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Super Connectivity of Line Graphs and Digraphs

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Abstract The h-super connectivity κ_h and the h-super edge-connectivity λ_h are more refined network reliability indices than the connectivity and the edge-connectivity. This paper shows that for a connected balanced digraph D and its line digraph L, if D is optimally super edge-connected, then $\kappa_1(L) = 2\lambda_1(D)$, and that for a connected graph G and its line graph L, if one of $\kappa_1(L)$ and $\lambda_2(G)$ exists, then $\kappa_1(L) = \lambda_2(G)$. This paper determines that $\kappa_1(B(d,n))$ is equal to $4d-8$ for $n=2$ and $d \geq 4$, and to $4d-4$ for $n \geq 3$ and $d \geq 3$, and that $\kappa_1(K(d,n))$ is equal to $4d-4$ for $d \geq 2$ and $n \geq 2$ except $K(2,2)$. It then follows that $B(d,n)$ and $K(d,n)$ are both super connected for any $d \geq 2$ and $n \geq 1$.

Keywords Line graphs, super connectivity, super edge-connectivity, de Bruijn digraphs, Kautz digraphs
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1 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph or strongly connected digraph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity $\kappa(G)$ or the edge-connectivity $\lambda(G)$ of G is an important measurement for fault-tolerance of the network. In general, the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. It is well known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. A graph (digraph) G is called to be maximally connected if $\kappa(G) = \delta(G)$ and maximally edge-connected if $\lambda(G) = \delta(G)$. One might be interested in more refined indices of reliability. As more refined indices than the connectivity and the edge-connectivity, the super connectivity and the super edge-connectivity were proposed in [1,2]. A graph (digraph) G is super connected if every minimum vertex-cut isolates a vertex of G. A super edge-connected graph (digraph) is similarly defined. Since then it has been found that many well-known graphs are super connected or super edge-connected. In particular, Soneoka[10] showed that the de Bruijn digraph $B(d,n)$ is super edge-connected for any $d \geq 2$ and $n \geq 1$; Fábrega and Fiol[5] proved that the Kautz digraph $K(d,n)$ is super edge-connected for any $d \geq 3$ and $n \geq 2$.

A quite natural problem is that if a (strongly) connected (di)graph G is super connected or super edge-connected, then how many vertices or edges must be removed to disconnect G such that every (strongly) connected component of the resulting graph contains no isolated vertices. This problem results in the concept of the super (edge-) connectivity, introduced in [6].

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For a given nonnegative integer h, a vertex-set S of G is called an h-super vertex-cut if $G-S$ is not (strongly) connected and every (strongly) connected component contains at least $(h+1)$ vertices. In general, h-super vertex-cuts do not always exist. The h-super connectivity $\kappa_h(G)$ is the minimum cardinality of an h-super vertex-cut in G if h-super vertex-cuts exist, and, by convention, is ∞ otherwise. It is clear that $\kappa_0(G) = \kappa(G)$. It is easy to see if $\kappa_1(G) > \kappa(G)$ then G is super connected. Usually, we call a super vertex-cut and the super connectivity for a 1-super vertex-cut and the 1-super connectivity, respectively.

We can similarly define an h-super edge-cut and the h-super edge-connectivity $\lambda_h(G)$ for a (strongly) connected (di)graph G. It is also clear that if $\lambda_1(G) > \lambda(G)$, then G is super edge-connected. A (di)graph G is said to be optimally super edge-connected if $\lambda_1(G)$ exists and there is a minimum super edge-cut F such that $G - F$ has exactly two (strongly) connected components.

For λ_1, it has been extensively studied, (see, for example, [4,9,11,15]), and it is termed the restricted edge-connectivity denoted by the notation λ'. For κ_h, Esfahanian[3] determined $\kappa_1(Q_n) = 2n - 2$, and this was further generalized by Latifi et al.[8] to $\kappa_h(Q_n) = (n-h)2^h$ for $n \geq 3$ with $0 \leq h \leq \lfloor \frac{n}{2} \rfloor$, where Q_n is the n-cube. Up to now, however, we have not seen any results on κ_1 for digraphs in the literature.

