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Edge fault tolerance analysis of a class of interconnection networks ☆

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Abstract

Fault tolerant measures have played an important role in the reliability of an interconnection network. Edge connectivity, restricted-edge-connectivity, extra-edge-connectivity and super-edge-connectivity of many well-known interconnection networks have been explored. In this paper, we study the 2-extra-edge connectivity of a special class of graphs $G(G_0, G_1; M)$ proposed by Chen et al. [Appl. Math. Comput. 140 (2003) 245–254]. Then by showing that several well-known interconnection networks such as hypercubes, twisted cubes, crossed cubes and Möbius cubes are all contained in this class. We show that their 2-extra-edge-connectivity are all not less than $3n - 4$ when their dimension n is not less than 4. That is, when $n \geqslant 4$, at least $3n - 4$ edges are to be removed to get any of an n-dimensional above networks disconnected provided that the removed edges does not isolate a vertex or an edge in the faulty networks. Compared with previous results, our result enhances the fault tolerant ability of above networks theoretically.

Keywords: Hypercube; Edge connectivity; Extra-edge-connectivity

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1. Introduction

For all the terminologies and notations not defined here, we follow [3]. For a graph $G = (V, E)$ and $S \subset V(G)$ or $S \subset G$, we use $N_G(S)(E_G(S))$ to denote the set of neighboring vertices (edges) of S in $G - S$. For any vertex v, we use $d(v)$ to denote the degree of v. If the degree of every vertex is identical to k, we call G a k-regular graph. The connectivity of a graph is the minimum number of vertices to be removed to get the graph disconnected. If the connectivity of a k-regular graph is k, we call G a k-regular maximum connected graph. The minimum length of all the cycles in G is called the girth of G, denoted by $g(G)$. In this paper, we use graph and interconnection networks interchangeably, nodes and vertices interchangeably and links and edges interchangeably.

Routing has been a popular topic in the study of computer networks. In particular, it is important to know whether the non-faulty nodes in a given network with faults still remain connected. Traditionally, connectivity and edge connectivity have been mainly used for measures of functionality of the system. For example, the minimum number of faulty links in an n-cube that results in the remaining nodes being disconnected is its edge connectivity n. But the only case that n faulty links can disconnect an n-cube is that all these n links are neighboring to a same node [12]. However, the probability that all n faulty links are neighbors of the same node is very small.

The use of forbidden faulty set [13] is motivated by the fact that the traditional graph connectivity model cannot correctly reflect network resilience of large systems. The vertices or edges in a forbidden faulty set cannot fail at the same time. By restricting the forbidden fault set to be the sets of neighboring edges of any spanning subgraph with not more than h-vertices in the faulty networks, Fàbrega and Fiol [8] introduced the *h-extra-edge-connectivity* of interconnection networks.

Definition 1.1 [8]. Given a graph G and a non-negative integer h, the h-extra-edge-connectivity $\lambda_h(G)$ of G is the minimum cardinality of a set of edges of G, if any, whose deletion disconnects G and every remaining component contains more than h vertices.

This is another important generalization of the traditional edge connectivity.

In this paper we investigate the 2-extra-edge-connectivity of a special class of graphs $G(G_0, G_1; M)$ proposed by [4].

Definition 1.2. Assume that t is a positive integer. Let G_1 and G_2 be two graphs with t vertices, and M be any arbitrary perfect matching between the vertices of G_1 and G_2; i.e., a set of t edges with one endpoint in G_1, and the other endpoint in G_2. The graph $G(G_1, G_2; M)$ is defined as a graph with the vertex set

$V(G(G_1, G_2;\ M)) = V(G_1) \cup V(G_2)$, and edge set $E(G(G_1, G_2;\ M)) = E(G_1) \cup E(G_2) \cup M$.

For any vertex $u \in V(G)$, we use $\bar{e}(u) = (u, \bar{u})$ to denote the edge in M which is incident to u in $G(G_0, G_1;\ M)$, and we call $\bar{u}$ to be u's pair vertex.

Since the proposition of the class of graphs $G(G_0, G_1;\ M)$, much of its property has been studied. Many well-known interconnection networks such as hypercubes, twisted cubes, crossed cubes, and Möbius cubes all belong to this class, thus study of this class of graphs can lead to the findings of many important properties of these well-known interconnection networks.

