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Note

Connectivity of Cartesian product graphs ☆

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Abstract

Use v_i, κ_i, λ_i, δ_i to denote order, connectivity, edge-connectivity and minimum degree of a graph G_i for $i=1, 2$, respectively. For the connectivity and the edge-connectivity of the Cartesian product graph, up to now, the best results are $\kappa(G_1 \times G_2) \geqslant \kappa_1 + \kappa_2$ and $\lambda(G_1 \times G_2) \geqslant \lambda_1 + \lambda_2$. This paper improves these results by proving that $\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1\}$ and $\lambda(G_1 \times G_2) = \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}$ if G_1 and G_2 are connected undirected graphs; $\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1, 2\kappa_1 + \kappa_2, 2\kappa_2 + \kappa_1\}$ if G_1 and G_2 are strongly connected digraphs. These results are also generalized to the Cartesian products of n $(\geqslant 3)$ connected graphs and n strongly connected digraphs, respectively.

Keywords: Connectivity; Edge-connectivity; Cartesian product graphs

1. Introduction

We follow [7] for graph-theoretical terminology and notation not defined here. In this paper, a graph $G = (V, E)$ always means a connected undirected graph or strongly connected digraph with the vertex-set V and the edge-set E. For $x \in V(G)$, the symbol $N_G(x)$ denotes the set of neighbors of x if G is undirected; $N_G^+(x)$ and $N_G^-(x)$ denote the sets of out-neighbors and in-neighbors of x, respectively, if G is directed. The symbol $\delta(G)$ denotes the minimum degree of G, where $\delta(G) = \min\{\delta^+(G), \delta^-(G)\}$ if G is directed, and $\delta^+(G)$ and $\delta^-(G)$ are the minimum out-degree and the minimum in-degree of G, respectively. The symbols $\kappa(G)$ and $\lambda(G)$ denote the connectivity and the edge-connectivity of G, respectively. The well-known Whitney's inequality states that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$ for any graph G (see Theorem 4.4 in [7]). The connectivity is a basic concept in graph theory, but also an important measurement for reliability and fault tolerance in a network [6]. Let G_i be a graph. For short, we will write $v_i = |V(G_i)|$, $\delta_i = \delta(G_i)$, $\kappa_i = \kappa(G_i)$ and $\lambda_i = \lambda(G_i)$.

The Cartesian product is an important method to construct a bigger graph, and plays an important role in design and analysis of networks [6]. For the connectivity and the edge-connectivity of the Cartesian product, up to now, the best results are $\kappa(G_1 \times G_2) \geqslant \kappa_1 + \kappa_2$ and $\lambda(G_1 \times G_2) \geqslant \lambda_1 + \lambda_2$ (see, for example, [6,5,1,2,4]). This paper improves these

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results by proving that

(i) $\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1\}$ and $\lambda(G_1 \times G_2) = \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}$ if G_1 and G_2 are connected undirected graphs;
(ii) $\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1, 2\kappa_1 + \kappa_2, 2\kappa_2 + \kappa_1\}$ if G_1 and G_2 are strongly connected digraphs.

These results are also generalized to the Cartesian products of $n(\geqslant 3)$ connected graphs and n strongly connected digraphs, respectively.

The proofs of these results are in Sections 3 and 4. In the next section, some notations and lemmas will be recalled.

2. Some lemmas and notations

Let x and y be two distinct vertices in a graph $G = (V, E)$. The symbols $\zeta(G; x, y)$ and $\eta(G; x, y)$ denote the maximum numbers of internally-disjoint and, respectively, edge-disjoint (x, y)-paths in G; the symbols $\kappa(G; x, y)$ and $\lambda(G; x, y)$ denote the minimum numbers of vertices and, respectively, edges, whose deletion disconnects x and y in the remaining graph. The following two results are well-known (see Theorems 4.2, 4.3 and 4.5 in [7]).