In this paper, we consider the relationship between the h-super edge-connectivity of a graph and the super connectivity of its line graph. To be precise, we show that for a connected balanced digraph D and its line digraph L, if D is optimally super edge-connected, then $\kappa_1(L) = 2\lambda_1(D)$, and that for a connected graph G and its line graph L, if one of $\kappa_1(L)$ and $\lambda_2(G)$ exists, then $\kappa_1(L) = \lambda_2(G)$. We determine that $\kappa_1(B(d,n))$ is equal to $4d-8$ for $n = 2$ and $d \geq 4$, and to $4d-4$ for $n \geq 3$ and $d \geq 3$, and that $\kappa_1(K(d,n))$ is equal to $4d-4$ for $d \geq 2$ and $n \geq 2$ except $K(2,2)$. As consequences, we show that $B(d,n)$ and $K(d,n)$ are both super connected for any $d \geq 2$ and $n \geq 1$.

2 Line Digraphs

We follow[13] for graph-theoretical terminologies and notations unless otherwise stated. Let $D = (V,E)$ be a strongly connected digraph, in which parallel edges are not allowed. The line digraph of D, denoted by $L(D)$, or L for short, is a digraph with vertex set $V(L) = E(D)$, and a vertex (x,y) is adjacent to a vertex (w,z) in L if and only if $y = w$ in D. Many properties of line digraphs can be found in [12], one of which is the following lemma.

Lemma 2.1. *Let D be a digraph with order at least two. Then D is strongly connected if and only if the line digraph $L(D)$ is strongly connected.*

For a subset $E' \subseteq E(D)$, we use $D[E']$ to denote the edge-induced subgraph of D by E'. Let L_1 be a subgraph of $L(D)$ and $E_1 = V(L_1)$. Define $D_1 = D[E_1]$.

Lemma 2.2. *Using the above notations, we have that if L_1 is a strongly connected subgraph of L with at least two vertices, then the subgraph $D_1 \subseteq D$ is strongly connected.*

Proof. Assume that x and y are any two vertices of D_1. There is an edge e of D_1 such that x is incident with the edge e. Without loss of generality, we can denote the edge e by (x,z). If $z = y$, then x can reach y by the edge e. If $z \neq y$, then y is incident with another edge e'. Without loss of generality, we can assume the edge $e' = (w,y)$. So, $e = (x,z)$ and $e' = (w,y)$ are two vertices in L_1. Since L_1 is strongly connected, there is a directed path in L_1 from (x,z) to (w,y): $((x,z),(z,z_1),(z_1,z_2),\cdots,(z_k,w),(w,y))$. The corresponding edges $(x,z),(z,z_1),(z_1,z_2),\cdots,(z_k,w),(w,y)$ form a directed walk in D_1 from x to y: $(x,z,z_1,z_2,\cdots,z_k,w,y)$. So x can reach y in D_1.

On the other hand, there is also a directed path in L_1 from (w,y) to (x,z): $((w,y),$

$(y,y_1),(y_1,y_2),\cdots,(y_l,x),(x,z))$, from which we can construct a directed walk in D_1 from y to x: $(y,y_1,y_2,\cdots,y_l,x)$. So y can reach x. Therefore, x and y are strongly connected. So, D_1 is strongly connected.

A digraph D is called balanced if the out-degree $d_D^+(x)$ is equal to the in-degree $d_D^-(x)$ for any vertex x of D. The following property of balanced digraphs is useful, and the proof of which is simple; see Example 1.4.1 in [13].

Lemma 2.3. *If D is a balanced digraph then $|E(X,Y)|=|E(Y,X)|$ for any non-empty proper subsets X and $Y=V(D)\setminus X$, where $E(X,Y)$ denotes the set of edges from X to Y in D.*

Theorem 2.4. *Let D be a connected and balanced digraph and $L=L(D)$.*
(a) If $\kappa_1(L)$ exists, then $\lambda_1(D)\le \frac{1}{2}\kappa_1(L)$.
(b) If D is optimally super edge-connected, then $\kappa_1(L)=2\lambda_1(D)$.

