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Multiply-twisted Hypercube with Four or Less Dimensions is Vertex-transitive

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Abstract: P Kulasinghe and S Bettayeb showed that any multiply-twisted hypercube with five or more dimensions is not vertex-transitive. This note shows that any multiply-twisted hypercube with four or less dimensions is vertex-transitive, and that any multiply-twisted hypercube with three or larger dimensions is not edge-transitive.

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§1. Some Lemmas

We follow [1] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ is called to be vertex-transitive if for every pair of vertices $x, y \in V$, there exists an automorphism θ of G such that $y = \theta(x)$, and called to be edge-transitive if for every pair of edges $xy, uv \in V$, there exists an automorphism ϕ of G such that $\phi\{x, y\} = \{u, v\}$. From definition, the following lemma is true clearly.

Lemma 1 Let X be a non-empty subset of $V(G)$. Then for any automorphism θ of G, its restriction on X is an isomorphism between $G[X]$ and $G[\theta(X)]$, where $\theta(X) = \{y \in V(G) : y = \theta(x), x \in X\}$.

lemma 2 A graph G of order n is vertex-transitive if and only if all subgraphs of order $n-1$ of G are isomorphic.

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Proof (⇒) Suppose that G is vertex-transitive and x, y are any two vertices in G. Then, there exists an automorphism θ of G such that $y = \theta(x)$. Thus,

$$\theta(V(G-x)) = \theta(V(G))\backslash\{\theta(x)\} = \theta(V(G))\backslash\{y\} = V(G)\backslash\{y\} = V(G-y).$$

By Lemma 1, $G - x \cong G - y$.

(⇐) We first show that G is regular. Let $V(G) = \{x_1, x_2, \cdots, x_n\}$ with $d_G(x_i) = d_i$ for every $i = 1, 2, \cdots, n$ and $\{d_1, d_2, \cdots, d_n\}$ be a degree sequence of G with $d_1 \le d_2 \le \cdots \le d_n$. For any two vertices x and y in G, since $G - x \cong G - y$, the degree sequences $\{d_1^x, d_2^x, \cdots, d_{n-1}^x\}$ of $G - x$ and $\{d_1^y, d_2^y, \cdots, d_{n-1}^y\}$ of $G - y$ are the same. It is clear that

$$\sum_{i=1}^{n} d_i - 2d_G(x) = \sum_{i=1}^{n-1} d_i^x = \sum_{i=1}^{n-1} d_i^y = \sum_{i=1}^{n} d_i - 2d_G(y).$$

As a result from the above equalities, we have that $d_G(x) = d_G(y)$. By the arbitrary choice of x and y, G is regular.

We now show that G is vertex-transitive. Arbitrarily choose two vertices u and v in G. Then $G - u \cong G - v$ by our hypothesis. Let θ be an isomorphism from $G - u$ to $G - v$, and let

$$\Theta : G \to G,$$
$$x \mapsto \Theta(x) = \begin{cases} \theta(x), & x \in V(G-u) \\ v, & x = u. \end{cases}$$

To prove that Θ is an automorphism of G, it is sufficient to show that

$$xy \in E(G) \Leftrightarrow ((\Theta(x), \Theta(y)) \in E(G), \quad \forall\, x, y \in V(G).$$

Indeed, if $xy \in E(G-u)$, then since θ is an isomorphism from $G-u$ to $G-v$, $(\Theta(x), \Theta(y)) = (\theta(x), \theta(y)) \in E(G-u)$. If $xy \notin E(G-u)$, i.e., $x = u$, $y \in V(G-u)$, then $d_{G-u}(y) = d-1$, where d is the regularity of G. Since $G-u \cong G-v$, $d_{G-v}(\theta(y)) = d-1$. Thus, $\theta(y) \in N_G(v)$(otherwise, $d_{G-v}(\theta(y)) = d$). Thus, $(\Theta(x), \Theta(y)) = (v, \theta(y)) \in E(G)$.

