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Cycles in folded hypercubes☆

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Abstract

This work investigates important properties related to cycles of embedding into the folded hypercube FQ_n for $n \geq 2$. The authors observe that FQ_n is bipartite if and only if n is odd, and show that the minimum length of odd cycles is $n+1$ if n is even. The authors further show that every edge of FQ_n lies on a cycle of every even length from 4 to 2^n; if n is even, every edge of FQ_n also lies on a cycle of every odd length from $n+1$ to 2^n-1.

MSC: 05C40

Keywords: Folded hypercube; Pancyclic; Edge-pancyclic; Interconnection networks

1. Introduction

As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties [1,2]. The n-dimensional hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $x_1x_2\cdots x_n$ on the set $\{0, 1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly one bit. We use the symbol $d_H(x, y)$ to denote the Hamming distance between two vertices u and v in Q_n, that is, the number of different bits in the corresponding strings of both vertices. Clearly, $d_H(x, y) = d_{Q_n}(x, y)$, where $d_{Q_n}(x, y)$ denotes the distance between two vertices x and y in Q_n, i.e., the length of the shortest xy-path in Q_n.

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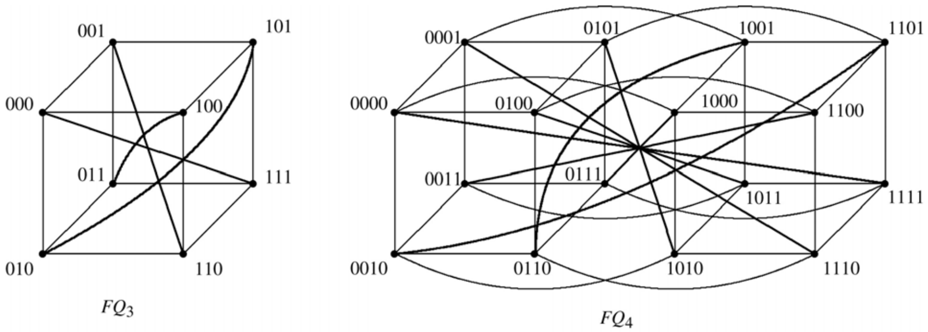


Fig. 1. FQ_3 and FQ_4. (Thick lines represent the complementary edges.)

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [3], is a graph obtained from the hypercube Q_n by adding an edge, called a complementary edge, between any two vertices $x = (x_1x_2 \cdots x_n)$ and $\bar{x} = (\bar{x}_1\bar{x}_2 \ldots \bar{x}_n)$, where $\bar{x}_i = 1 - x_i$. The graphs shown in Fig. 1 are FQ_3 and FQ_4, respectively.

It has been shown that FQ_n is $(n+1)$-regular $(n+1)$-connected. FQ_n is also superior to Q_n in some properties. For example, it has diameter $\lceil \frac{n}{2} \rceil$, about half the diameter of Q_n [3]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n. In particular, there are $n+1$ internally disjoint paths of length at most $\lceil \frac{n}{2} \rceil + 1$ between any pair of vertices in FQ_n, the deletion of less than $\lceil \frac{n}{2} \rceil - 2$ vertices or edges does not increase the diameter of FQ_n, and the deletion of up to n vertices or edges increases it by at most one [4,5]. These properties mean that interconnection networks modelled by FQ_n are extremely robust. As a result, the study of the folded hypercube has recently attracted the attention of many researchers [6–8].

In [9], Li et al. proved that every edge of Q_n $(n \geq 2)$ lies on a cycle of every even length from 4 to 2^n. In this work, we first observe that FQ_n is bipartite if and only if n is odd, and show that the minimum length of even cycles in FQ_n is 4 and that the minimum length of odd cycles is $n+1$ if n is even. Then, using the Li et al. result, we further show that every edge of FQ_n lies on a cycle of every even length from 4 to 2^n; moreover, every edge of FQ_n also lies on a cycle of odd length from $n+1$ to $2^n - 1$ if n is even.

The proofs of our main results are in Section 3. In the next section, we explore some new topological properties of FQ_n.