Proof. We first note that D is strongly connected by Lemma 2.3.

(a) Assume $\kappa_1(L)$ exists and that E_0 is a super vertex-cut of L with $|E_0|=\kappa_1(L)$. Then $L-E_0$ is partitioned into several strongly connected components $L_1,L_2,\cdots,L_t$ with $|V(L_i)|\ge 2$ for $i=1,2,\cdots,t$. Let D_i be the edge-induced subgraph of D by $E_i=V(L_i)$ and $V_i=V(D_i)$ for $i=1,2,\cdots,t$. So V_i is nonempty clearly. By Lemma 2.2, D_i is strongly connected. Since $|E_i|=|V(L_i)|\ge 2$ and there is at most one loop at every vertex in D_i, $|V_i|\ge 2$ for $i=1,2,\cdots,t$.

We claim $V_i\cap V_j=\emptyset$ for $i,j\in\{1,2,\cdots,t\}$ and $i\ne j$. Suppose to the contrary that $y\in V_i\cap V_j$ for some $i,j\in\{1,2,\cdots,t\}$ and $i\ne j$. Since D_i is strongly connected and $|V_i|\ge 2$, there are vertices $x,z\in V_i$ such that $(x,y),(y,z)\in E(D_i)$ (maybe $x=z$). Then (x,y) and (y,z), as vertices, are in L_i. Similarly, there are vertices $x',z'\in V_j$ such that $(x',y),(y,z')\in E(D_j)$ (maybe $x'=z'$). Then (x',y) and (y,z'), as vertices, are in L_j. By the construction of L, there is an edge $((x,y),(y,z'))$ from $V(L_i)$ to $V(L_j)$ and an edge $((x',y),(y,z))$ from $V(L_j)$ to $V(L_i)$. So, $L_i\cup L_j$ is strongly connected, a contradiction. Therefore, $V_i\cap V_j=\emptyset$ for any $i,j\in\{1,2,\cdots,t\}$ and $i\ne j$.

Let $\overline{V}_1=V(D)\setminus V_1$. Then the edge-sets $E(V_1,\overline{V}_1)$ and $E(\overline{V}_1,V_1)$ are both super edge-cuts in D and hence $\lambda_1(D)$ exists.

To prove $2\lambda_1(D)\le\kappa_1(L)$, we first show $E(V_1,\overline{V}_1)\cup E(\overline{V}_1,V_1)\subseteq E_0$. Suppose to the contrary that there is an edge $(x,y)\in E(V_1,\overline{V}_1)\cup E(\overline{V}_1,V_1)$ but $(x,y)\notin E_0$. Without loss of generality, we suppose $(x,y)\in E(V_1,\overline{V}_1)$. So as a vertex of L, $(x,y)\in E-E_0$. But E_0 is a minimum super vertex-cut of L, so the vertex (x,y) is in and only in one of the t strongly connected components of $L-E_0$. Without loss of generality, assume that (x,y) is in L_1, then by the definition of D_1, the edge (x,y) is in D_1, contradicting to our assumption that $(x,y)\in E(V_1,\overline{V}_1)$. So $E(V_1,\overline{V}_1)\cup E(\overline{V}_1,V_1)\subseteq E_0$.

Since D is balanced, $|E(V_1,\overline{V}_1)|=|E(\overline{V}_1,V_1)|$ by Lemma 2.3. So we have

$$2\lambda_1(D)\le 2|E(V_1,\overline{V}_1)|=|E(V_1,\overline{V}_1)\cup E(\overline{V}_1,V_1)|\le|E_0|=\kappa_1(L). \tag{1}$$