Lemma 1 (*Menger's Theorem*). *Let G be a connected undirected graph or a strongly connected digraph. Then, for any $x, y \in V(G)$,*

(i) $\kappa(G; x, y) = \zeta(G; x, y)$ *if* $(x, y) \notin E(G)$;
(ii) $\lambda(G; x, y) = \eta(G; x, y)$.

Lemma 2 (*Menger–Whitney's Theorem*). *Let G be a connected undirected graph or a strongly connected digraph. Then*

(i) $\kappa(G) \geqslant k$ *if and only if* $\zeta(G; x, y) \geqslant k$ *for every* $x, y \in V(G)$;
(ii) $\lambda(G) \geqslant k$ *if and only if* $\eta(G; x, y) \geqslant k$ *for every* $x, y \in V(G)$.

For $x \in V(G)$ and $W = \{w_1, w_2, \ldots, w_k\} \subset V(G - x)$, if there exist k (x, w_i)-paths $W_1, W_2, \ldots, W_k$, any two of which have only the vertex x in common, then the set of paths $F_k(x, W) = \{W_1, W_2, \ldots, W_k\}$ is called an (x, W)-fan in G. Equally, if there exist k (w_i, x)-paths $U_1, U_2, \ldots, U_k$, any two of which have only the vertex x in common, then the set of paths $F_k(W, x) = \{U_1, U_2, \ldots, U_k\}$ is called a (W, x)-fan in G. The following lemma insures the existence of these fans if $\kappa(G) \geqslant k$, found first by Dirac [3].

Lemma 3. *Let G be a connected undirected graph or a strongly connected digraph. If $\kappa(G) \geqslant k$, then for any vertex x of G and a set $W = \{w_1, w_2, \ldots, w_k\}$ of any k distinct vertices of $G - x$, there are an (x, W)-fan $F_k(x, W)$ and a (W, x)-fan $F_k(W, x)$ of G.*

Let $G_i = (V_i, E_i)$ be a digraph for each $i = 1, 2$. The Cartesian product $G_1 \times G_2$ of G_1 and G_2 is a digraph with $V(G_1 \times G_2) = V_1 \times V_2$. There is a directed edge from a vertex x_1x_2 to another y_1y_2 in $G_1 \times G_2$, $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$, if and only if either $x_1 = y_1$ and $(x_2, y_2) \in E_2$, or $x_2 = y_2$ and $(x_1, y_1) \in E_1$. The Cartesian product of two undirected graphs can be defined similarly. From definition, the following fact can be verified easily.

Lemma 4. *$G_1 \times G_2 \cong G_2 \times G_1$ and $\delta(G_1 \times G_2) = \delta_1 + \delta_2$ for any graphs G_1 and G_2.*

The following observations and notations are very useful for the proofs of some results on the Cartesian product. If $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, then for any $a \in V_1$ and $b \in V_2$, $H_1 \times \{b\}$ and $\{a\} \times H_2$ are subgraphs of $G_1 \times G_2$, denoted by H_1b and aH_2, respectively. In particular, if $P = (x_1, v_1, v_2, \ldots, v_m, y_1)$ is an (x_1, y_1)-path in G_1, then for any $b \in V_2$, $Pb = (x_1b, v_1b, v_2b, \ldots, v_mb, y_1b)$ is an (x_1b, y_1b)-path from the vertex x_1b to the vertex y_1b in $G_1 \times G_2$. Similarly, if $W = (x_2, u_1, u_2, \ldots, u_l, y_2)$ is an (x_2, y_2)-path in G_2, then for any $a \in V_1$, $aW = (ax_2, au_1, au_2, \ldots, au_l, ay_2)$ is an (ax_2, ay_2)-path from the vertex ax_2 to the vertex ay_2 in $G_1 \times G_2$. If $x = x_1x_2$ and $y = y_1y_2$ are two vertices in $G_1 \times G_2$, then $Q = Px_2 \cup y_1W$ is an (x, y)-path from x to y in $G_1 \times G_2$. Such a path will be expressed as

$$Q : x = x_1x_2 \overset{Px_2}{\to} y_1x_2 \overset{y_1W}{\to} y_1y_2 = y.$$

3. Connectivity of Cartesian products

Lemma 5. *Let $G_i = (V_i, E_i)$ be a strongly connected digraph or a connected undirected graph for each $i = 1, 2$. Then $\kappa(G_1 \times G_2) \geqslant w$ if and only if*