2. New properties of FQ_n

A vertex u in Q_n is said to be odd or even if the sum of its bits is odd or even. Let $X = \{u : u \text{ is odd}\}$ and $Y = \{u : u \text{ is even}\}$. Then $\{X, Y\}$ is a bipartition of Q_n, clearly. Conversely, if two vertices in Q_n are in the same part of a bipartition of Q_n, then they have the same parity.

Theorem 2.1. *FQ_n is a bipartite graph if and only if n is odd.*

Proof. Since FQ_n is obtained from Q_n by adding 2^{n-1} complementary edges, to prove the theorem, it is sufficient to consider complementary edges. Let $\{X, Y\}$ be a bipartition of Q_n. Since any vertex u and its complement $\bar{u}$ in Q_n have different parity if and only if n is odd and, hence, any complementary edge in FQ_n joins two vertices in different parts of $\{X, Y\}$, it follows that FQ_n is a bipartite graph if and only if n is odd. The theorem follows. □

Theorem 2.2. *The length of any cycle in FQ_n that contains exactly two complementary edges is even.*

Proof. Let $C = (v, \bar{v}, v_1, \ldots, v_m, \bar{u}, u, u_1, \ldots, u_n, v)$ be a cycle in FQ_n that contains exactly two complementary edges $e_1 = (u, \bar{u})$ and $e_2 = (v, \bar{v})$. Clearly, $v \neq u$ and, hence, $\bar{v} \neq \bar{u}$ since $e_1 \neq e_2$.

If FQ_n is bipartite, then we are done. We now assume FQ_n is not bipartite. By Theorem 2.1, n is even and, hence, u and $\bar{u}$ have the same parity. Let $\{X, Y\}$ be a bipartition of Q_n. Then, u and $\bar{u}$ are in the same part of $\{X, Y\}$. Similarly, v and $\bar{v}$ are also in the same part of $\{X, Y\}$. Since C contains exactly two complementary edges, all of the other edges in C are in Q_n. If $u, \bar{u}, v, \bar{v}$ are in the same part, then the lengths of the sections $(\bar{v}, v_1, \ldots, v_m, \bar{u})$ and $(u, u_1, \ldots, u_n, v)$ are even. If $u, \bar{u}, v, \bar{v}$ are not in the same part, then the lengths of the two sections are odd. It follows that the length of C is even in both cases. □

Theorem 2.3. *The length of any cycle in FQ_n that contains exactly one complementary edge is at least $n+1$. Moreover, any complementary edge and any vertex lie on a common cycle of length $n+1$ containing the unique complementary edge in FQ_n.*

Proof. Assume that $e = (u, \bar{u})$ is a complementary edge in FQ_n and that C is a cycle in FQ_n containing exactly one complementary edge e. Then $P = C - e$ is a $u\bar{u}$-path in Q_n. Clearly, $d_{Q_n}(u, \bar{u}) = n$ and, hence, the length of any $u\bar{u}$-path in Q_n is at least n. Thus, the length of P is at least n and so the length of the cycle C is at least $n + 1$.

Let P' be a shortest $u\bar{u}$-path in Q_n and let v be any vertex in FQ_n. If $v \in \{u, \bar{u}\}$, then $P' + e$ is a cycle of length exactly $n + 1$ and contains the unique complementary edge e in FQ_n. We now assume $v \notin \{u, \bar{u}\}$. Since $d_H(u, v) + d_H(\bar{u}, v) = n$, there are a shortest vu-path R and a shortest $v\bar{u}$-path R' in Q_n. Then the sum of their lengths is equal to n. Clearly, $R \cup R' + u\bar{u}$ is a cycle of length $n + 1$ and contains the vertex v and the unique complementary edge. The theorem follows. □

Theorem 2.4. *Every shortest path between any two distinct vertices in FQ_n contains at most one complementary edge.*

