Distance Domination Numbers of Generalized de Bruijn and Kautz Digraphs

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Abstract

The distance $\ell$-domination number $\gamma_{\ell}(G)$ of a strongly connected digraph $G$ is the minimum number $\gamma$ for which there is a set $D \subseteq V(G)$ with cardinality $\gamma$ such that any vertex $v \notin D$ can be reached within distance $\ell$ from some vertex in $D$. In this paper, we establish a lower bound and an upper bound for $\gamma_{\ell}$ of a generalized de Bruijn digraph and a generalized Kautz digraph, and also give a sufficient condition for these digraphs whose $\gamma_{2}$ are equal to the lower bounds. As a consequence, for the de Bruijn digraph $B(d, k)$, we determine that $\gamma_{2}(B(d, k)) = \left\lfloor \frac{d^k}{d^2 + d + 1} \right\rfloor$. At the end of this paper, we conjecture $\gamma_{2}(K(d, k)) = \left\lfloor \frac{d^k}{d^2 + d + 1} \right\rfloor$.

Key words: Operation research, distance domination numbers, domination numbers, generalized de Bruijn digraphs, generalized Kautz digraphs

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1 Introduction

Let $G(V, A)$ be a strongly connected digraph with vertex set $V(G)$ and the arc set $A(G)$. If there is an arc $(x, y)$ from $x$ to $y$, then the vertex $x$ is called a predecessor of $y$ and $y$ is called...
a successor of $x$. The distance $d_G(x, y)$ from the vertex $x$ to the vertex $y$ is the minimum length of an $(x, y)$-directed path in $G$. The maximum distance between any two vertices in $G$ is called the diameter of $G$. For a vertex $x$ in $G$, the $k$-th out-neighborhood of $x$, $O_k(x)$, is defined by $\{ y \in V(G) : d_G(x, y) = k \}$, and $O_1(v)$ is usually called the out-neighborhood of $x$ in $G$. Let $S$ be a proper subgraph of $G$ or a nonempty subset of $V(G)$, we use the symbol $O_\ell(S)$ to denote $\bigcup_{s \in S} O_\ell(s)$. For terminology and notation not given here, the reader is referred to [2].

Let $\ell$ be a positive integer. A subset $D \subset V(G)$ is called a distance $\ell$-dominating set of $G$ if $v \in O_\ell(D)$ for all $v \notin D$, where $1 \leq k \leq \ell$. The distance $\ell$-domination number of $G$, $\gamma_\ell(G)$, is the minimum cardinality of a distance $\ell$-dominating set of $G$.

Slater [7] termed a distance $\ell$-dominating set as an $\ell$-basis and also gave an interpretation for an $\ell$-basis in terms of communication networks. Since then many researchers pay much attention to this subject, for example, see [3, 6, 7, 8]. Although digraphs have many applications, there are only a few studies for domination on digraphs [1, 5].

The concept of distance domination in graphs finds applications in many situations and structures which give rise to graphs. A minimum $\ell$-dominating set in $G$ may be used to locate a minimum number of facilities (such as utilities, police stations, waste disposal dumps, hospitals, blood banks, transmission towers) such that every intersection is within $\ell$ city block of a facility.

One is interested in determining the exact value of $\gamma_\ell(G)$ for a given strongly connected digraph $G$ and positive integer $\ell$. From definition, $\gamma_\ell(G) = 1$ for any positive integer $\ell$ not less than the diameter of $G$. Thus let $\ell$ be any positive integer less than the diameter of $G$ below. Clearly, $\gamma_1(G)$ is the classic domination number $\gamma(G)$ of $G$. However, Barkauskas and Host [1] showed that the problem of determining $\gamma(G)$ is NP-hard for a general graph $G$.

In this paper, we consider a generalized de Bruijn digraph $G_B(n, d)$ and a generalized Kautz digraph $G_K(n, d)$ with $d \geq 2$ and $n \geq d$. They have the same vertex set $\{0, 1, 2, \cdots, n-1\}$. The edge set of $G_B(n, d)$ is $\{(x, y) : y \equiv dx + i \pmod{n}, 0 \leq i < d\}$, while the edge set of $G_K(n, d)$ is $\{(x, y) : y \equiv -dx - i \pmod{n}, 0 < i \leq d\}$. In particular, for any positive integer $k$, $G_B(d^k, d)$ and $G_K(d^k + d^{k-1}, d)$ are the well-known de Bruijn digraph $B(d, k)$ and the Kautz digraph $K(d, k)$, respectively.