(i) $\zeta(G_1 \times G_2; xa, xb) \geqslant w$ *for any* $x \in V_1$, $a, b \in V_2$, *and*
(ii) $\zeta(G_1 \times G_2; xa, ya) \geqslant w$ *for any* $x, y \in V_1$, $a \in V_2$.

Proof. We only need to show that the sufficiency holds for digraphs. By Lemma 2, it is sufficient to show that $\zeta(G_1 \times G_2; xa, yb) \geqslant w$ for any $xa, yb \in V(G_1 \times G_2)$, where $x, y \in V_1$, $a, b \in V_2$.

If $x = y$ or $a = b$, then $\zeta(G_1 \times G_2; xa, yb) \geqslant w$ holds clearly by our hypothesis. Suppose that $x \neq y$ and $a \neq b$. Then $(xa, yb) \notin E(G_1 \times G_2)$ below. It is sufficient to prove that there is an (xa, yb)-path in $G_1 \times G_2 - S$ for any $S \subset V(G_1 \times G_2)\backslash\{xa, yb\}$ with $|S| < w$.

Choose $x_1, x_2, \ldots, x_{\delta_1} \in N^+_{G_1}(x)$ and $a_1, a_2, \ldots, a_{\delta_2} \in N^+_{G_2}(a)$. Without loss of generality, suppose $x_i \neq y$, $i = 1, 2, \ldots, \delta_1$, $a_j \neq b$, $j = 1, 2, \ldots, \delta_2$ (if, for example, $x_1 = y$, we replace $\{x_1a, x_1b\}$ with singleton $\{x_1a\}$ in (1)). Then $(\delta_1 + \delta_2)$ pairs of vertices

$$\begin{aligned}&\{x_1a, x_1b\}, \{x_2a, x_2b\}, \ldots, \{x_{\delta_1}a, x_{\delta_1}b\},\\ &\{xa_1, ya_1\}, \{xa_2, ya_2\}, \ldots, \{xa_{\delta_2}, ya_{\delta_2}\}\end{aligned} \tag{1}$$

are disjoint. By our hypotheses, Lemmas 1 and 4, we have that

$$\begin{aligned}|S| < w &\leqslant \min_{x,a,b} \zeta(G_1 \times G_2; xa, xb)\\ &= \min_{x,a,b} \kappa(G_1 \times G_2; xa, xb)\\ &\leqslant \delta(G_1 \times G_2)\\ &= \delta_1 + \delta_2,\end{aligned}$$

which implies that there exists at least one pair in (1) that is not in S. Without loss of generality, suppose that $\{xa_1, ya_1\}$ is not in S. Because of our hypothesis that $\zeta(G_1 \times G_2; xa_1, ya_1) \geqslant w$ and $\zeta(G_1 \times G_2; ya_1, yb) \geqslant w$, there exist an (xa_1, ya_1)-path P_1 and a (ya_1, yb)-path P_2 in $G_1 \times G_2 - S$. Thus, $G_1 \times G_2 - S$ contains an (xa, yb)-walk $W = xa \to xa_1 \overset{P_1}{\to} ya_1 \overset{P_2}{\to} yb$, which contains an (xa, yb)-path. □

Theorem 1. *Let $G_i = (V_i, E_i)$ be a connected undirected graph for each $i = 1, 2$. Then*

$$\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1\}.$$

Proof. By symmetry, we only need to show that

$$\zeta(G_1 \times G_2; xa, ya) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1\} \text{ for any } x, y \in V_1 \text{ and } a \in V_2.$$