Chen et al. studied the super-connectivity and super-edge-connectivity of the class of graphs $G(G_0, G_1;\ M)$ and they obtained the following results.

Lemma 1.3 [4]. *Assume that t is a positive integer. Let G_1 and G_2 be two k-regular maximum-edge-connected graphs with t vertices, and M be any perfect matching between $V(G_1)$ and $V(G_2)$. Then $G(G_1, G_2;\ M)$ is $(k + 1)$-regular super-edge-connected if and only if* (1) $t > k + 1$ *or* (2) $t = k + 1$ *with* $k = 0$.

Lemma 1.4 [4]. *Assume that t is a positive integer. Let G_1 and G_2 be two k-regular maximum connected graphs with t vertices, and M be any perfect matching between $V(G_1)$ and $V(G_2)$. Then $G(G_1, G_2;\ M)$ is $(k + 1)$-regular super-connected if and only if* (1) $t > k + 1$ *or* (2) $t = k + 1$ *with* $k = 0, 1, 2$.

The remaining of this article is organized as follows. In Section 2, we study a special class of graphs in $G(G_0, G_1;\ M)$ and obtain their 2-extra-edge-connectivity. Then in Section 3, by proving that hypercubes, twisted cubes, Möbius cubes and crossed cubes all belong to this special class of graphs, we obtain their 2-extra-edge-connectivity.

2. Fault tolerance analysis of $G(G_0, G_1;\ M)$

Lemma 2.1. *Suppose G_0 and G_1 are two k-regular maximum-connected graphs with the same order, $g(G_0) \geqslant 4$ and $g(G_1) \geqslant 4$, M is a perfect matching between $V(G_0)$ and $V(G_1)$. Then $G = G(G_0, G_1;\ M)$ is $(k + 1)$-regular maximum connected and $g(G) \geqslant 4$.*

Proof

(1) Applying Lemma 1.4, we immediately obtain that G is $(k + 1)$-regular maximum connected.
(2) We only need to prove that there is no triangle in $G(G_0, G_1;\ M)$. Since $g(G_0) \geqslant 4$ and $g(G_1) \geqslant 4$, there is no triangle in G_0 and G_1. It is sufficient

to show that every edge in M is not contained in a triangle. Suppose $e = (u_0, u_1)$ is any edge in M. Since M is a perfect matching, there is no vertex adjacent to u_0 and u_1 simultaneously, so e cannot be contained in a triangle. Thus there is no triangle in G, $g(G) \geqslant 4$. □

To simplify the impression of our results, we introduce the following definition.

Definition 2.2. For a graph G, we define the *cn*-number of G to be the least integer l such that any two vertices in G share at most l common neighboring vertices, and we denote this number by $cn(G)$.

Lemma 2.3. *For any two graphs G_0 and G_1 with the same order, M is a perfect matching between $V(G_0)$ and $V(G_1)$. If $cn(G_0) = cn(G_1) = 2$, then $cn(G(G_0, G_1; M)) = 2$.*

Proof. It is obvious that $cn(G(G_0, G_1; M)) \geqslant \max\{cn(G_0), cn(G_1)\} = 2$, so we only need to prove that for any two vertices x, y in $V(G(G_0, G_1; M))$, x and y have at most two common neighboring vertices.

(1) If x and y are both in $V(G_0)(V(G_1))$. Since M is a perfect matching, there is no vertices in $G_1(G_0)$ which is adjacent to both x and y. So the common neighbors of x and y all lie in $G_0(G_1)$. Thus x and y have at most two common neighbors since $cn(G_0) = cn(G_1) = 2$.
(2) If $x \in V(G_0)(V(G_1))$ and $y \in V(G_1)(V(G_0))$, then the common neighbors of x and y must be x's or y's pair vertex. Since M is a perfect matching, both x and y have only one pair vertex. Thus x and y have at most two common neighbors. □

Lemma 2.4. *Suppose G_0 and G_1 are two k-regular ($k \geqslant 3$) graphs with t vertices; M is a perfect matching between $V(G_0)$ and $V(G_1)$. $g(G_0) \geqslant 4$ and $g(G_1) \geqslant 4$. $cn(G_0) = cn(G_1) = 2$. $F \subset E(G)$, $|F| \leqslant 3k - 2$ and there is no isolated vertex or isolated edge in $G - F$, $F_0 = F \cap E(G_0)$, $F_1 = F \cap E(G_1)$ and $F_m = F \cap M$. Then any vertex in $G_0 - F_0(G_1 - F_1)$ is connected to $G_1 - F_1(G_0 - F_0)$ in $G - F$.*

Proof. Without loss of generality, we only need to prove that any vertex in $G_1 - F_1$ is connected to $G_0 - F_0$ in $G - F$.

