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Paths in Möbius cubes and crossed cubes ☆

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Abstract

The Möbius cube MQ_n and the crossed cube CQ_n are two important variants of the hypercube Q_n. This paper shows that for any two different vertices u and v in $G \in \{MQ_n, CQ_n\}$ with $n \geqslant 3$, there exists a uv-path of every length from $d_G(u, v) + 2$ to $2^n - 1$ except for a shortest uv-path, where $d_G(u, v)$ is the distance between u and v in G. This result improves some known results.

Keywords: Möbius cubes; Crossed cubes; Hypercubes; Path; Hamilton-connected

1. Introduction

Let $G = (V, E)$ be a graph. For two vertices $u, v \in V$, a path joining u and v is called a uv-path, and the distance between u and v is the length of a shortest uv-path, denoted by $d_G(u, v)$. The diameter $D(G)$ of G is the maximal value of distances between all pairs of vertices in G. A graph G is Hamilton-connected if there is a uv-path containing all vertices for every pair of vertices u and v in G.

The hypercube network Q_n has been proved to be one of the most popular interconnection networks. The Möbius cube MQ_n and the crossed cube CQ_n are two important variants of Q_n. Because of many attractive features superior to the hypercube, such as

$$D(MQ_n^1) = \left\lceil \frac{n+1}{2} \right\rceil, \qquad D(MQ_n^0) = \left\lceil \frac{n+2}{2} \right\rceil,$$

and

$$D(CQ_n) = \left\lceil \frac{n+1}{2} \right\rceil,$$

the Möbius cube and the crossed cube have been extensively investigated in the literature (see, for example, [1–21]). In particular, Fan [9] and Huang et al. [14], independently, showed that MQ_n is Hamilton-connected and contains a cycle of every length from 4 to 2^n, Fan et al. [10] and Ma and Xu [18], independently, showed that every edge of CQ_n lies on a cycle of every length from 4 to 2^n. In this paper, we improve these results by showing the following theorem.

Theorem. *If $n \geqslant 3$ then for any two different vertices u and v in $G \in \{MQ_n, CQ_n\}$, there exists a uv-path of every length from $d_G(u, v) + 2$ to $2^n - 1$.*

When our manuscript was submitted to Information Processing Letters, one of the referees told us there had been one manuscript submitted to ICPP titled "Complete path embeddings in hypercubes and crossed

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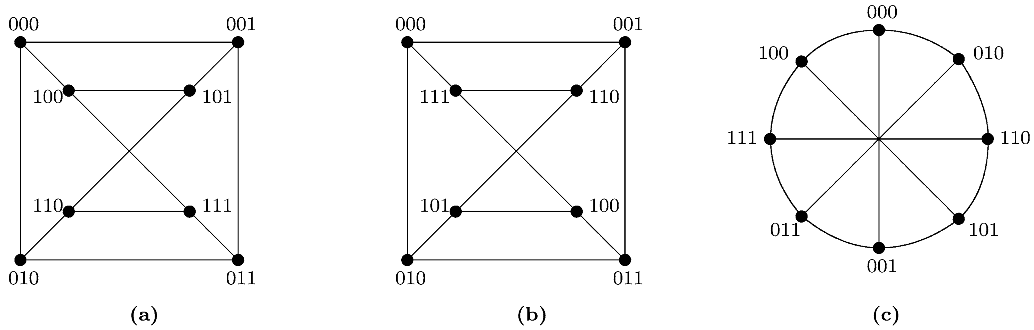


Fig. 1. (a) MQ_3^0, (b) MQ_3^1, (c) a symmetric drawing of MQ_3^0.

cubes" by Fan et al., in which the result on CQ_n has been obtained. However, we have not read the manuscript yet so far. In despite of the different structures of MQ_n and CQ_n, we find that the ways used in the proofs of our results on them are similar. Therefore, we give here the proof of the theorem on MQ_n in detail.