Since $\kappa(G_1) = \kappa_1$, by Lemma 2, there exist κ_1 internally-disjoint (x, y)-paths $P_1, P_2, \ldots, P_{\kappa_1}$ in G_1. Choose $u_1, u_2, \ldots, u_{\delta_2} \in N_{G_2}(a)$. We can construct $(\kappa_1 + \delta_2)$ internally-disjoint (xa, ya)-paths $R_1, R_2, \ldots, R_{\kappa_1+\delta_2}$ in $G_1 \times G_2$ as follows.

$$R_i = xa \overset{P_ia}{\to} ya, \quad i = 1, 2, \ldots, \kappa_1;$$
$$R_{\kappa_1+j} = xa \to xu_j \overset{P_1u_j}{\to} yu_j \to ya, \quad j = 1, 2, \ldots, \delta_2.$$

It follows that $\zeta(G_1 \times G_2; xa, ya) \geqslant \kappa_1 + \delta_2 \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1\}$. □

By induction we can obtain the following corollary immediately.

Corollary 1. *Let $G_1, G_2, \ldots, G_n$ be connected undirected graphs. Then,*

$$\kappa(G_1 \times G_2 \times \cdots \times G_n) \geqslant \sum_{i=1}^{n} \delta_i - \max_{1 \leqslant i \leqslant n} \{\delta_i - \kappa_i\}.$$

Proof. We proceed by induction on $n \geqslant 2$. Clearly, the assertion holds if $n = 2$ since it is a version of Theorem 1. Assume the induction hypothesis for $n - 1$ with $n > 2$. Let

$$H = G_1 \times G_2 \times \cdots \times G_{n-1}, \quad \delta_H = \delta(H), \quad \kappa_H = \kappa(H), \quad v_H = v(H).$$

By the induction hypothesis, we have that

$$\delta_H - \kappa_H \leqslant \max_{1 \leqslant i \leqslant n-1} \{\delta_i - \kappa_i\}.$$

It follows that

$$\begin{aligned} \kappa(G_1 \times G_2 \times \cdots \times G_n) &\geqslant \delta_H + \delta_n - \max\{\delta_H - \kappa_H, \delta_n - \kappa_n\} \\ &\geqslant \delta_H + \delta_n - \max\left\{\max_{1 \leqslant i \leqslant n-1} \{\delta_i - \kappa_i\}, \delta_n - \kappa_n\right\} \\ &= \sum_{i=1}^{n} \delta_i - \max_{1 \leqslant i \leqslant n} \{\delta_i - \kappa_i\} \end{aligned}$$

as desired. □

Theorem 2. *Let $G_i = (V_i, E_i)$ be a strongly connected digraph for each $i = 1, 2$. Then $\kappa(G_1 \times G_2) \geqslant \min\{\kappa_1 + \delta_2, \kappa_2 + \delta_1, 2\kappa_1 + \kappa_2, 2\kappa_2 + \kappa_1\}$.*

Proof. Let $d = \min\{\delta_1 - \kappa_1, \delta_2 - \kappa_2, \kappa_1, \kappa_2\}$. It is sufficient to prove that

$$\zeta(G_1 \times G_2; xa, ya) \geqslant \kappa_1 + \kappa_2 + d \quad \text{for any } x, y \in V_1 \text{ and } a \in V_2.$$

Let $P_1, \ldots, P_{\kappa_1}$ be κ_1 internally-disjoint (x, y)-paths in G_1. Then, $|N^-_{G_1}(y) \backslash \bigcup_{i=1}^{\kappa_1} V(P_i)| \geqslant \delta_1 - \kappa_1 \geqslant d$. Choose $W = \{w_1, w_2, \ldots, w_d\} \subseteq N^-_{G_1}(y) \backslash \bigcup_{i=1}^{\kappa_1} V(P_i)$ and an (x, W)-fan $F_d(x, W) = \{W_1, W_2, \ldots, W_d\}$ in G_1 (such a fan exists for $d \leqslant \kappa_1$ by Lemma 3).