$\forall u \in V(G_1) - F_1$, if $\bar{e}(u) \notin F_m$ then we are done. So we suppose that $\bar{e}(u) \in F_m$. Since u cannot be an isolated vertex in $G - F$, there exist a vertex $v \in N_{G_1}(u)$ such that $(u, v) \notin F_1$, if $\bar{e}(v) \notin F_m$, then we are done, so we suppose that $\bar{e}(v) \in F_m$. Since (u, v) cannot be an isolated edge in $G - F$, there exist an edge $e \in E_{G_1}((u, v)) - F_1$. The edge e may be incident to u or v, but since whether it is incident to u or v makes no difference to the following proof, we may suppose that $e = (v, w)$ for simplicity, if $\bar{e}(w) \notin M$, then we are done, so we suppose $\bar{e}(w) \in M$. Since $g(G_1) \geqslant 4$, w cannot be neighboring to u. Since $cn(G_1) = 2$, $|N_{G_1}(u) \cap N_{G_1}(w)| \leqslant 2$.

Case I. $N_{G_1}(u) \cap N_{G_1}(w) = \{v, x\}$.

*Subcase I.*1. Both (u, x) and (w, x) belong to F.

Let $X = N_{G_1}(u, v, w) - \{x\}$. Since $g(G_1) \geqslant 4$, v has no common neighbors with u or w. $|X| = 3k - 6$ since u and w have exactly two common neighbors. Let $\overline{X} = \{\bar{e}(x_i)|x_i \in X\}$, then $|\overline{X}| = |X|$ since M is a perfect matching. It is easy to see that $\overline{X} \cap \{\bar{e}(u), \bar{e}(v), \bar{e}(w), (u, x), (v, x)\} = \phi$ and $\{\bar{e}(u), \bar{e}(v), \bar{e}(w), (u, x), (v, x)\} \subset F$, $|\overline{X} \cap F| \leqslant |F| - 5 \leqslant 3k - 7 < |\overline{X}|$. So at least one edge of $\overline{X}$ does not belong to F, which means that u can be connected to $G_0 - F_0$ in this case.

*Subcase I.*2. At least one of (u, x) and (w, x) does not belong to F.

Thus u is connected to x in $G - F$, if $\bar{e}(x) \notin F$, then we are done, so we suppose that $\bar{e}(x) \in F$.

Since $g(G_1) \geqslant 4$ and $cn(G_1) = 2$, $|N_{G_1}(u, v, w, x)| = 4k - 8$. We define $Y = \{\bar{e}(y)|y \in N_{G_1}(u, v, w, x)\}$. Then $|Y| = 4k - 8$ since M is perfect matching. It is easy to see that $Y \cap \{\bar{e}(u), \bar{e}(v), \bar{e}(w), \bar{e}(x)\} = \phi$ and $\{\bar{e}(u), \bar{e}(v), \bar{e}(w), \bar{e}(x)\} \subset F$. So $|Y \cap F| \leqslant |F| - 4 \leqslant 3k - 6 < |Y|$ (when $k \geqslant 3$). At least one edge of Y does not belong to F, so u can be connected to $G_0 - F_0$.

Case II. $N_{G_1}(u) \cap N_{G_1}(w) = \{v\}$.

Let $Z = \{\bar{e}(z)|z \in N_{G_1}(u, v, w)\}$, then $|Z| = |N_{G_1}(u, v, w)| = 3k - 4$. Since $\{\bar{e}(u), \bar{e}(v), \bar{e}(w)\} \subset F$ and $\{\bar{e}(u), \bar{e}(v), \bar{e}(w)\} \cap Z = \phi$, $|Z \cap F| \leqslant |F| - 3 \leqslant 3k - 5 < |Z|$. Thus at least one edge of Z does not belong to F, which means that u can be connected to $G_0 - F_0$. □