(b) Assume that D is optimally super edge-connected. Then $\lambda_1(D)$ exists and there is a super edge-cut F of D with $|F|=\lambda_1(D)$ such that $G-F$ has exactly two strongly connected components, say D_1 and D_2. Let $X=V(D_1)$ and $Y=V(D_2)$. Then $|X|\ge 2$ and $|Y|\ge 2$. Without loss of generality, assume $F=E(X,Y)$. Since D is a connected balanced digraph, $E(Y,X)$ is also a super edge-cut and $|E(Y,X)|=|F|=|E(X,Y)|$ by Lemma 2.3. By Lemma 2.1, $L(D_1)$ and $L(D_2)$ are strongly connected and disjoint. Since D_1 and D_2 are strongly connected with at least two vertices, D_1 and D_2 both have at least two edges, which implies that $L(D_1)$ and $L(D_2)$ both contain at least two vertices. So $E(X,Y)\cup E(Y,X)$ is a super vertex-cut of L. It follows that $\kappa_1(L)$ exists and

$$\kappa_1(L)\le|E(X,Y)\cup E(Y,X)|=2|E(X,Y)|=2\lambda_1(D). \tag{2}$$

Combining (1) and (2) yields $\kappa_1(L) = 2\lambda_1(D)$. The theorem then follows.

3 de Bruijn and Kautz Digraphs

For an integer n, the n-th iterated line digraph of D is recursively defined as $L^n(D) = L(L^{n-1}(D))$ with $L^0(D) = D$. In the design of communication networks, line digraphs of some special digraphs are often used as the topology, for they meet many requirements such as small delays and high reliability [12]. The well-known de Bruijn networks and Kautz networks are two of such examples.

For any integers $d \geq 2$ and $n \geq 1$, the de Bruijn digraph $B(d,n)$ can be defined by the $(n-1)$-th iterated line digraph of K_d^+, where K_d^+ is a complete digraph of order d plus a loop at every vertex, that is, $B(d,1) = K_d^+$ and $B(d,n) = L^{n-1}(K_d^+) = L(B(d,n-1))$; the Kautz digraph $K(d,n)$ can be defined by the $(n-1)$-th iterated line digraph of K_{d+1}, where K_{d+1} is a complete digraph of order $d+1$, that is, $K(d,1) = K_{d+1}$ and $K(d,n) = L^{n-1}(K_{d+1}) = L(K(d,n-1))$. In this section, we will determine the super connectivity κ_1 of $B(d,n)$ and $K(d,n)$.

Lemma 3.1[14]**.** *The removal of the efhes incident with the two end-vertices of a pair of symmetric edges of $B(d,n)$ or $K(d,n)$ results in exactly two strongly connected components.*

Lemma 3.2[14]**.** *For any de Bruijn digraph $B(d,n)$ with $n \geq 1$ and $d \geq 2$,*

$$\lambda_1(B(d,n)) = \begin{cases} \infty, & \text{for } n=1 \text{ and } 2 \leq d \leq 3, \text{ or } \quad n=d=2; \\ 2d-4, & \text{for } n=1 \text{ and } d \geq 4; \\ 2d-2, & \text{otherwise.} \end{cases}$$

Theorem 3.3. *For the de Bruijn digraph $B(d,n)$ with $n \geq 1$ and $d \geq 2$,*

$$\kappa_1(B(d,n)) = \begin{cases} \infty, & \text{for } n=1, \text{ or} \\ & \quad n=2 \text{ and } 2 \leq d \leq 3, \text{ or} \\ & \quad n=3 \text{ and } d=2; \\ 4d-8, & \text{for } n=2 \text{ and } d \geq 4; \\ 4d-4, & \text{otherwise.} \end{cases}$$

Proof. Note that $B(d,n) = L(B(d,n-1))$. It is easy to check that $\kappa_1(B(d,n))$ does not exist when d and n take some small values. Combining Theorem 2.4 and Lemma 3.2, we can immediately obtain the theorem if we can prove that $B(d,n)$ is optimally super edge-connected. To this end, by Lemma 3.2, we only need to show that there is a minimum super edge-cut F such that $B(d,n) - F$ contains exactly two strongly connected components when $\lambda_1(B(d,n))$ exists. Choose a pair of symmetric edges with end-vertices $\{x,y\}$. Let $X = \{x,y\}$, $\overline{X} = V(B(d,n)) \setminus X$ and $F = E(X, \overline{X})$. Then $B(d,n) - X$ contains exactly two strongly connected components by Lemma 3.1. Thus, we only need to check that $|F| = \lambda_1(B(d,n))$.