Choose $U = \{u_1, u_2, \ldots, u_{\kappa_2}\} \subseteq N^+_{G_2}(a)$ and a (U, a)-fan $F_{\kappa_2}(U, a) = \{U_1, U_2, \ldots, U_{\kappa_2}\}$ in G_2. Note that $|N^+_{G_2}(a) \backslash U| \geqslant \delta_2 - \kappa_2 \geqslant d$. Choose $T = \{v_1, v_2, \ldots, v_d\} \subseteq N^+_{G_2}(a) \backslash U$ and a (T, a)-fan $F_d(T, a) = \{T_1, T_2, \ldots, T_d\}$ in G_2 (such a fan exists for $d \leqslant \kappa_2$ by Lemma 3). We now construct $(\kappa_1 + \kappa_2 + d)$ internally-disjoint (xa, ya) as follows:

$$\begin{aligned} R_i &= xa \overset{P_i a}{\rightarrow} ya, \quad i = 1, 2, \ldots, \kappa_1; \\ R_{\kappa_1 + j} &= xa \rightarrow xu_j \overset{P_1 u_j}{\rightarrow} yu_j \overset{yU_j}{\rightarrow} ya, \quad j = 1, 2, \ldots, \kappa_2; \\ R_{\kappa_1 + \kappa_2 + l} &= xa \rightarrow xv_l \overset{W_l v_l}{\rightarrow} w_l v_l \overset{w_l T_l}{\rightarrow} w_l a \rightarrow ya, \quad l = 1, 2, \ldots, d. \end{aligned}$$

It follows that $\zeta(G_1 \times G_2; xa, ya) \geqslant \kappa_1 + \kappa_2 + d$. □

Corollary 2. *Let $G_1, G_2, \ldots, G_n$ be strongly connected digraphs. Then*

(i) $\kappa(G_1 \times G_2 \times \cdots \times G_n) \geqslant \sum_{i=1}^{n} \kappa_i + \min_{1 \leqslant i \leqslant n} \{\delta_i - \kappa_i, \ \kappa_i\}$;
(ii) $\kappa(G_1 \times G_2 \times \cdots \times G_n) \geqslant \sum_{i=1}^{n} \delta_i - \max_{1 \leqslant i \leqslant n} \{\delta_i - \kappa_i\}$ *if* $\delta_i \leqslant 2\kappa_i$.

4. Edge-connectivity of Cartesian products

Lemma 6. *Let $G_i = (V_i, E_i)$ be a strongly connected digraph or a connected undirected graph for each $i = 1, 2$. Then $\lambda(G_1 \times G_2) \geqslant w$ if and only if*

(i) $\eta(G_1 \times G_2; xa, ya) \geqslant w$ *for any* $x, y \in V_1, a \in V_2$, *and*
(ii) $\eta(G_1 \times G_2; xa, xb) \geqslant w$ *for any* $x \in V_1, a, b \in V_2$.

Proof. We only need to prove the sufficiency. Furthermore, it is sufficient to prove that $\eta(G_1 \times G_2; xa, yb) \geqslant w$ for any $x, y \in V_1$ and $a, b \in V_2$. In fact, if $x = y$ or $a = b$, the sufficiency holds by our hypothesis. Suppose $x \neq y$ and $a \neq b$ below. By Lemma 2, it is sufficient to prove $\lambda(G_1 \times G_2; xa, yb) \geqslant w$. Indeed, let $B \subseteq E(G_1 \times G_2)$ such that $\lambda(G_1 \times G_2; xa, yb) = |B|$ and $G_1 \times G_2 - B$ contains no (xa, yb)-path. If $|B| < w$, then by our hypothesis, $G_1 \times G_2 - B$ contain a (xa, ya)-path P_1 and a (ya, yb)-path P_2. Thus,

$$xa \overset{P_1}{\to} ya \overset{P_2}{\to} yb$$

is an (xa, yb)-walk in $G_1 \times G_2 - B$, which contains an (xa, yb)-path, a contradiction. □

The next lemma is simple but useful in the proof of Theorem 3.