Lemma 2.5. *Suppose G_0 and G_1 are two k-regular ($k \geqslant 3$) maximum connected graphs. $|V(G_0)| = |V(G_1)| = t$. M is a perfect matching between $V(G_0)$ and $V(G_1)$. Let $G = G(G_0, G_1; M)$. If $g(G_0) \geqslant 4$ and $g(G_1) \geqslant 4$, then $\lambda_1(G) = 2k$.*

Proof

(1) $\forall e = (x, y) \in E(G)$. For any vertex $u \in V(G) - \{x, y\}$, $|E_G(u) \cap E_G(e)| \leqslant 1 (< k + 1) = d_G(u)$ since $g(G) \geqslant 4$ by Lemma 2.1. So there is no isolated vertex in $G - E_G(e)$. $\lambda_1(G) \leqslant |E_G(e)| = 2k$.

(2) $\forall F \subset E(G)$, $|F| \leqslant 2k - 1$ and there is no isolated vertex in $G - F$. In the following we will prove that $G - F$ is connected.

Let $F_0 = F \cap E(G_0)$, $F_1 = F \cap E(G_1)$ and $F_m = F \cap M$. It is clear that F_0, F_1 and F_m are disjoint to each other. So at least one of F_0 and F_1 is strictly less than k. Without loss of generality, we suppose that $|F_0| < k$. $\lambda(G_0) \geqslant \kappa(G_0) = k$ since G_0 is k-regular maximum connected. Then $G_0 - F_0$ is connected since $|F_0| < \lambda(G_0)$.

In the following we will prove that any vertex in $G_1 - F_1$ is connected to $G_0 - F_0$ in $G - F$.

For any vertex $u_1 \in V(G_1)$, if $\bar{e}(u_1) \notin M$, then we are done. So we suppose that $\bar{e}(u_1) \in M$. Since there is no isolated vertex in $G - F$, there exist a vertex $v_1 \in N_{G_1}(u_1)$ such that $(u_1, v_1) \notin F_1$. If $\bar{e}(v_1) \notin F_m$, then we are done. So we suppose $\bar{e}(v_1) \in F_m$. Let $X = \{\bar{e}(u) \in M | u \in N_{G_1}((u_1, v_1))\}$. Then $|X| = |N_{G_1}((u_1, v_1))|$ since M is a compete matching. $|N_{G_1}((u_1, v_1))| = 2k - 2$ since $g(G_1) \geqslant 4$. Since $\bar{e}(u_1)$ and $\bar{e}(v_1)$ cannot be in X, $|X \cap F| \leqslant |F| - 2 \leqslant 2k - 3 < |X|$. At least one edge of X does not belong to F. Thus u_1 can be connected to $G_0 - F_0$. □

Theorem 2.6. *Suppose G_0 and G_1 are two k-regular ($k \geqslant 3$) maximum-connected graphs with t vertices; M is a perfect matching between $V(G_0)$ and $V(G_1)$. Let $G = G(G_0, G_1; M)$. If (1) $cn(G_0) = cn(G_1) = 2$; (2) $g(G_0) \geqslant 4$ and $g(G_1) \geqslant 4$; (3) $\lambda_1(G_0) = 2k - 2$ and $\lambda_1(G_1) = 2k - 2$, then $\lambda_2(G) = 3k - 1$.*

Proof

(1) Let $T = x \to z \to y$ be a path of length 2 between x and y. By Lemma 2.1, $g(G) \geqslant 4$, so any vertex u not in T can have at most two neighboring vertex in T, that is, $|E_G(u) \cap E_G(T)| \leqslant 2 < k + 1 = d_G(u)$ when $k \geqslant 2$. And for any edge e not in T $|E_G(e) \cap E_G(T)| \leqslant 3 < 2k = |E_G(e)|$ (when $k \geqslant 2$), so there is no isolated vertex or isolated edge in $G - E_G(T)$. It is easy to see that $|E_G(T)| = (k + 1 - 1) + (k + 1 - 2) + (k + 1 - 1) = 3k - 1$. Thus $\lambda_2(G) \leqslant 3k - 1$ when $k \geqslant 2$.

(2) For any edge subset $F \subset E(G)$, $|F| \leqslant 3k - 2$ and there is no isolated vertex or isolated edge in $G - F$. In the following we will prove that $G - F$ is connected.

Let $F_0 = F \cap E(G_0)$, $F_1 = F \cap E(G_1)$ and $F_m = F \cap M$.

