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Panconnectivity of locally twisted cubes[☆]

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Abstract

The locally twisted cube LTQ_n which is a newly introduced interconnection network for parallel computing is a variant of the hypercube Q_n. Yang et al. [X. Yang, G.M. Megson, D.J. Evans, Locally twisted cubes are 4-pancyclic, Applied Mathematics Letters 17 (2004) 919–925] proved that LTQ_n is Hamiltonian connected and contains a cycle of length from 4 to 2^n for $n \geq 3$. In this work, we improve this result by showing that for any two different vertices u and v in LTQ_n ($n \geq 3$), there exists a uv-path of length l with $d(u, v) + 2 \leq l \leq 2^n - 1$ except for a shortest uv-path.

MSC: 05C38; 90B10

Keywords: Locally twisted cubes; Hypercubes; Path; Hamiltonian connected

1. Introduction

An interconnection network is usually represented by an undirected simple graph $G = (V, E)$, where V and E are the vertex set and the edge set, respectively, of G. For two vertices $u, v \in V$, a path joining u and v is called a uv-path, and the distance between u and v is the length of a shortest uv-path, denoted as $d_G(u, v)$, or simply $d(u, v)$. The diameter $D(G)$ of G is the maximum distance between any two vertices of G. A Hamiltonian cycle of G is a cycle that contains every vertex of G exactly once. A graph G is Hamiltonian if G contains a Hamiltonian cycle.

Recently, properties stronger than that of the Hamiltonian are considered in many network topologies. A graph G is pancyclic if G contains a cycle of length k for each k satisfying $4 \leq k \leq |V|$. Panconnectivity is another Hamiltonian-like property. A graph G is panconnected if for any two distinct vertices u and v of G and for each integer k satisfying $d(u, v) \leq k \leq |V| - 1$, there is a uv-path of length k in G. If a graph G is panconnected, then clearly it is pancyclic. The Hamiltonian-like properties of many interconnection networks have been investigated in the literature (see, for example, [1–5]).

The hypercube network Q_n has been proved to be one of the most popular interconnection networks. The locally twisted cube LTQ_n is a variant of Q_n, proposed by Yang et al. [6]. It has many attractive features superior to those of the hypercube, such as $D(LTQ_n) = \lceil \frac{n+3}{2} \rceil$ for $n \geq 5$. In particular, Yang et al. [1] showed that LTQ_n is

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Hamiltonian connected and contains a cycle of every length from 4 to 2^n for $n \geq 3$. In this work, we investigate the panconnectivity of LTQ_n. Since there is no uv-path of length $d(u, v) + 1$ between some two vertices u and v in LTQ_n—for example, there is no uv-path of length 3 between 0000 and 1010 in LTQ_4—we prove the following theorem.

Theorem. *For any two different vertices u and v in $LTQ_n (n \geq 3)$, there exists a uv-path of length l with $d(u, v) + 2 \leq l \leq 2^n - 1$.*

The proof of the theorem is in Section 3. Section 2 recalls the definition of the n-dimensional locally twisted cube LTQ_n.

2. Locally twisted cubes

An n-dimensional locally twisted cube LTQ_n ($n \geq 2$), proposed first by Yang et al. [6], has 2^n vertices. Each vertex is an n-string on $\{0, 1\}$. Two vertices $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ are adjacent if and only if one of the following conditions is satisfied:

(1) There is an integer $1 \leq k \leq n - 2$ such that
  (a) $x_k = \bar{y}_k$ ($\bar{y}_k$ is the complement of y_k in $\{0, 1\}$),
  (b) $x_{k+1} = y_{k+1} + x_n$, and
  (c) all the remaining bits of x and y are identical.
  If so, y is called the kth-dimensional neighbor of x, denoted by $y = N_k(x)$.
(2) There is an integer $k \in \{n - 1, n\}$ such that x and y differ only in the kth bit. If so, y is called the kth-dimensional neighbor of x, denoted by $y = N_k(x)$.