Conversely, if $\Theta(x)\Theta(y) \in E(G)$, then we can similarly show that $xy \in E(G)$. Thus, Θ is an automorphism of G, and $\Theta(u) = v$. By the arbitrary choice of u and v, G is vertex-transitive.

Lemma 3 (Theorem 14.12 in [4]) Every edge-transitive graph with no isolated vertices is vertex-transitive or bipartite.

§2. Main Results

As a topological architecture of interconnection networks, the n-dimensional multiply-twisted hypercube, also called the crossed cube and denoted by CQ_n, was first proposed by Efe [2, 3] as a variation on the hypercube Q_n. In the recent years, it has received much attention of researchers because it has been regarded as an attractive alternative to the hypercube. The crossed cube and the hypercube have many same and different properties [2, 3, 6]. The following lemma is useful to us here.

Lemma 4([2]) For all $n \geq 2$ and all values of l with $4 \leq l \leq 2^n$, CQ_n contains cycles of length l.

However, Kulasinghe and Bettayeb [5] showed that the following result.

Lemma 5 *For all $n > 4$, CQ_n is not vertex-transitive.*

In this section, we consider CQ_n for $n \leq 4$. CQ_1 and CQ_2 are isomorphic to Q_1 and Q_2, which are a complete graph of order two and a cycle of length four, respectively. CQ_3 and CQ_4 are shown in Figure 1 and Figure 2, respectively.

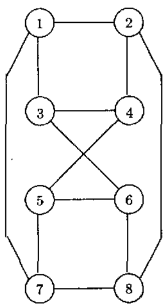

Figure 1 CQ_3

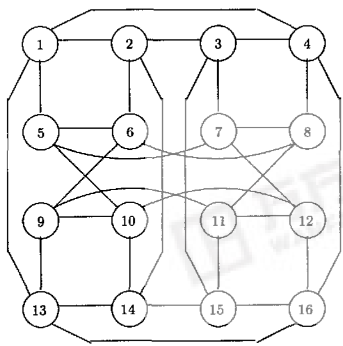

Figure 2 CQ_4

Theorem 1 CQ_n is vertex-transitive for $n \leq 4$.

Proof It is a simple observation that both CQ_1 and CQ_2 are vertex-transitive. It is also easily to be verified that all subgraphs with order 7 of CQ_3 are isomorphic. By Lemma 2, CQ_3 is vertex-transitive.

To prove that CQ_4 is vertex-transitive, by Lemma 2, it is sufficient to show that the subgraph $CQ_4 - k$ is isomorphic to $CQ_4 - 1$ for every $k = 2, 3, \cdots, 16$. To this aim, we express the vertex labels of CQ_4 as the following matrix.

$$A = (a_{ij}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

It is easy to be observed from Figure 2 that each of the subgraphs $CQ_4 - 4$, $CQ_4 - 13$ and $CQ_4 - 16$ is isomorphic to $CQ_4 - 1$. In fact, for every $k = 4, 13, 16$, let σ_k be a permutation on the vertex-set CQ_4 such that $\sigma_k(1) = k$ whose images are specified by the following matrices, respectively,

$$A_4 = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \\ 12 & 11 & 10 & 9 \\ 16 & 15 & 14 & 13 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad A_{16} = \begin{pmatrix} 16 & 15 & 14 & 13 \\ 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

in which $\sigma_k(a_{ij}) = a_{k_{ij}}$, where $A_k = (a_{k_{ij}})$. It is not difficult to be verified that the restriction of σ_k to the vertex-set of $CQ_4 - 1$ is an isomorphism from $CQ_4 - 1$ to $CQ_4 - k$ for every $k = 4, 13, 16$.