Case I. At least one of $G_0 - F_0$ and $G_1 - F_1$ is connected.

Without loss of generality, we suppose that $G_0 - F_0$ is connected. By Lemma 2.4, any vertex $u \in V(G_1)$ can be connected to $G_0 - F_0$ in $G - F$. Thus $G - F$ is connected in this case.

Case II. Both $G_0 - F_0$ and $G_1 - F_1$ are disconnected.

Since both G_0 and G_1 are k-regular maximum-connected, $\kappa(G_0) = \kappa(G_1) = \lambda(G_0) = \lambda(G_1) = k$. Then $|F_0| \geqslant k$ and $|F_1| \geqslant k$ since $G_0 - F_0$ and $G_1 - F_1$ are disconnected. Thus $|F_0| \leqslant |F| - |F_1| \leqslant 2k - 2$ and $|F_1| \leqslant 2k - 2$.

*Subcase II.*1. $|F_0| = 2k - 2$ or $|F_1| = 2k - 2$.

Without loss of generality, we suppose that $|F_0| = 2k - 2$, so $|F_1| = k$ and $|F_m| = 0$.

Since $\lambda_1(G_1) \geqslant 2k - 2 > k$, G_1 is super-edge-connected. thus there exist a vertex u_1 in $V(G_1)$ such that $E_{G_1}(u_1) = F_1$. Since $\lambda(G_1 - u_1) \geqslant \kappa(G_1 - u_1) \geqslant$

$k-1>0$ (when $k \geqslant 2$), $G_1-u_1=(G_1-F_1)-u_1$ is connected. Suppose $\bar{e}(u_1)=(u_1,u_0)$. Since there is no isolated vertex and isolated edge in $G-F$, there exist a vertex $v_0 \in N_{G_0}(u_0)$ such that $(u_0,v_0) \notin F_0$. Since $|F_m|=0$, $\bar{e}(u_0) \notin F$ and $\bar{e}(v_0) \notin F$. Thus u_1 can be connected to G_1-u_1 in $G-F$. And by Lemma 2.4, any vertex in G_0-F_0 is connected to G_1-F_1 in $G-F$. Thus $G-F$ is connected.

Subcase II.2. $k \leqslant |F_0| < 2k-2$ and $k \leqslant |F_1| < 2k-2$.

If there is no isolated vertex in G_0-F_0 or G_1-F_1, then G_0-F_0 and G_1-F_1 are both connected since $\lambda_1(G_0)=2k-2$ and $\lambda_1(G_1)=2k-2$. So there exist a vertex $u_0 \in V(G_0)$ and a vertex $v_1 \in V(G_1)$ such that $N_{G_0}(u_0) \subset F_0$ and $N_{G_1}(v_1) \subset F_1$, that is, u_0 is an isolated vertex G_0-F_0 and v_1 is an isolated vertex in G_1-F_1. Since $\lambda(G_0-u_0) \geqslant \kappa(G_0-u_0) \geqslant k-1$ and $|F_0 \cap E(G_0-u_0)| = |F_0|-k \leqslant k-2$, $(G_0-u_0)-F_0$ is connected. Similarly, we can prove that $(G_1-v_1)-F_1$ is connected. Since there is no isolated vertex and edge in $G-F$, u_0 and v_1 cannot be adjacent to each other. Thus u_0 is connected to $(G_1-v_1)-F_1$ and v_1 is connected to $(G_0-u_0)-F_0$ in $G-F$.

$|F_m|=|F|-|F_0|-|F_1| \leqslant k-2$. Since there are $t-2>k-2$ edges of M between $(G_0-u_0)-F_0$ and $(G_1-v_1)-F_1$, $(G_0-u_0)-F_0$ and $(G_1-v_1)-F_1$ can be connected to each other. Thus $G-F$ is connected in this case. □

3. Applications

Topologies of many interconnection networks can be viewed as $G(G_0,G_1;M)$ for some k-regular graphs G_0 and G_1, such as hypercubes, twisted cubes, crossed cubes and Möbius cubes. These networks are derived by changing the connection of some hypercube edges according to some specified rules, and thus, have many attractive properties as the same as the hypercube's. Moreover, they have many advantages over the hypercube. In particular, they have a diameter of approximately a half of the hypercubes's diameter. Thus, each of these networks is regarded as an attractive alternative to the hypercube and has attracted many researcher's interest. In this section, we prove that all these networks belong to the class of graphs studied in Section 2, and thus obtain the particular results on their 2-extra-edge-connectivity.