2. Möbius cubes

The n-dimensional Möbius cube MQ_n, proposed first by Cull and Larson [3–5], has 2^n vertices. Each vertex is an n-string on $\{0, 1\}$. A vertex $X = x_1x_2 \ldots x_n$ connects to a vertex Y_i ($i = 1, 2, \ldots, n$) by an edge if Y_i satisfies one of the following rules:

$$Y_i = \begin{cases} x_1 \ldots x_{i-1}\bar{x}_i x_{i+1} \ldots x_n & \text{if } x_{i-1} = 0, \\ x_1 \ldots x_{i-1}\bar{x}_i \bar{x}_{i+1} \ldots \bar{x}_n & \text{if } x_{i-1} = 1, \end{cases}$$

where $\bar{x}_i$ is the complement of x_i in $\{0, 1\}$.

More informally, a vertex X connects to a neighbor that differs in x_i if $x_{i-1} = 0$, and to a neighbor that differs in x_i through x_n if $x_{i-1} = 1$. The connection between X and Y_i is undefined when $i = 1$, so we can assume x_0 is either equal to 0 or equal to 1, which gives us slightly different network topologies. If $x_0 = 0$, the network is denoted by MQ_n^0; and if $x_0 = 1$, the network is denoted by MQ_n^1. Fig. 1 shows MQ_3^0 and MQ_3^1, where (c) is a symmetric drawing of MQ_3^0.

According to the above definition, it is not difficult to see that MQ_n^0 (resp. MQ_n^1) can be recursively constructed from MQ_{n-1}^0 and MQ_{n-1}^1 by adding 2^{n-1} edges. For any vertex $X = x_1x_2 \ldots x_{n-1}$ in MQ_{n-1}^0 or MQ_{n-1}^1, we construct a new vertex $X' = x_1'x_2' \ldots x_n'$, where $x_{i+1}' = x_i$ for $i = 1, 2, \ldots, n-1$, then assigning $x_1' = 0$ if X is in MQ_{n-1}^0, or $x_1' = 1$ if X is in MQ_{n-1}^1. So MQ_n^0 can be constructed by connecting all pairs of vertices that differ only in the first bit, and MQ_n^1 can be constructed by connecting all pairs of vertices that differ in the first through the nth bits. For short, we denote $MQ_n = L \oplus R$, where $L \cong MQ_{n-1}^0$ and $R \cong MQ_{n-1}^1$.

Similarly, it has been shown by Efe [7] that the crossed cube CQ_n can also express $CQ_n = L \oplus R$, where $L \cong CQ_{n-1}^0$ and $R \cong CQ_{n-1}^1$.

Lemma. *Let u and v be two vertices in $G \in \{MQ_n, CQ_n\}$ with $n \geqslant 3$. Then $d_G(u, v) = d_L(u, v)$ if both u and v are in L, and $d_G(u, v) = d_R(u, v)$ if both u and v are in R.*

Proof. Notice that the first bits of the vertices in L (or R) are 0 (or 1). An exact minimal routing algorithm given in [5] on MQ_n and [7] on CQ_n can determine a shortest path between u and v, in which the first bits of all vertices are 0 (resp. 1) if u and v are in L (resp. R). The lemma follows. □

3. Proof of theorem

We prove the theorem by induction on $n \geqslant 3$.

For $n = 3$, since each graph G in $\{MQ_3^0, MQ_3^1, CQ_3\}$ is isomorphic to the vertex symmetric graph in Fig. 1, we only need to prove that for the vertex $u = 000$ and $v \in \{100, 111, 011, 001\}$ in G, there exists a uv-path of length ℓ with $d_G(u, v) + 2 \leqslant \ell \leqslant 7$. All uv-paths of required length are constructed as follows.