When $n=1$, $B(d,1) = K_d^+$. Thus, when $d \geq 4$, $|F| = |E(X,\overline{X})| = 2d-4 = \lambda_1(B(d,1))$ since there are loops at x and y respectively.

When $n \geq 2$ and $d \geq 3$, since there are no loops at x and y, $|F| = |E(X,\overline{X})| = 2d-2 = \lambda_1(B(d,n))$.

Corollary 3.4. *The de Bruijn digraph $B(d,n)$ is super connected for any $d \geq 2$ and $n \geq 1$.*

Proof. Since $B(d,1)$ is a complete digraph of order d with a loop at every vertex, it is clear that $B(d,1)$ is super connected for any $d \geq 2$. It is easy to see that $B(2,2), B(3,2)$ and $B(2,3)$

are super connected. By Theorem 3.3, for $d \geq 4$ and $n = 2$, $\kappa_1(B(d,2)) = 4d - 8 > d - 1 = \kappa(B(d,n))$, which means that $B(d,2)$ is super connected for $d \geq 4$. Similarly, for $d \geq 2$ and $n \geq 3$ except $B(2,3)$, $\kappa_1(B(d,n)) = 4d - 4 > d - 1 = \kappa(B(d,n))$, which means that $B(d,n)$ is super connected for $d \geq 2$ and $n \geq 3$.

Lemma 3.5[14]**.** *For any Kautz digraph $K(d,n)$ with $d \geq 2$ and $n \geq 1$, $\lambda_1(K(d,n)) = 2d-2$ except $K(2,1)$.*

Note that $K(d,n) = L(K(d,n-1))$. Combining Theorem 2.4, Lemma 3.1 and Lemma 3.5, we immediately obtain a similar result for $K(d,n)$.

Theorem 3.6. *$\kappa_1(K(d,n)) = 4d - 4$ for $d \geq 2, n \geq 2$ except $K(2,2)$.*

Corollary 3.7. *The Kautz digraph $K(d,n)$ is super connected for any $d \geq 2$ and $n \geq 1$.*

4 Line Graphs

Let $G = (V,E)$ be a simple undirected graph. The line graph of G, denoted by $L(G)$, or L for short, is an undirected graph, in which $V(L(G)) = E(G)$ and two distinct vertices are linked by an edge if and only if they are adjacent as edges of G.

It is not always true that for a strongly connected digraph G, if $\lambda_h(G)$ exists, then there is a minimum super edge-cut F of G such that $G - F$ contains exactly two strongly connected components, but it is not hard to see that for an undirected graph, the following result holds.

Lemma 4.1. *For a connected graph G, if $\lambda_h(G)$ exists, then for any minimum h-super edge-cut of G, $G - F$ contains exactly two connected components, whereby, it is optimally h-super edge-connected.*

Clearly, Lemma 2.1 and Lemma 2.2 are valid for an undirected graph. By Lemma 4.1, if $\lambda_2(G)$ exists and F is a minimum 2-super edge-cut of G, then the two connected components of $G - F$ both contain at least three vertices, which implies each connected component contains at least two edges. So their line graphs are connected and both contain at least two vertices. Therefore, $L(F)$ corresponds to a super vertex-cut of $L(G)$. If L_1 is a subgraph of $L(G)$ and connected, then the edge-induced subgraph $G[E_1]$ of G is connected too, where $E_1 = V(L_1)$. In the same manner as the proof of the digraph case, the following theorem can be obtained, and the detailed proof is omitted here.

Theorem 4.2. *If G is a connected undirected graph without parallel edges or loops, then $\kappa_1(L)$ exists if and only if $\lambda_2(G)$ exists. Moreover, if one of $\lambda_2(G)$ and $\kappa_1(L)$ exists, then $\kappa_1(L) = \lambda_2(G)$.*

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