Proof. Suppose that $P = (u, u_1, \ldots, u_i, \bar{u}_i, u_{i+1}, \ldots, u_j, \bar{u}_j, \ldots, v)$ is a shortest uv-path in FQ_n containing more than one complementary edge, where $(u_i, \bar{u}_i)$ and $(u_j, \bar{u}_j)$ are the two complementary edges that first occur in P in the order from u to v. Then the section $(\bar{u}_i, \ldots, u_j)$ of P contains no complementary edge and is a shortest $\bar{u}_i u_j$-path. Since $d_H(u_i, \bar{u}_j) = d_H(\bar{u}_i, u_j)$, the section $(u_i, \bar{u}_i, u_{i+1}, \ldots, u_j, \bar{u}_j)$ of P is not a shortest $u_i\bar{u}_j$-path, whereby P is not the shortest path, a contradiction. The theorem follows. □

Theorem 2.5. *Let u and v be two vertices in FQ_n ($n \geq 2$). If $d_H(u, v) \leq \lfloor \frac{n}{2} \rfloor$, then any shortest uv-path in FQ_n contains no complementary edges. If $d_H(u, v) > \lceil \frac{n}{2} \rceil$, then any shortest uv-path in FQ_n contains exactly one complementary edge.*

Proof. Suppose $d_H(u, v) = h \leq \lfloor \frac{n}{2} \rfloor$; then $v \neq \bar{u}$ since $d_H(u, \bar{u}) = n$. Suppose to the contrary that there is a shortest uv-path P in FQ_n containing a complementary edge. By Theorem 2.4, P contains exactly one complementary edge, say $e = (x, \bar{x})$. Let $d_H(u, x) = i$ and $d_H(\bar{x}, v) = m$. Then $d_{FQ_n}(u, x) = d_H(u, x) = i < \lfloor \frac{n}{2} \rfloor$, $d_{FQ_n}(\bar{x}, v) = d_H(\bar{x}, v) = m < \lfloor \frac{n}{2} \rfloor$ and $i + m + 1 = h \leq \lfloor \frac{n}{2} \rfloor$. Since $d_H(u, x) + d_H(u, \bar{x}) = n$, we have $d_H(u, \bar{x}) = n - i$. Also since $d_H(u, v) + d_H(\bar{x}, v) \geq d_H(u, \bar{x})$, we can deduce a contradiction as follows:

$$\left\lfloor \frac{n}{2} \right\rfloor \geq d_H(u,v) \geq d_H(u,\bar{x}) - d_H(\bar{x},v) = n-i-m > n-h \geq \left\lceil \frac{n}{2} \right\rceil.$$

Thus, P contains no complementary edges.

If $d_H(u,v) > \lceil \frac{n}{2} \rceil$, since $d_{FQ_n}(u,v) \leq \lceil \frac{n}{2} \rceil$, any shortest path between u and v must contain a complementary edge. By Theorem 2.4, the theorem follows. □

Remarks. If n is odd and $d_H(u,v) = \lceil \frac{n}{2} \rceil$, then there are a shortest uv-path containing no complementary edge and a shortest uv-path containing exactly one complementary edge in FQ_n.

Theorem 2.6. *If FQ_n contains an odd cycle, then any shortest odd cycle contains exactly one complementary edge and the length is $n+1$.*

Proof. If FQ_n contains an odd cycle, then by Theorem 2.1 n is even and any odd cycle must contain a complementary edge. Moreover, any odd cycle must contain at least one edge that is not a complementary edge since any two complementary edges are not adjacent. Assume that $C = (v_0, v_1, \ldots, v_{2l}, v_0)$ is a shortest odd cycle of length $2l+1$ in FQ_n, where the edge $e_l = (v_l, v_{l+1})$ is not a complementary edge. Consider two paths in C, $P_1 = (v_0, v_1, \ldots, v_l)$ and $P_2 = (v_{l+1}, \ldots, v_{2l}, v_0)$. If P_1 is not shortest in FQ_n and P' is, then the length of any cycle C' in $P_1 \cup P'$ is less than $2l+1$, so C' is an even cycle. Thus, the lengths of P_1 and P' have the same parity. By a similar argument, we can prove that the lengths of P' and $P_2 + e_l$ have the same parity. It follows that the lengths of P_1 and $P_2 + e_l$ have the same parity, which means that C is an even cycle, a contradiction. Therefore, P_1 is a shortest path. Similarly, P_2 is also a shortest path. By Theorem 2.4, C contains at most two complementary edges. Since C is an odd cycle, C contains exactly one complementary edge by Theorem 2.2.