3.1. Hypercubes

The n-dimensional binary hypercube Q_n is a graph whose vertex set $V(Q_n)$ consists of all binary sequence of length n on the set $\{0,1\}$, and two vertices $u=u_{n-1}u_{n-2}\cdots u_0$ and $v=v_{n-1}v_{n-2}\cdots v_0$ are linked by an edge if and only if u and v differ in exactly one coordinate, i.e. $\sum_{i=1}^{n}|u_i-v_i|=1$. The Hamming distance of any two vertices u, v in $V(Q_n)$ is defined to be:

$H(u,v) = \sum_{i=1}^{n}|u_i - v_i|$. The distance between two vertices $d(u,v)$ is equal to its Hamming distance.

The hypercube Q_n is an n-regular bipartite graph with 2^n vertices. It is the best well-known and popular topological structure of interconnection networks because of its attractive properties [10]. Saad and Schultz [14] proved that Q_n is n-connected. Esfahanian [6] determined $\kappa'(Q_n) = 2n - 2$ for $n \geqslant 3$ and $\lambda'(Q_n) = 2n - 2$ for $n \geqslant 2$.

Let $iQ_{n-1}(i \in \{0,1\})$ be the spanning subgraph of all the vertices in Q_n with the leftmost bit i, then $0Q_{n-1}$ and $1Q_{n-1}$ are two $n-1$ dimensional hypercubes and $Q_n = G(0Q_{n-1}, 1Q_{n-1}; M)$ for a particular perfect matching M between $V(G_0)$ and $V(G_1)$.

Corollary 3.1. *$\lambda_2(Q_n) = 3n - 5$ for $n \geqslant 4$.*

Proof. Q_2 is isomorphic to C_4, so $g(Q_2) = 4$ and $cn(Q_2) = 2$. Applying Lemma 2.1 and Lemma 2.3 recursively, we obtain that Q_n is a n-regular maximum-connected graph, $cn(Q_n) = 2$ and $g(Q_n) \geqslant 4$ when $n \geqslant 2$, applying Lemma 2.5, we may obtain that $\lambda_1(Q_n) = 2n - 2$ for $n \geqslant 3$. Then applying Theorem 2.6, we may obtain that $\lambda_2(Q_n) = 3n - 4$ when $n \geqslant 4$. □

3.2. Twisted cubes

The n-dimensional twisted cube. TQ_n, is derived by changing the connectivity of some of the n-cube's links. Construction methods proposed by [1,2] and [7] differ slightly. In [1,2], the construction method applies only to an odd-dimensioned TQ_n. A link connecting nodes X $(= x_{n-1} \cdots x_i x_{i-1} \cdots x_0)$ and $Y(= y_{n-1} y_i y_{i-1} \cdots y_0)$ is said to span dimension i if and only if $x_i \neq y_i$. A parity function, P, is defined for the dimension i link as: $P_i(X) = x_i \oplus x_{i-1} \oplus \cdots \oplus x_0$ where $\oplus$ is the *EXCLUSIVE-OR* operation. Construction starts with an n-cube, where $n = 2k + 1$, $k \geqslant 0$. For each node X $(= x_{n-1} \cdots x_i x_{i-1} \cdots x_0)$ in the n-cube, if $P_{2j-2}(X) = 0$ $(0 \leqslant j \leqslant \frac{n+1}{2})$, relocate the link spanning dimension $2j - 1$ to node Y where $y_{2j} y_{2j-1} = \bar{x}_{2j} \bar{x}_{2j-1}$ and $y_i = x_i$ for $i \neq 2j$ or $2j - 1$ (the link label $2j - 1$ is retained even though the relocated link now spans two dimensions). Construction is complete when all such links are relocated. The second construction method proposed by [7] is similar but applied to only one 2-cube in the $TQ_n (n \geqslant 3)$. In this section, we only discuss twisted cubes proposed in [1,2].