Lemma 7. *Let G be a λ-edge-connected undirected graph and let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_s\} (s \leqslant \lambda)$ be two disjoint sets of vertices of G. Two disjoint fans $F_s(x, X)$ and $F_s(y, Y)$ has common vertices with G in exactly $X \cup Y$. Let $G' = G \cup F_s(x, X) \cup F_s(y, Y)$, then there are λ edge-disjoint (x, y)-paths in G'.*

Proof. Let B be an edge subset of G' such that $|B| < \lambda$. Then at least one of the (x, x_i)-paths $(1 \leqslant i \leqslant \lambda)$ in $F_s(x, X)$, say (x, x_1)-path P_1, remains intact after the removal of B. And assume (y, y_1)-path P_2 is intact in $F_s(y, Y) - B$. Because G is λ-edge-connected, $G - B$ is still connected, and there is an (x_1, y_1)-path in $G - B$. This path, together with P_1 and P_2, forms an (x, y)-path in $G' - B$. Thus $\eta(G'; x, y) = \lambda(G'; x, y) \geqslant \lambda$. □

Theorem 3. *Let $G_i = (V_i, E_i) \neq K_1$ be a connected undirected graph for each $i = 1, 2$. Then*

$$\lambda(G_1 \times G_2) = \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}.$$

Proof. Clearly, $\lambda(G_1 \times G_2) \leqslant \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}$. We only need to prove $\lambda(G_1 \times G_2) \geqslant \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}$. Note that if $\delta_1 = \lambda_1$ and $\delta_2 = \lambda_2$ then the conclusion holds clearly by the known result. Without loss of generality, suppose that $\delta_1 > \lambda_1$. By Lemmas 2 and 6, it is sufficient to prove that

$$\eta(G_1 \times G_2; xa, ya) \geqslant \min\{\delta_1 + \delta_2, \lambda_1 v_2, \lambda_2 v_1\}, \quad \forall x, y \in V_1, \quad a \in V_2. \tag{2}$$

The main idea of the proof is to find edge-disjoint subgraphs containing xa and ya of $G_1 \times G_2$, each of which has several edge-disjoint (xa, ya)-paths. By summing the number of paths over those subgraphs, we obtain the desired result.

The first subgraph H_0 of $G_1 \times G_2$ is obtained as follows. Select λ_1 edge-disjoint (x, y)-paths $P_1, P_2, \ldots, P_{\lambda_1}$ (if xy is an edge in G_1, then choose $P_1 = xy$) in G_1. Let $H'_0 = \bigcup_{i=1}^{\lambda_1} P_i$, then $H_0 = H'_0 a$ is a subgraph of $G_1 \times G_2$. By the construction of H_0, it has λ_1 edge-disjoint (xa, ya)-paths.

Let X and Y be the sets of $\delta_1 - \lambda_1$ neighbors of x and y in $G_1 - E(H'_0)$, respectively. We may assume $X \cap Y = \emptyset$, otherwise, let $z \in X \cap Y$, then xzy is yet another (x, y)-path besides $P_i (1 \leqslant i \leqslant \lambda_1)$, and may add this path to H'_0 in the previous step. Let $B = \{b_1, b_2, \ldots, b_{\delta_2}\}$ be the set of δ_2 neighbors of a vertex a in G_2, and let $C = V_2 - \{a\} - B = \{c_1, c_2, \ldots, c_{v_2 - \delta_2 - 1}\}$.

