

$$\pi_{n-2p,n} = \begin{cases} 3, & \text{if } n \geqslant 10p+1; \\ 4, & \text{if } 6p+1 \leqslant n < 10p-4 \text{ or } n = 10p-3; \\ 6, & \text{if } 4p+1 \leqslant n \leqslant 4p + \left\lceil \frac{1}{3}(2p-4) \right\rceil - 1; \end{cases}$$

and $3 \leqslant \pi_{n-2p,n} \leqslant 4$ if $n = 10p, 10p-1, 10p-2, 10p-4$.

Proof Indeed, from Lemma 1 and Lemma 2, we have

$$\pi_{n-2p-1,n} \geqslant \pi_{n-2p,n} \geqslant \frac{4p+2n-4}{n-2p},$$

which gives

$$\pi_{n-2p-1,n} \geqslant \pi_{n-2p,n} \geqslant 2 + \left\lceil \frac{8p-4}{n-2p} \right\rceil. \tag{2}$$

Moreover, if n is odd, we could get

$$\pi_{n-2p-1,n} \geqslant \pi_{n-2p,n} \geqslant 2 + \left\lceil \frac{8p-2}{n-2p} \right\rceil. \tag{3}$$

Combining (2), (3) with (1), we have that for any $p \geqslant 1$,

$$\pi_{n-2p,n} = \begin{cases} 3, & \text{if } n \geqslant 10p+1; \\ 4, & \text{if } 6p+1 \leqslant n < 10p-4 \text{ or } n = 10p-3; \\ 6, & \text{if } 4p+1 \leqslant n \leqslant 4p + \left\lceil \frac{1}{3}(2p-4) \right\rceil - 1; \end{cases}$$

and $3 \leqslant \pi_{n-2p,n} \leqslant 4$ if $n = 10p, 10p-1, 10p-2, 10p-4$. □

Lemma 4 For any p, there is a partition of the set $\{1,2,\cdots,4p\}$ into $2p$ pairs in which $4p$ and $3p$ are in one pair, so that the set of difference between the pairs is twice the set $\{1,2,\cdots,p\}$.

Proof We prove this lemma by constructing all possible partitions according to the following two cases, respectively.

(ⅰ) Case p is odd.

The pairs $\{i, p+1-i\}$, $1 \leqslant i \leqslant \frac{1}{2}(p-1)$, have even differences $2,4,\cdots,p-1$.

The pairs $\{p+1+i, 2p-i\}$, $0 \leqslant i \leqslant \frac{1}{2}(p-3)$, have even differences $2,4,\cdots,p-1$.

The pair $\left\{\frac{1}{2}(p+1), \frac{1}{2}(3p+1)\right\}$ has odd difference p.

The pairs $\{2p+i+1, 3p-1-i\}, 0 \leqslant i \leqslant \frac{1}{2}(p-3)$, have odd differences $1, 3, \cdots, p-2$.

The pairs $\{3p+i, 4p-i\}, 0 \leqslant i \leqslant \frac{1}{2}(p-1)$, have odd differences $1, 3, \cdots, p-2, p$.

(ⅱ) Case p is even.

The pairs $\{i, p+1-i\}, 1 \leqslant i \leqslant \frac{1}{2}p$, have odd differences $1, 3, \cdots, p-1$.

The pairs $\{p+i, 2p+1-i\}, 1 \leqslant i \leqslant \frac{1}{2}p$, have odd differences $1, 3, \cdots, p-1$.

The pairs $\{2p+i, 3p-i\}, 1 \leqslant i \leqslant \frac{1}{2}p-1$, have even differences $2, 4, \cdots, p-2$.

The pairs $\{3p-1+i, 4p+1-i\}, 1 \leqslant i \leqslant \frac{1}{2}p$, have even differences $2, 4, \cdots, p$.

The pair $\left\{2p+\frac{1}{2}p, 3p+\frac{1}{2}p\right\}$ has even difference p.

It is clear that all these pairs partition the set $\{1, 2, \cdots, 4p\}$. □

2 Proof

In this section we will complete the proof of Theorem 1. The following notion will appear in the proof.