TQ_n can be recursively defined as follows: TQ_1 is a complete graph K_2 with two vertices labelled by 0 and 1. Let n be an odd integer and $n \geqslant 3$. We decompose vertices of TQ_n into four sets V^{00}, V^{01}, V^{10} and V^{11}, where V^{ij} consists of those vertices x with $x_{n-1} = i$ and $x_{n-2} = j$. For each $ij \in \{00, 01, 10, 11\}$, the induced subgraph of V^{ij} in TQ_n is isomorphic to TQ_{n-2}, denoted by TQ_{n-2}^{ij}. Edges which connect these four subtwisted cubes can be described as follows: Any vertex $x = x_{n-1}x_{n-2} \cdots x_1x_0$ with $P_{n-3}(x) = 0$ is connected to $\bar{x}_{n-1}\bar{x}_{n-2} \cdots x_1x_0$ and $\bar{x}_{n-1}x_{n-2} \cdots x_1x_0$; or is connected to $x_{n-1}\bar{x}_{n-2} \cdots x_1x_0$ and $\bar{x}_{n-1}x_{n-2} \cdots x_1x_0$, if $P_{n-3}(x) = 1$.

From the above definition, $TQ^{00}_{n-2} \cup TQ^{10}_{n-2}$ and $TQ^{01}_{n-2} \cup TQ^{11}_{n-2}$ are isomorphic to $TQ_{n-2} \otimes K_2$ (where $TQ_{n-2} \otimes K_2$ denotes the cartesian product graph of G and H). Moreover, the edges joining $TQ^{00}_{n-2} \cup TQ^{10}_{n-2}$ and $TQ^{01}_{n-2} \cup TQ^{11}_{n-2}$ form a perfect matching of TQ_n. Thus, TQ_n can be viewed as $G(TQ_{n-2} \otimes K_2; TQ_{n-2} \otimes K_2; M)$ for some perfect matching M, where TQ_1 is a complete graph K_2 with two vertices labelled by 0 and 1.

Corollary 3.2. *For an odd integer* $n \geqslant 5$, $\lambda_2(TQ_n) = 3n - 4$.

Proof. $TQ_1 \otimes K_2 = G(TQ_1, TQ_1; M)$ is isomorphic to the 4-cycle C_4. So its 2-regular maximum connected. $cn(TQ_1 \otimes K_2) = 2$ and $g(TQ_1 \otimes K_2) = 4$. Since $TQ_n = G(TQ_{n-2} \otimes K_2;\ TQ_{n-2} \otimes K_2;\ M)$ and $TQ_n \otimes K_2 = G(TQ_n, TQ_n;\ M)$, applying Lemma 2.1 and Lemma 2.3 recursively. We may obtain that when $n \geqslant 3$, the following propositions hold.

(1) TQ_n is n-regular maximum connected and $TQ_n \otimes K_2$ is $(n+1)$-regular maximum connected;
(2) $cn(TQ_n) = cn(TQ_n \otimes K_2) = 2$;
(3) $g(TQ_n) \geqslant 4$ and $g(TQ_n \otimes K_2) \geqslant 4$.

By Lemma 2.5, we may obtain that $\lambda_1(TQ_n \otimes K_2) = 2k$ and $\kappa_1(TQ_{n+2}) = 2k + 2$ for $n \geqslant 3$. Then by applying Theorem 2.6, we may obtain $\lambda_2(TQ_n \otimes K_2) = 3n - 1$ and $\lambda_2(TQ_{n+2}) = 3n + 2$ when $n \geqslant 3$. That is, for an odd integer $n \geqslant 5$, $\lambda_2(TQ_n) = 3n - 4$. □

3.3. Crossed cubes

The n-dimensional crossed cube, CQ_n, is an interconnection network defined inductively in the following way. CQ_1 is a complete graph on two nodes with labels 0 and 1. For $n > 1$, CQ_n contains CQ^0_{n-1} and CQ^1_{n-1}, where all binary node addresses in a CQ^m_k are prefixed by m. Node $x = 0x_{n-2} \cdots x_0$ in CQ^0_{n-1} is linked to node $y = 1y_{n-2} \cdots y_0$ in CQ^1_{n-1} if and only if $x_{n-2} = y_{n-2}$ for even n, and $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i < \lfloor (n-1)/2 \rfloor$. Two binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair-related, denoted by $x \sim y$, if and only if $(x, y) \in \{(00,00),(10,10),(01,11),(11,01)\}$. From the above definition, it follows that every node in CQ_n with a leading 0 bit has exactly one neighbor with a leading 1 bit and vice versa. The CQ_n has a recursive structure and is a connected, regular graph. Efe [15] and Haq [9] proved that CQ_n can be viewed as

$G(CQ_{n-1}, CQ_{n-1}; M)$ for some perfect matching M. Kulasinghe [11] proved that CQ_n is n-connected.