The paths of different lengths between 000 and 100 with distance one are listed as follows:

$P_3 = \langle 000, 001, 101, 100\rangle$,

$P_4 = \langle 000, 001, 011, 111, 100\rangle$,

$P_5 = \langle 000, 001, 101, 110, 111, 100\rangle$,

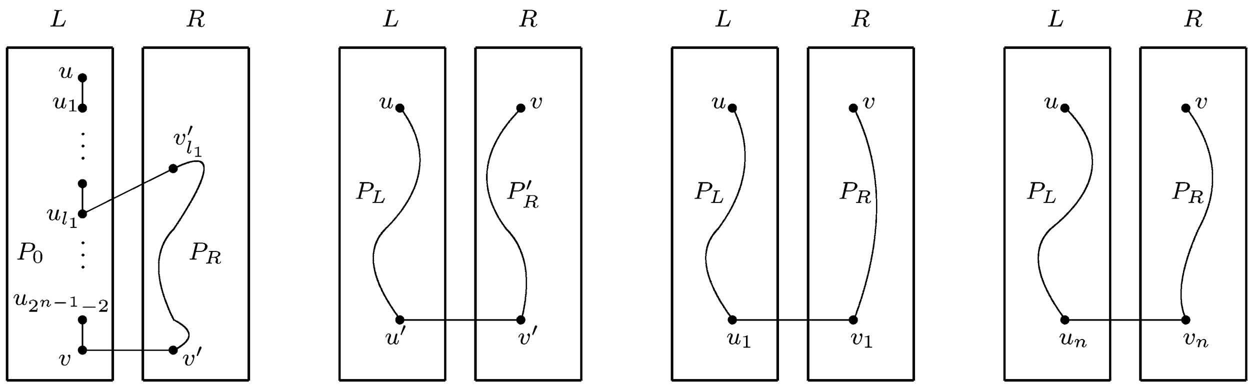
$P_6 = \langle 000, 010, 011, 111, 110, 101, 100\rangle$,

$P_7 = \langle 000, 010, 110, 101, 001, 011, 111, 100\rangle$.

The paths of different lengths between 000 and 001 with distance one are listed as follows:

$P_3 = \langle 000, 010, 011, 001\rangle$,

$P_4 = \langle 000, 010, 110, 101, 001\rangle$,



(a) Case 1 $2^{n-1} \le \ell \le 2^n - 1$

(b) case 2 $d(u,v) \ge 2$ $d(u,v)+2 \le \ell \le 2^{n-1}$

(c) case 2 $d(u,v) \ge 2$ $2^{n-1}+1 \le \ell \le 2^n - 1$

(d) case 2 $d(u,v) = 1$ $5 \le \ell \le 2^n - 1$

Fig. 2. Illustrations for the proof of theorem.

$P_5 = \langle 000, 010, 110, 111, 011, 001\rangle$,

$P_6 = \langle 000, 010, 011, 111, 100, 101, 001\rangle$,

$P_7 = \langle 000, 010, 110, 101, 100, 111, 011, 001\rangle$.

The paths of different lengths between 000 and 111 with distance two are listed as follows:

$P_4 = \langle 000, 100, 101, 110, 111\rangle$,

$P_5 = \langle 000, 100, 101, 001, 011, 111\rangle$,

$P_6 = \langle 000, 010, 110, 101, 001, 011, 111\rangle$,

$P_7 = \langle 000, 100, 101, 001, 011, 010, 110, 111\rangle$.

The paths of different lengths between 000 and 011 with distance two are listed as follows:

$P_4 = \langle 000, 010, 110, 111, 011\rangle$,

$P_5 = \langle 000, 010, 110, 101, 001, 011\rangle$,

$P_6 = \langle 000, 010, 110, 101, 100, 111, 011\rangle$,

$P_7 = \langle 000, 010, 110, 111, 100, 101, 001, 011\rangle$.

Assume that the conclusion holds for any k with $3 \leqslant k < n$. Let u and v be any two distinct vertices in $G = L \oplus R$. We complete the proof by the following two cases.