For brevity, a graph of order n and maximum degree Δ will be called an (n, Δ)-graph. A circulant $C(n; a_1, a_2, \cdots, a_k), 1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant \frac{n}{2}$, is a graph of order n with the set of vertices $\{0, 1, 2, \cdots, n-1\}$, where any vertex i is joined to $i \pm a_s \pmod n$ for every s, $1 \leqslant s \leqslant k$. Note that this graph is $(2k-1)$-regular if $a_k = \frac{n}{2}$, and $(2k)$-regular otherwise.

We define a graph $G(h_1, h_2; \pm S)$ obtained from a complete K_n with the set of vertices $\{0, 1, \cdots, n-1\}$ minus all edges of a circulant graph $C(n; h_1, h_1+1, \cdots, h_2)$ then plus (+) or minus (−) a specified edge sets S, where $h_1 < h_2 \leqslant \frac{n}{2}$. We call an edge in $G(h_1, h_2; +S)$ a new edge if it is in S, and an old edge otherwise. It is easy to see that the subgraph of $G(h_1, h_2; +S)$ induced by all old edges is vertex-transitive since K_n and $C(n; h_1, h_1+1, \cdots, h_2)$ is vertex-transitive.

Given a graph G with the set of vertices $\{0, 1, 2, \cdots, n-1\}$, the difference of an edge $\{x, y\}$ is defined as $|x-y|$ or $n-|x-y|$. Clearly, if an edge $\{x, y\}$ has difference d then all edges in $\{(x+i) \pmod n, (y+i) \pmod n\}$ have difference d, and they are the only edges of G having this difference.

Proof of Theorem 1 In order to prove Theorem 1, by Lemma 3 we only need to prove that

$$\pi_{n-2p,n} \leqslant 3 \tag{4}$$

for $n=10p, 10p-1, 10p-2, 10p-4$. To the end, we need to only construct an $(n, n-2p)$-graph G such that $\pi(G) \leqslant 3$, which will be done according to the following four cases, $n = 10p, 10p-1, 10p-2, 10p-4$, respectively. We will here give the proof for $n = 10p$ in detail, and omit others because of a similar argument.

If $n = 10p$, then $n - 2p = 8p$. We need to construct a $(10p, 8p)$-graph G such that $\pi(G) \leqslant 3$. Consider such a graph G as $G(1, p; +S)$, where S, consisting of $5p$ edges, is as follows:

$$\begin{aligned} S = \{ & \{0,p\}, & \{1,p+1\}, & \quad \cdots, & \{p-1,2p-1\}, \\ & \{2p,3p\}, & \{2p+1,3p+1\}, & \quad \cdots, & \{3p-1,4p-1\}, \\ & \{4p,5p\}, & \{4p+1,5p+1\}, & \quad \cdots, & \{5p-1,6p-1\}, \\ & \{6p,7p\}, & \{6p+1,7p+1\}, & \quad \cdots, & \{7p-1,8p-1\}, \\ & \{8p,9p\}, & \{8p+1,9p+1\}, & \quad \cdots, & \{9p-1,10p-1\}\}. \end{aligned}$$

It is not difficult to verify that G is $(8p)$-regular. Thus, $\pi_{8p,10p} \leqslant \pi(G)$.

From the construction of G, we note that the difference r of any edge in G satisfies $p \leqslant r \leqslant 5p$ and an edge of G has difference p if and only if it is new. Moreover, the difference d between any pair of nonadjacent vertices satisfies $1 \leqslant d \leqslant p$, that is, for a vertex i of G, the vertex not adjacent to i is $i+d$ or $i-d$ for each $d = 1, 2, \cdots, p$, where $i+d$ and $i-d$ are taken modulo n.

In order to obtain the upper bound 3 of $\pi(G)$, we define a routing R of G. If two vertices i and j are adjacent in G, then the path in R is defined as the edge between them. If i and j are not adjacent in G with difference d ($1 \leqslant d \leqslant p$), then $j = i+d$ or $j = i-d$, suppose $j = i+d$ and so the paths $R(i, i+d)$ and $R(i+d, i)$ in R is defined as follows.