Corollary 3.3. $\lambda_2(CQ_n) = 3n - 4$ *for* $n \geqslant 4$.

Proof. Since $CQ_n = G(CQ_{n-1}, CQ_{n-1}; M)$ for some perfect matching M, then $CQ_2 = G(K_2, K_2; M)$ is isomorphic to a four-cycle C_4. So its 2-regular maximum-connected, $g(CQ_2) = 4$ and $cn(CQ_2) = 2$. Applying Lemma 2.1 and Lemma 2.3 recursively, we may obtain that CQ_n is a $(n+1)$-regular maximum-connected, $g(CQ_n) \geqslant 4$ and $cn(CQ_n) = 2$ for each integer $n \geqslant 2$.

Applying Lemma 2.5, we may obtain that $\lambda_1(CQ_n) = 2n - 2$ for $n \geqslant 3$. Applying Theorem 2.6, we may obtain that $\lambda_2(CQ_n) = 3n - 4$ for $n \geqslant 4$. □

3.4. Möbius cubes

The n-dimensional Möbius cube MQ_n, proposed by Cull and Larson [5], is such a graph, whose vertex-set is the same as the vertex-set of Q_n, the vertex $X = x_1x_2 \cdots x_n$ connects to n other vertices X_i, $(1 \leqslant i \leqslant n)$, where each X_i satisfies one of the following equations:

$$X_i = x_1x_2\cdots x_{i-1}\bar{x}_ix_{i+1}\cdots x_n \quad \text{if } x_{i-1} = 0 = x_1x_2\cdots x_{i-1}\bar{x}_i\bar{x}_{i+1}\cdots\bar{x}_n \quad \text{if } x_{i-1} = 1.$$

From the above definition, X connects to X_i by complementing the bit x_i if $x_{i-1} = 0$ or by complementing all bits of $x_i, \ldots, x_n$ if $x_{i-1} = 1$. The connection between X and X_1 is undefined, so we can assume x_0 is neither equal to 0 or equal to 1, which gives us slightly different network topologies. If we assume $x_0 = 0$, we call the network a '0-Möbius cube'; and if we assume $x_0 = 1$, we call the network a '1-Möbius cube', denoted by $0\text{-}MQ_n$ and $1\text{-}MQ_n$, respectively.

From the above definition, $0\text{-}MQ_n$ and $1\text{-}MQ_n$ can be recursively constructed from a $0\text{-}MQ_{n-1}$ and $1\text{-}MQ_{n-1}$ by adding a perfect matching, where $0\text{-}MQ_1$ and $1\text{-}MQ_1$ are a complete graph K_2 with two vertices labelled by 0 and 1.

In the following, when we use $x\text{-}MQ_n$ in a proposition, we mean that the proposition hold for both $0\text{-}MQ_n$ and $1\text{-}MQ_n$.

Corollary 3.4. $\lambda_2(x\text{-}MQ_n) = 3n - 4$ *for* $n \geqslant 4$.

Proof. From the above statement, we can see that both $0\text{-}MQ_2$ and $1\text{-}MQ_2$ can be viewed as $G(K_2, K_2; M)$ for some perfect matching M, so they are both isomorphic to the 4-cycle C_4. So $g(x\text{-}MQ_2) = 4$, $x\text{-}MQ_2$ are 2-regular maximum-connected and $cn(x\text{-}MQ_2) = 2$. Applying Lemma 2.1 and Lemma 2.3 recursively, we may obtain that $x\text{-}MQ_n$ is n-regular maximum-connected, $g(x\text{-}MQ_n) \geqslant 4$ and $cn(x\text{-}MQ_n) = 2$ for each integer $n \geqslant 2$.

By Lemma 2.5, we may obtain that $\lambda_1(x\text{-}MQ_n) = 2n - 2$ for $n \geqslant 3$. Applying Theorem 2.6, we may obtain that, $\lambda_2(x\text{-}MQ_n) = 3n - 4$ for any integer $n \geqslant 4$. □

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