Similarly, we can also easily see that each of the subgraphs $CQ_4 - 3$, $CQ_4 - 14$ and $CQ_4 - 15$ is isomorphic to $CQ_4 - 2$, each of $CQ_4 - 8$, $CQ_4 - 9$ and $CQ_4 - 12$ is isomorphic to $CQ_4 - 5$, and each of $CQ_4 - 7$, $CQ_4 - 10$ and $CQ_4 - 11$ is isomorphic to $CQ_4 - 6$.

We now show that each of $CQ_4 - 2$, $CQ_4 - 5$ and $CQ_4 - 6$ is isomorphic to $CQ_4 - 1$. In fact, to do this, for every $k = 2, 5, 6$, we choose the permutation σ_k such that its images are specified by the following matrices, respectively,

$$A_2 = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 6 & 5 & 8 & 7 \\ 10 & 9 & 12 & 11 \\ 14 & 13 & 16 & 15 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 5 & 6 & 9 & 10 \\ 1 & 2 & 13 & 14 \\ 3 & 4 & 15 & 16 \\ 7 & 8 & 11 & 12 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 6 & 5 & 10 & 9 \\ 2 & 1 & 14 & 13 \\ 4 & 3 & 16 & 15 \\ 8 & 7 & 12 & 11 \end{pmatrix}.$$

It is also not difficult to be verified that the restriction of σ_k to the vertex-set of $CQ_4 - 1$ is an isomorphism from $CQ_4 - 1$ to $CQ_4 - k$ for every $k = 2, 5, 6$.

The proof of the theorem is complete.

Theorem 2 For all $n \geq 3$, CQ_n is not edge-transitive.

Proof It is clear that both CQ_1 and CQ_2 are edge-transitive.

We now show that CQ_3 is not edge-transitive. Suppose to the contrary that CQ_3 is edge-transitive. Then for given two edges $a = 12$ and $b = 13$ in CQ_3, there exists an automorphism ϕ of CQ_3 such that $\phi\{1, 3\} = \{1, 2\}$. We can deduce a contradiction.

1° If $\phi(1) = 1, \phi(3) = 2$, then $\phi(4) = 4, \phi(2) = 3$ since the four vertices $1, 3, 4, 2$ form a cycle of length four, Thus the vertices $\phi(1), \phi(3), \phi(4), \phi(2)$ must also form a cycle of length four. Since $4 = \phi(4), 3 = \phi(2), 6, 5$ form a cycle of length four, then $4, 2, \phi^{-1}(6), \phi^{-1}(5)$ form a cycle of length four. As a result, $\phi^{-1}(6) = 3, \phi^{-1}(5) = 1$, that is, $\phi(3) = 6,\ \phi(1) = 5$, which contradicts the hypothesis that ϕ is an automorphism.

2° If $\phi(1) = 2, \phi(3) = 1$, then, similarly, we have that $\phi(4) = 3, \phi(2) = 4$ and obtain a contradiction: $\phi^{-1}(5) = 1, \phi^{-1}(6) = 3$.

We now prove that CQ_4 is not edge-transitive. In fact, let $a = 12$ and $b = 15$ be two edges in

CQ_4. Suppose to the contrary that CQ_4 is edge-transitive. Then there exists an automorphism ϕ of CQ_4 such that $\phi\{1,5\} = \{1,2\}$. We can deduce a contradiction.

1° If $\phi(1) = 1$, $\phi(2) = 5$, then $\phi(6) = 6$, $\phi(5) = 2$. Thus, the vertices 5, 6, 9, 10 form a cycle of length four, and so the vertices $\phi(5) = 2$, $\phi(6) = 6$, $\phi(9)$, $\phi(10)$ also form a cycle of length four, from which we deduce a contradiction: $\phi(9) = 5, \phi(10) = 1$.

2° If $\phi(1) = 5$, $\phi(2) = 1$, then, similarly, we can deduce a contradiction: $\phi(9) = 1$, $\phi(10) = 5$.

For $n \geq 5$, if CQ_n is edge-transitive, then by Lemma 3 CQ_n is vertex-transitive or bipartite, which contradicts Lemma 4 or Lemma 5.

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