Case 1. Both u and v are in L or R. Without loss of generality, we may assume u and v are in L. By lemma, we have $d_G(u,v) = d_L(u,v)$.

For $d_L(u,v) + 2 \leqslant \ell \leqslant 2^{n-1} - 1$, by the induction hypothesis, there exists a uv-path of length ℓ in $L \subset G$.

Suppose that $2^{n-1} \leqslant \ell \leqslant 2^n - 1$. We can write $\ell = \ell_1 + \ell_2 + 2$ where $1 \leqslant \ell_1 \leqslant 2^{n-1} - 2$ and $2^{n-1} - 3 \leqslant \ell_2 \leqslant 2^{n-1} - 1$. Let $P_0 = \langle u, u_1, u_2, \ldots, u_{2^{n-1}-2}, v\rangle$ be a uv-path of length $2^{n-1} - 1$ in L. Let u'_i be the neighbor of u_i in R and v' be the neighbor of v in R. Since $2^{n-1} - 3 > D(R) + 2$, by the induction hypothesis, there is a $u'_{l_1}v'$-path P_R of length l_2 in R. Hence $P = \langle u, u_1, u_2, \ldots, u_{l_1}, u'_{l_1}, P_R, v', v\rangle$ is a uv-path of length ℓ in G (see Fig. 2(a)).

Case 2. $u \in L$ and $v \in R$.

We first assume $d_G(u,v) \geqslant 2$. There is a uv-path P_0 of length $d_G(u,v)$ in G. Then there is an edge $u'v'$ in P_0 with $u' \in L$ and $v' \in R$. Let $P(u,u')$ be the segment of P_0 between u and u'. Let $P(v',v)$ be the segment of P_0 between v' and v. It is clear that $P(u,u')$ is a shortest path between u and u', and $P(v',v)$ is a shortest path between v' and v. By lemma, we may assume $P(u,u') \subset L$ and $P(v',v) \subset R$. We use ℓ' and ℓ'' to denote the lengths of $P(u,u')$ and $P(v',v)$, respectively. Noting that $d_G(u,v) = \ell' + \ell'' + 1$ and $d_G(u,v) \geqslant 2$, we have $\ell' \geqslant 1$ or $\ell'' \geqslant 1$. We may assume $\ell' \geqslant 1$.

For $d_G(u,v) + 2 \leqslant \ell \leqslant 2^{n-1}$, we can write $\ell = \ell_1 + \ell'' + 1$ where $d_G(u,u') + 2 \leqslant \ell_1 \leqslant 2^{n-1} - 1$. By the induction hypothesis, there exists a uu'-path P_L of length ℓ_1 in L. Then $P\langle u, P_L, u', v', P'_R, v\rangle$ is a uv-path of length ℓ in G (see Fig. 2(b)).

For $2^{n-1} + 1 \leqslant \ell \leqslant 2^n - 1$, we can write $\ell = \ell_1 + \ell_2 + 1$ where $D(L) + 2 \leqslant \ell_1 \leqslant 2^{n-1} - 1$, $D(R) + 2 \leqslant \ell_2 \leqslant 2^{n-1} - 1$. Choose $u_1 \in L$ such that $u_1 \neq u$ and the neighbor v_1 of u_1 in R is different from v. By the induction hypothesis, there exist a uu_1-path P_L of length ℓ_1 in L and a v_1v-path P_R of length ℓ_2 in R. Then $P\langle u, P_L, u_1, v_1, P_R, v\rangle$ is a uv-path of length ℓ in G (see Fig. 2(c)).

We now assume $d_G(u,v) = 1$ and, without loss of generality, assume $u = 0u_2u_3 \ldots u_n \in L$ and $v = 1v_2v_3 \ldots v_n \in R$. Only in this case, we construct a uv-path of length ℓ depending on $G = MQ_n$ or $G = CQ_n$.