Let $\mathcal{P}_1$ be a partition of the set $\{1, 2, \cdots, 4p\}$ into pairs that $4p$ and $3p$ are in one pair, so that the set of differences between the pairs is twice $\{1, 2, \cdots, p\}$ (such a partition exists by Lemma 4). For each $d = 1, 2, \cdots, p$, let $\{x_d, y_d\}$ and $\{x'_d, y'_d\}$ be two different pairs in $\mathcal{P}_1$ with difference d, for example, such that $y_d - x_d = d = y'_d - x'_d$. Define

$$\left.\begin{aligned} R(i, i+d) &= (i, i+p+y_d, i+d), \\ R(i+d, i) &= (i+d, i+p+y'_d, i), \end{aligned}\right\} \quad \text{for each} \quad i = 0, 1, \cdots, n-1, \tag{5}$$

where values are taken modulo n. Since $i+p+1 \leqslant i+p+y_d, i+p+y'_d \leqslant i+5p, 1 \leqslant d \leqslant p$, the paths defined by (5) exist in G. Furthermore, since $y_d \neq y'_d$, two paths in (5) are internally disjoint for each $i = 0, 1, \cdots, n-1$.

In order to prove $\pi(G, R) \leqslant 3$, we need to only show that any edge of G is used in paths in (5) at most once since each edge results in a load of two to itself. Note that for any given $d (1 \leqslant d \leqslant p)$, two paths in (5) use four different edge differences $p+y_d, p+x_d, p+x'_d, p+y'_d$ between $p+1$ and $5p$, which are specified by two different pairs $\{x_d, y_d\}$ and $\{x'_d, y'_d\}$ in $\mathcal{P}_1$ with the same difference d. Moreover, for two different d and d', the edge differences specified by the edges in (5) are different. Thus, for a fixed i $(0 \leqslant i \leqslant n-1)$ and any d $(1 \leqslant d \leqslant p)$, the edge of G with difference r $(p+1 \leqslant r \leqslant 5p)$ is used exactly once by these paths of R by our choice of $\mathcal{P}_1$. Owning to all the edges with difference $r(p+1\leqslant r\leqslant 5p-1)$ in (5), we have that $n-r\notin[p+1,5p]$. Since the subgraph induced by all old edges is vertex-transitive, each old edge with difference $r(p+1\leqslant r\leqslant 5p-1)$ is used in paths in (5) at most once.

The edges with difference $5p$ and $n-5p$ are likely to be the same, but is used in paths (5) at most twice. We now prove that each of such edges is used in the paths in (5) at most once. Suppose to the contrary that the edge $(i,i+5p)$ is used in paths in (5) twice. Then $d=p$ and one of y_p, y'_p in (5) is $4p$. Without loss of generality, suppose $y_p=4p$, then $x_p=3p$ since the pair $\{4p,3p\}$ appears in the same pair in $\mathcal{P}_1$. Thus, it is easy to be verified that two paths in (5) passing through the edge $(i,i+5p)$ are $R(i,i+p)=(i,i+5p,i+p)$ and $R(i+5p,i+6p)=(i+5p,i,i+6p)$, respectively, where two pairs of vertices both $\{i,i+p\}$ and $\{i+5p,i+6p\}$ are not adjacent in G. But this is impossible since at most one of the two pairs of vertices is adjacent in G. So the edges with difference $5p$ is used in paths in (5) at most once. Thus we get $\pi(G,R)\leqslant 3$.

Summing up the above discussions, we show $\pi_{8p,10p}\leqslant 3$, which implies that the inequality in (4) holds for $n=10p$. □

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顶点最大度被限制的图的边转发指数

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摘要： 对于给定的 n 阶连通图 G，一个路由选择 R 是指 G 中的 $n(n-1)$ 条路集，其中每个有序点对都有路集中的一条路连接. 图 G 关于 R 的边转发指数 $\pi(G,R)$ 是 R 中路经过一条边的最大条数. 图 G 的边转发指数 $\pi(G)$ 是 G 关于任何路由选择 R 的边转发指数 $\pi(G,R)$ 的最小值. 符号 $\pi_{\Delta,n}$ 表示所有顶点数为 n，最大度至多为 Δ 的图中最小边转发指数. 当 $n\geqslant 4p+1$，且 $n\notin\left[4p+\left\lceil\frac{1}{3}(2p-1)\right\rceil-1,6p\right]$ 时，其中 $p\geqslant 1$，确定了 $\pi_{n-2p,n}$ 的值.

关键词： 转发指数；点转发指数；边转发指数；路由选择