Next, we will construct a series of subgraphs of $G_1 \times G_2$ by the following way, which will be call *Method A* for convenience. Take $x_1, x_2, \ldots, x_s \in X$, $y_1, y_2, \ldots, y_s \in Y$ and $b \in B$, where $0 \leqslant s \leqslant \lambda_1 - 1$. The subgraph H is composed by the union of $G_1 b$ and $2(s + 1)$ paths: $xa \to xb$, $xa \to x_i a \to x_i b (1 \leqslant i \leqslant s)$, $ya \to yb$ and $ya \to y_i a \to y_i b (1 \leqslant i \leqslant s)$, as illustrated in Fig. 1. By Lemma 7, H has $s + 1$ edge disjoint (xa, ya)-paths.

If $\delta_1 - \lambda_1 = |X| = |Y| \leqslant \delta_2(\lambda_1 - 1)$, namely $\delta_1 + \delta_2 \leqslant (\delta_2 + 1)\lambda_1$, we can partition X and Y into δ_2 disjoint set $X_1, X_2, \ldots, X_{\delta_2}$ and $Y_1, Y_2, \ldots, Y_{\delta_2}$, respectively, such that $0 \leqslant |X_i| = |Y_i| \leqslant \lambda_1 - 1$. By applying Method A to X_i, Y_i

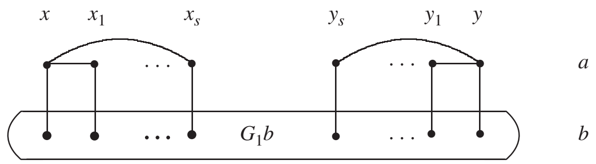

Fig. 1. An illustration for Method A.

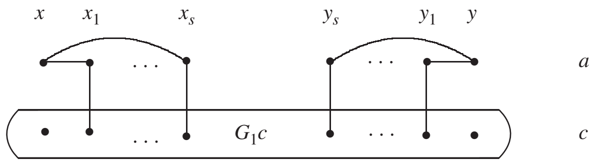

Fig. 2. An illustration for Method B.

and b_i, we construct δ_2 subgraphs $H_i (1 \leqslant i \leqslant \delta_2)$, each of which has $|X_i| + 1$ edge-disjoint (xa, ya)-paths. It is easy to see that the subgraphs $H_0, H_1, \ldots, H_{\delta_2}$ are mutually edge-disjoint. Thus in all there are at least

$$\begin{aligned}\lambda_1 + \sum_{i=1}^{\delta_2} (|X_i| + 1) &= \lambda_1 + \sum_{i=1}^{\delta_2} |X_i| + \delta_2 \\ &= \lambda_1 + (\delta_1 - \lambda_1) + \delta_2 \\ &= \delta_1 + \delta_2\end{aligned}$$

edge-disjoint (xa, ya)-paths.

If $|X| = |Y| > \delta_2(\lambda_1 - 1)$, the construction of the first $\delta_2 + 1$ subgraphs is the same as before, with slight difference that we always choose $|X_i| = |Y_i| = \lambda_1 - 1$ when we apply Method A. Let

$$X' = X \Bigg\backslash \bigcup_{i=1}^{\delta_2} X_i \quad \text{and} \quad Y' = Y \Bigg\backslash \bigcup_{i=1}^{\delta_2} Y_i.$$

Clearly, $X' \neq \emptyset$ and $Y' \neq \emptyset$. With X', Y' and C, we introduce *Method B* to find more subgraphs of $G_1 \times G_2$. Take $x_1, x_2, \ldots, x_s \in X'$, $y_1, y_2, \ldots, y_s \in Y'$ and $c \in C$, where $0 \leqslant s \leqslant \lambda_1$. Let P_{ac} be an ac-path in G_2. As illustrated in Fig. 2, the subgraph H is the union of G_1c and $2s$ paths: $xa \to x_i a \overset{x_i P_{ac}}{\to} x_i c (1 \leqslant i \leqslant s)$ and $ya \to y_i a \overset{y_i P_{ac}}{\to} y_i c (1 \leqslant i \leqslant s)$. By Lemma 7, H has s edge-disjoint (xa, ya)-paths.