By Theorem 2.3, the length of C is at least $n+1$. Also by Theorem 2.3, any complementary edge lies on a cycle of length $n+1$ in FQ_n. This shows that the minimum length of any odd cycle is exactly $n+1$. □

3. Embedding cycles into FQ_n

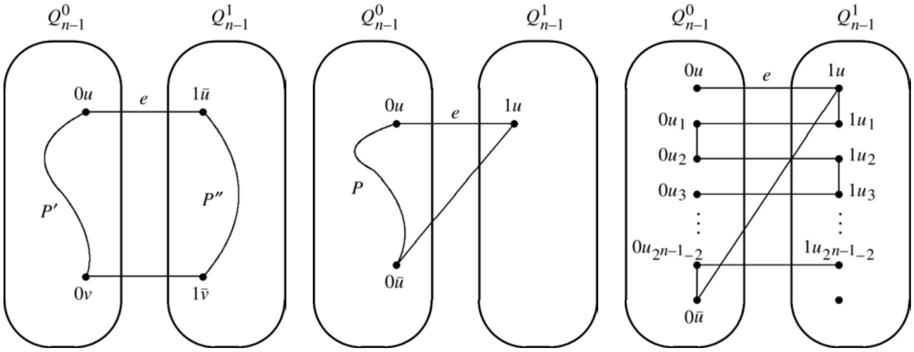
For convenience, we use Q^0_{n-1} and Q^1_{n-1} to denote the two $(n-1)$-subcubes of Q_n induced by the vertices with the leftmost bit 0 and 1, respectively. Then, all of 2^{n-1} complementary edges of FQ_n join $0u$ and $1\bar{u}$ between Q^0_{n-1} and Q^1_{n-1} for any $u \in V(Q_{n-1})$.

Lemma 3.1 (*Li et al. [9]*). *If $n \geq 2$, then*

(a) *for any two different vertices x and y in Q_n there exists an xy-path of length l with $d_{Q_n}(x,y) \leq l \leq 2^n - 1$, where l and $d_{Q_n}(x,y)$ have the same parity;*
(b) *every edge of Q_n lies on a cycle of every even length from 4 to 2^n.*

Theorem 3.2. *Every complementary edge in FQ_n lies on a cycle of every even length from 4 to 2^n for $n \geq 2$.*

Proof. Let e be a complementary edge in FQ_n. Without loss of generality, we may assume that $e = (0u, 1\bar{u})$ with $0u \in V(Q^0_{n-1})$ and $1\bar{u} \in V(Q^1_{n-1})$. Choose an edge $(0u, 0v)$ in Q^0_{n-1} and a corresponding edge $(1\bar{u}, 1\bar{v})$ in Q^1_{n-1}. We want to prove that the edge $e = (0u, 1\bar{u})$ lies on a cycle of length l in FQ_n with $4 \leq l \leq 2^n$ and l is even.



(a) Illustration for Theorem 3.2.

(b) Illustration for case 2 of Theorem 3.3. $n+1 \le l \le 2^{n-1}+1$

(c) Illustration for case 2 of Theorem 3.3. $2^{n-1}+3 \le l \le 2^n - 1$

Fig. 2. Illustrations for the proofs in Section 3.

We can express $l = l' + l'' + 2$ where $1 \le l' \le 2^{n-1} - 1$, $1 \le l'' \le 2^{n-1} - 1$ and both l' and l'' are odd integers. By Lemma 3.1(a), there exist a path P' of length l' in Q^0_{n-1} joining $0u$ and $0v$ and a path P'' of length l'' in Q^1_{n-1} joining $1\bar{u}$ and $1\bar{v}$. Then $P' + (0v, 1\bar{v}) + P'' + (1\bar{u}, 0u)$ is a cycle containing the edge e in FQ_n, whose length is equal to l (see Fig. 2(a)). The theorem follows. □

Theorem 3.3. *If n is odd, then every edge of FQ_n lies on a cycle of every even length from 4 to 2^n for $n \ge 3$. If n is even, then every edge of FQ_n lies on a cycle of every even length from 4 to 2^n and every odd length from $n+1$ to $2^n - 1$.*

Proof. If n is odd, then FQ_n is a bipartite graph by Theorem 2.1. By Lemma 3.1(b) and Theorem 3.2, the first assertion follows. We now prove the second assertion.