Assume $G = MQ_n$. For $5 \leqslant \ell \leqslant 2^n - 1$, we can write $\ell = \ell_1 + \ell_2 + 1$ where $3 \leqslant \ell_1 \leqslant 2^{n-1} - 1$ and $\ell_2 = 1$ or $3 \leqslant \ell_1 \leqslant 2^{n-1} - 1$ and $3 \leqslant \ell_2 \leqslant 2^{n-1} - 1$. Let $u_n = 0u_2 \ldots u_{n-1}\bar{u}_n$ be a neighbor of u in L and $v_n = 1v_2 \ldots v_{n-1}\bar{v}_n$ be a neighbor of v in R. It is clear that $u_n v_n \in E(MQ_n)$ because $uv \in E(MQ_n)$. By the induction hypothesis, there exist a uu_n-path P_L of length ℓ_1 in L and a $v_n v$-path P_R of length ℓ_2 in R. Then $P = \langle u, P_L, u_n, v_n, P_R, v\rangle$ is a uv-path of length ℓ in MQ_n (see Fig. 2(d)).

For $\ell = 3, 4$, noting $v = 1u_2u_3 \ldots u_n$ if $G = MQ_n^0$ and $v = 1\bar{u}_2\bar{u}_3 \ldots \bar{u}_n$ if $G = MQ_n^1$, then

$$P = \langle u = 0u_2u_3 \ldots u_n, 0u_2u_3 \ldots u_{n-1}\bar{u}_n, \\ 1u_2u_3 \ldots u_{n-1}\bar{u}_n, 1u_2u_3 \ldots u_n = v\rangle$$

and

$$P = \langle u = 0u_2u_3 \ldots u_n, 0u_2u_3 \ldots u_{n-1}\bar{u}_n, \\ 1\bar{u}_2\bar{u}_3 \ldots \bar{u}_{n-1}u_n, 1\bar{u}_2\bar{u}_3 \ldots \bar{u}_n = v\rangle$$

are uv-paths of length 3 in MQ_n^0 and MQ_n^1, respectively.

$$P = \begin{cases} \langle u = 00u_3 \ldots u_n, 01u_3 \ldots u_n, 01\bar{u}_3 \ldots \bar{u}_n, \\ 11\bar{u}_3 \ldots \bar{u}_n, 10u_3 \ldots u_n = v\rangle \\ \quad \text{if } u_2 = 0, \\ \langle u = 01u_3 \ldots u_n, 01\bar{u}_3 \ldots \bar{u}_n, 00\bar{u}_3 \ldots \bar{u}_n, \\ 10\bar{u}_3 \ldots \bar{u}_n, 11u_3 \ldots u_n = v\rangle \\ \quad \text{if } u_2 = 1 \end{cases}$$

and

$$P = \begin{cases} \langle u = 00u_3 \ldots u_n, 01u_3 \ldots u_n, 10\bar{u}_3 \ldots \bar{u}_n, \\ 11u_3 \ldots u_n, 11\bar{u}_3 \ldots \bar{u}_n = v\rangle \\ \quad \text{if } u_2 = 0, \\ \langle u = 01u_3 \ldots u_n, 00u_3 \ldots u_n, 11\bar{u}_3 \ldots \bar{u}_n, \\ 10u_3 \ldots u_n, 11\bar{u}_3 \ldots \bar{u}_n = v\rangle \\ \quad \text{if } u_2 = 1 \end{cases}$$

are uv-paths of length 4 in MQ_n^0 and MQ_n^1, respectively.

If $G = CQ_n$, a uv-path of length ℓ in CQ_n can be constructed by the similar argument, and omitted here for details.

Remark. Our result is optimal in the sense that there is no uv-path of length $d_{MQ_4^0}(u, v) + 1$ between $u = 0001$ and $v = 1000$ in MQ_4^0, which means that theorem does not always hold for any integers $n \geqslant 3$ and $\ell = d_{MQ_4^0}(u, v) + 1$ and any two vertices u and v in MQ_4^0.

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