According to the above definition, it is not difficult to see that LTQ_n can be recursively defined as follows. LTQ_2 is a graph consisting of four vertices labelled with 00, 01, 10, and 11, respectively connected by four edges (00, 01), (00, 10), (10, 11) and (01, 11). For $n \geq 3$, LTQ_n is constructed from two disjoint copies of LTQ_{n-1} by adding 2^{n-1} edges as follows. Let $0LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of one copy of LTQ_{n-1} with 0, let $1LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of the other copy of LTQ_{n-1} with 1, and connect each vertex $x = 0x_2x_3 \ldots x_n$ of $0LTQ_{n-1}$ with the vertex $1(x_2 + x_n)x_3 \ldots x_n$ of $1LTQ_{n-1}$ by an edge, where '+' represents the modulo 2 addition. For short, we write $LTQ_n = L \oplus R$, where $L \cong 0LTQ_{n-1}$ and $R \cong 1LTQ_{n-1}$.

Lemma. *Let u and v be two vertices in LTQ_n with $n \geq 3$. Then $d_{LTQ_n}(u, v) = d_L(u, v)$ if both u and v are in L. Similarly, $d_{LTQ_n}(u, v) = d_R(u, v)$ if both u and v are in R.*

Proof. Notice that the first bits of the vertices in L (or R) are 0 (or 1). An exact minimal routing Algorithm 4.1 given in [6] can determine a shortest path between u and v (see Theorem 4.2 in [6]), in which the first bits of all vertices are 0 (resp. 1) if u and v are in L (resp. R). The lemma follows.

3. Proof of theorem

We prove the theorem by induction on $n \geq 3$.

For $n = 3$, since LTQ_3 is vertex symmetric from Fig. 1, we only need to prove that for the vertex $u = 000$ and $v \in \{001, 111, 110, 010\}$ in LTQ_3, there exists a uv-path of length l with $d(u, v) + 2 \leq l \leq 7$. All uv-paths of required length are constructed as follows.

The paths of different lengths between 000 and 001 are listed as follows:

$P_1 = \langle 000, 001\rangle$

$P_3 = \langle 000, 010, 011, 001\rangle$

$P_4 = \langle 000, 010, 110, 111, 001\rangle$

$P_5 = \langle 000, 010, 011, 101, 111, 001\rangle$

$P_6 = \langle 000, 100, 110, 111, 101, 011, 001\rangle$

$P_7 = \langle 000, 100, 101, 011, 010, 110, 111, 001\rangle$.

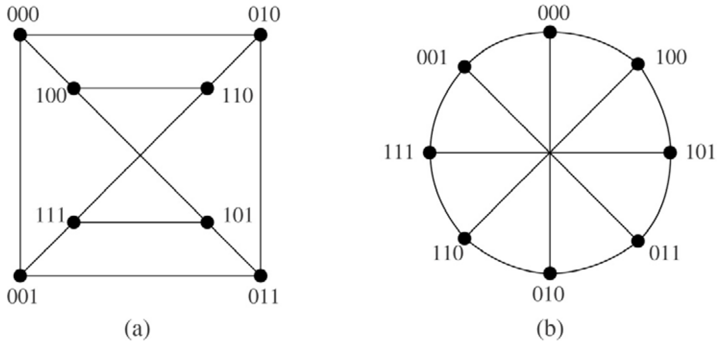

Fig. 1. (a) An ordinary drawing of LTQ_3; (b) a symmetric drawing of LTQ_3.

The paths of different lengths between 000 and 010 are listed as follows:

$P_1 = \langle 000, 010\rangle$
$P_3 = \langle 000, 100, 110, 010\rangle$
$P_4 = \langle 000, 100, 101, 011, 010\rangle$
$P_5 = \langle 000, 100, 101, 111, 110, 010\rangle$
$P_6 = \langle 000, 100, 110, 111, 001, 011, 010\rangle$
$P_7 = \langle 000, 100, 101, 011, 001, 111, 110, 010\rangle$.