If n is even, then FQ_n is not a bipartite graph by Theorem 2.1. By Lemma 3.1(b) and Theorem 3.2, every edge lies on a cycle of every even length from 4 to 2^n inclusive.

We now show that every edge of FQ_n lies on a cycle of any odd length l with $n+1 \le l \le 2^n - 1$ when n is even. Let e be any edge in FQ_n. There are two cases.

Case 1. The edge $e = (u, \bar{u})$ is a complementary edge. In this case, u and $\bar{u}$ belong to the same part of a bipartition of Q_n, $d_{Q_n}(u, \bar{u}) = n$ and $\frac{1}{2}((l-1) - n) = \frac{1}{2}((l-1) - d_{Q_n}(u, \bar{u}))$ is an integer. By Lemma 3.1(a), there is a path of length $l-1$ joining u and $\bar{u}$ with $n \le (l-1) \le 2^n - 2$. Choose P to be such a path of even length $(l-1)$ with $n \le (l-1) \le 2^n - 2$ in Q_n. Thus, $P + e$ is an odd cycle of length l in FQ_n with $n+1 \le l \le 2^n - 1$.

Case 2. The edge $e = (u, v)$ is not a complementary edge. In this case, u and v belong to different parts of a bipartition of Q_n. Without loss of generality, we may assume $e = (0u, 1u)$; then $(1u, 0\bar{u})$ is a complementary edge in FQ_n and $d_H(0u, 0\bar{u}) = n - 1$.

Since l and $(n-1)$ are odd, $\frac{1}{2}((l-2) - (n-1)) = \frac{1}{2}((l-2) - d_{Q_n}(0u, 0\bar{u}))$ is an integer. By Lemma 3.1(a), there is a path P of odd length $(l-2)$ joining $0u$ and $0\bar{u}$ in Q^0_{n-1} with $n-1 \le (l-2) \le 2^{n-1} - 1$. In particular, we use P_0 to denote such a path of length $2^{n-1} - 1$ joining $0u$ and $0\bar{u}$ in Q^0_{n-1}.

If $n+1 \le l \le 2^{n-1} + 1$, then $P + (0u, 1u, 0\bar{u})$ is an odd cycle of length l containing the edge e in FQ_n with $n+1 \le l \le 2^{n-1} + 1$ (see Fig. 2(b)).

If $2^{n-1} + 3 \le l \le 2^n - 1$, let $l' = l - 2^{n-1} - 1$; then $2 \le l' \le 2^{n-1} - 2$. We may assume $P_0 = (0u, 0u_1, 0u_2, \ldots, 0u_{2^{n-1}-2}, 0\bar{u})$. For every edge $(0u_i, 0u_{i+1})$ in Q^0_{n-1}, there is a corresponding edge $(1u_i, 1u_{i+1})$ in Q^1_{n-1}. Thus $C = (0u, 0u_1, 1u_1, 1u_2, 0u_2, 0u_3, 1u_3, \ldots, 0u_{l'-1}, 1u_{l'-1}, 1u_{l'}, 0u_{l'}, 0u_{l'+1}, 0u_{l'+2}, \ldots, 0u_{2^{n-1}-2}, 0\bar{u}, 1u, 0u)$ is a cycle containing the edge e in FQ_n whose length is equal to $2^{n-1} + 1 + l' = l$ (see Fig. 2(c)).

The theorem follows. □

4. Conclusion

In this work, we obtain some new properties of FQ_n. We show that every edge of FQ_n lies on a cycle of every even length from 4 to 2^n for $n \geq 2$; furthermore, every edge of FQ_n also lies on a cycle of every odd length from $n + 1$ to $2^n - 1$ if n is even. The result that an odd cycle can be embedded into FQ_n when n is even shows that FQ_n is superior to Q_n in view of the cycle embedding capability.

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