Now, we can continue finding subgraphs. Each time, take λ_1 (or less if there are not so many) unused vertices from X' and Y', respectively, take one vertex from C and apply Method B to construct a subgraph of $G_1 \times G_2$. First assume $\delta_2(\lambda_1 - 1) < |X| = |Y| \leqslant \delta_2(\lambda_1 - 1) + (v_2 - \delta_2 - 1)\lambda_1$, namely $(\delta_2 + 1)\lambda_1 < \delta_1 + \delta_2 \leqslant v_2 \lambda_1$. The process will end when we use up the vertices of X' and Y'. So the total number of edge-disjoint (xa, ya)-paths in all subgraphs is at least

$$\begin{aligned}\lambda_1 + \sum_{i=1}^{\delta_2} (|X_i| + 1) + |X'| &= \lambda_1 + \sum_{i=1}^{\delta_2} |X_i| + |X'| + \delta_2 \\ &= \lambda_1 + (\delta_1 - \lambda_1) + \delta_2 \\ &= \delta_1 + \delta_2.\end{aligned}$$

If $|X| = |Y| > \delta_2(\lambda_1 - 1) + (v_2 - \delta_2 - 1)\lambda_1$, namely $\delta_1 + \delta_2 > v_2\lambda_1$, the process will terminate when the vertices in C are exhausted. In this situation, the number of edge-disjoint (xa, ya)-paths is at least

$$\begin{aligned}\lambda_1 + \sum_{i=1}^{\delta_2}(|X_i| + 1) + (v_2 - \delta_2 - 1)\lambda_1 &= \lambda_1 + \delta_2\lambda_1 + (v_2 - \delta_2 - 1)\lambda_1 \\ &= v_2\lambda_1.\end{aligned}$$

Summing the above discussion, the inequality (2) holds, and so theorem follows. □

Corollary 3. *Let $G_1, G_2, \ldots, G_n$ be connected undirected graphs. Then*

$$\lambda(G_1 \times G_2 \times \cdots \times G_n) = \min\left\{\sum_{i=1}^{n}\delta_i,\ \min_{1\leqslant i\leqslant n}\{v_1 \cdots v_{i-1}\lambda_i v_{i+1} \cdots v_n\}\right\}.$$

Proof. We proceed by induction on $n \geqslant 2$. The assertion is true for $n = 2$ by Theorem 3. Suppose that $n \geqslant 3$ and the assertion holds for $n - 1$. It is clear that

$$\begin{aligned}(\delta_1 + \delta_2 + \cdots + \delta_{n-1})v_n &\geqslant (\delta_1 + \delta_2 + \cdots + \delta_{n-1}) \cdot (1 + \delta_n) \\ &> \delta_1 + \delta_2 + \cdots + \delta_{n-1} + \delta_n.\end{aligned}$$

It follows that

$$\begin{aligned}&\lambda(G_1 \times G_2 \times \cdots \times G_n) \\ &\quad = \min\{\delta(G_1 \times \cdots \times G_{n-1}) + \delta_n, \lambda(G_1 \times \cdots \times G_{n-1})v_n, v(G_1 \times \cdots \times G_{n-1})\lambda_n\} \\ &\quad = \min\left\{\sum_{i=1}^{n}\delta_i, \min\left\{\sum_{i=1}^{n-1}\delta_i,\ \min_{1\leqslant i\leqslant n-1}\{v_1 \cdots v_{i-1}\lambda_i v_{i+1} \cdots v_{n-1}\}\right\} v_n,\ v_1 \cdots v_{n-1}\lambda_n\right\} \\ &\quad = \min\left\{\sum_{i=1}^{n}\delta_i,\ \min_{1\leqslant i\leqslant n}\{v_1 \cdots v_{i-1}\lambda_i v_{i+1} \cdots v_n\}\right\}.\end{aligned}$$

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