The paths of different lengths between 000 and 111 are listed as follows:

$P_2 = \langle 000, 001, 111\rangle$
$P_4 = \langle 000, 001, 011, 101, 111\rangle$
$P_5 = \langle 000, 001, 011, 010, 110, 111\rangle$
$P_6 = \langle 000, 100, 101, 011, 010, 110, 111\rangle$
$P_7 = \langle 000, 001, 011, 010, 110, 100, 101, 111\rangle$.

The paths of different lengths between 000 and 110 are listed as follows:

$P_2 = \langle 000, 100, 110\rangle$
$P_4 = \langle 000, 100, 101, 111, 110\rangle$
$P_5 = \langle 000, 100, 101, 011, 010, 110\rangle$
$P_6 = \langle 000, 100, 101, 011, 001, 111, 110\rangle$
$P_7 = \langle 000, 100, 101, 111, 001, 011, 010, 110\rangle$.

Thus, the theorem holds for $n = 3$. Assume the conclusion holds for k with $3 \le k < n$. Let u and v be any two vertices in $LTQ_n = L \oplus R$. We complete the proof with the following two cases.

Case 1. Both u and v are in L or R. Without loss of generality, we may assume u and v are in L.

For $d_{LTQ_n}(u, v) + 2 = d_L(u, v) + 2 \le l \le 2^{n-1} - 1$, by the induction hypothesis, there exists a uv-path of length l in $L \subset LTQ_n$.

Suppose that $2^{n-1} \le l \le 2^n - 1$. We can write $l = l_1 + l_2 + 2$ where $0 \le l_1 \le 2^{n-1} - 2$ and $D(R) + 2 \le 2^{n-1} - 2 \le l_2 \le 2^{n-1} - 1$. Let $P_0 = \langle u = u_0, u_1, u_2, \ldots, u_{2^{n-1}-2}, v\rangle$ be a uv-path of length $2^{n-1} - 1$ in L. Let u'_i be the neighbor of u_i in R and v' be the neighbor of v in R. By the induction hypothesis, there is a $u'_{l_1}v'$-path P_R of length l_2 in R. Hence $P = \langle u, u_1, u_2, \ldots, u_{l_1}, u'_{l_1}, P_R, v', v\rangle$ is a uv-path of length l in LTQ_n (see Fig. 2(a)).

Case 2. $u = 0u_2u_3 \ldots u_n \in L$ and $v = 1v_2v_3 \ldots v_n \in R$.

We first assume $d(u, v) = 1$. Then $v = 1(u_2 + u_n)u_3 \ldots u_n$.

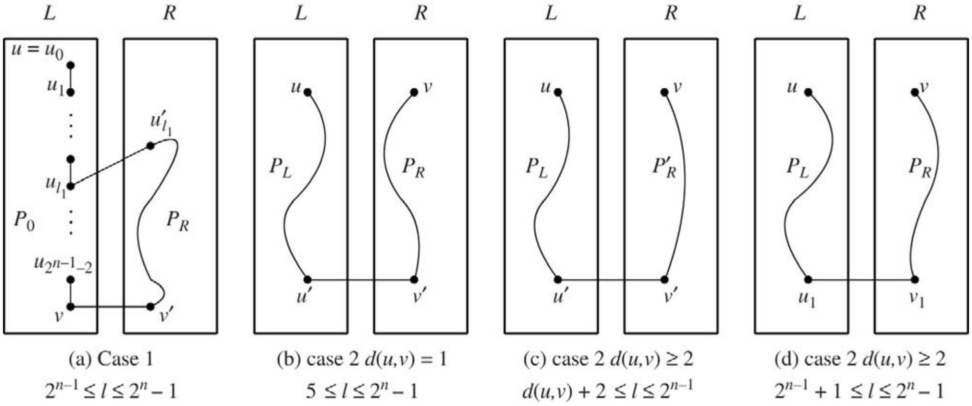

Fig. 2. Illustrations for the proof of the theorem.

Thus

$$P = \langle u = 0u_2 \ldots u_n, 0u_2 \ldots \bar{u}_{n-1}u_n, 1(u_2 + u_n)u_3 \ldots \bar{u}_{n-1}u_n, \\ 1(u_2 + u_n)u_3 \ldots u_{n-1}u_n = v\rangle$$

is a uv-path of length 3 in LTQ_n.

If $u_n = 0$,

$$P = \langle u = 0u_2u_3 \ldots u_n, 0u_2u_3 \ldots \bar{u}_n, 1(u_2 + \bar{u}_n)u_3 \ldots \bar{u}_n, \\ 1(u_2 + \bar{u}_n)u_3 \ldots u_n, 1(u_2 + u_n)u_3 \ldots u_n = v\rangle$$

is a uv-path of length 4 in LTQ_n.

If $u_n = 1$,

$$P = \langle u = 0u_2u_3 \ldots u_n, 0u_2u_3 \ldots \bar{u}_n, 1(u_2 + \bar{u}_n)u_3 \ldots \bar{u}_n, \\ 1(u_2 + u_n)u_3 \ldots \bar{u}_n, 1(u_2 + u_n)u_3 \ldots u_n = v\rangle$$

is a uv-path of length 4 in LTQ_n.

For $5 \le l \le 2^n - 1$, we can write $l = l_1 + l_2 + 1$ where $3 \le l_1 \le 2^{n-1} - 1$ and $l_2 = 1$ or $3 \le l_1 \le 2^{n-1} - 1$ and $3 \le l_2 \le 2^{n-1} - 1$. Let $u' = 0u_2u_3 \ldots \bar{u}_{n-1}u_n$ be a neighbor of u in L and $v' = 1(u_2 + u_n)u_3 \ldots \bar{u}_{n-1}u_n$ be a neighbor of v in R. It is clear that $u'v' \in E(LTQ_n)$. By the induction hypothesis, there exist a uu'-path P_L of length l_1 in L and a $v'v$-path P_R of length l_2 in R. Then $P = \langle u, P_L, u', v', P_R, v\rangle$ is a uv-path of length l in LTQ_n (see Fig. 2(b)).

We now assume $d(u, v) \ge 2$. Let P_0 be a uv-path of length $d(u, v)$ in LTQ_n. Then there is an edge $u'v'$ in P_0 with $u' \in L$ and $v' \in R$. Let $P(u, u')$ be the segment of P_0 between u and u'. Let $P(v', v)$ be the segment of P_0 between v' and v. It is clear that $P(u, u')$ is a shortest path between u and u', and $P(v', v)$ is a shortest path between v' and v. By lemma, we may assume $P(u, u') \subset L$ and $P(v', v) \subset R$. We use l' and l'' to denote the lengths of $P(u, u')$ and $P(v', v)$, respectively. Noting that $d_{LTQ_n}(u, v) = l' + l'' + 1$ and $d_{LTQ_n}(u, v) \ge 2$, we have $l' \ge 1$ or $l'' \ge 1$. We may assume $\ell' \ge 1$.

For $d(u, v) + 2 \le l \le 2^{n-1}$, we can write $l = l_1 + l'' + 1$ where $d(u, u') + 2 \le l_1 \le 2^{n-1} - 1$. By the induction hypothesis, there exists a uu'-path P_L of length l_1 in L. Then $P\langle u, P_L, u', v', P'_R, v\rangle$ is a uv-path of length l in LTQ_n (see Fig. 2(c)).

For $2^{n-1} + 1 \le l \le 2^n - 1$, we can write $l = l_1 + l_2 + 1$ where $D(L) + 2 \le l_1 \le 2^{n-1} - 1$, $D(R) + 2 \le l_2 \le 2^{n-1} - 1(D(LTQ_3) = 2)$. Choose $u_1 \in L$ such that $u_1 \ne u$ and the neighbor v_1 of u_1 in R is different from v. By the induction hypothesis, there exist a uu_1-path P_L of length l_1 in L and a v_1v-path P_R of length l_2 in R. Then $P\langle u, P_L, u_1, v_1, P_R, v\rangle$ is a uv-path of length l in LTQ_n (see Fig. 2(d)). □

We can obtain the results in [1] from the theorem.

Corollary. *For $n \ge 3$, LTQ_n is Hamiltonian connected and pancyclic.*

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