Kikuchi and Shibata [5] have considered $\gamma(G_B(n, d))$ and $\gamma(G_K(n, d))$. This motivates us to consider $\gamma_\ell(G_B(n, d))$ and $\gamma_\ell(G_K(n, d))$ for $\ell \geq 2$. In this paper, we establish a lower bound and an upper bound for $\gamma_\ell(G_B(n, d))$ and $\gamma_\ell(G_K(n, d))$, and also give a sufficient condition for $\gamma_2(G_B(n, d))$ and $\gamma_2(G_K(n, d))$ attaining the lower bounds. As consequence, we determine that $\gamma_2(B(d, k)) = \left[ \frac{d^2}{d^2 + d + 1} \right]$. At the end of this paper, we conjecture $\gamma_2(K(d, k)) = \left[ \frac{d^k + d^{k-1}}{d^k + d + 1} \right]$.
\section{\( \ell \)-domination numbers of \( G_B(n, d) \)}

Let \( \ell \) be any positive integer less than the diameter of \( G \) below.

**Lemma 2.1** For any positive integers \( n, d \) and \( \ell \) with \( d \geq 2 \) and \( n \geq d \),
\[
\left\lceil \frac{n}{1 + d + \cdots + d^\ell} \right\rceil \leq \gamma_\ell(G_B(n, d)) \leq \left\lceil \frac{n}{d^\ell} \right\rceil.
\]

**Proof** Let \( D \) be a minimum \( \ell \)-dominating set of \( G_B(n, d) \). From the definition of \( G_B(n, d) \), we have \( G_B(n, d) \) is \( d \)-regular and \( |D| + d|D| + \cdots + d^\ell|D| \geq n \), from which the lower bound of \( \gamma_\ell(G_B(n, d)) \) can be derived immediately.

To obtain the upper bound of \( \gamma_\ell(G_B(n, d)) \), let \( D = \{1, 2, \cdots, \left\lceil \frac{n}{d^\ell} \right\rceil \} \). Then the elements in
\[
O_\ell(D) = \{d^\ell, d^\ell + 1, \cdots, d^\ell + d^\ell \left\lceil \frac{n}{d^\ell} \right\rceil - 1 \pmod{n}\}
\]
is consecutive and \( |O_\ell(D)| = \left\lceil \frac{n}{d^\ell} \right\rceil d^\ell \geq n \). Thus \( D \) is a \( \ell \)-dominating set of \( G_B(n, d) \), and hence, \( \gamma_\ell(G_B(n, d)) \leq |D| = \left\lceil \frac{n}{d^\ell} \right\rceil \).

From the above proof, \( D = \{1, 2, \cdots, \left\lceil \frac{n}{d^\ell} \right\rceil \} \) is an \( \ell \)-dominating set, but not minimum. For example, \( \{1, 2, 3\} \) is a 2-dominating set of \( G_B(12, 2) \). Furthermore, \( \{1, 2\} \) is not a 2-dominating set of \( G_B(12, 2) \), whereas \( \gamma_2(G_B(12, 2)) = 2 \) because the vertex set \( \{2, 3\} \) is a minimum 2-dominating set of \( G_B(12, 2) \). We are interested in the conditions subject to which the lower bound of \( \gamma_\ell(G_B(n, d)) \) given in Lemma 2.1 can be attained. The following theorem gives such a sufficient condition for \( \ell = 2 \).

**Theorem 2.2** Let \( n \) and \( d \) be positive integers with \( d \geq 2 \) and \( n \geq d \), and let \( m = \left\lceil \frac{n}{1 + d + d^2} \right\rceil \). If there is a vertex \( x \in V(G_B(n, d)) \) satisfying
\[
(d - 1)x - (m - \ell_1) \equiv 0 \pmod{n} \quad (2.1)
\]
\[
(d^2 - d)x - (dm - \ell_2) \equiv 0 \pmod{n} \quad (2.2)
\]
for some nonnegative integers \( \ell_1 \leq dm \) and \( \ell_2 \leq d^2m \) with \( 0 \leq \ell_1 + \ell_2 \leq (1 + d + d^2)m - n \), then
\[
\gamma_2(G_B(n, d)) = m
\]

**Proof** From Lemma 2.1, we only need to construct a 2-dominating set of \( G_B(n, d) \) with cardinality \( m \) under the given conditions. Let
\[
D = \{x, x + 1, \cdots, x + m - 1\} \pmod{n} \subseteq V(G_B(n, d))
\]
For short, let \( D_1 = O_1(D) \) and \( D_2 = O_2(D) \). Then, from the definition of \( G_B(n, d) \), we have
\[
D_1 = \{dx, dx + 1, \cdots, dx + dm - 1\} \pmod{n},
\]
\[
D_2 = \{d^2x, d^2x + 1, \cdots, d^2x + d^2m - 1\} \pmod{n}.
\]
Since \( dm < d^2 m < n \), we have
\[
|D| = m, \ |D_1| = dm, \ |D_2| = d^2 m. \tag{2.3}
\]

To show that \( D \) is a 2-dominating set of \( G_B(n, d) \), it is sufficient to prove that \( |D \cup D_1 \cup D_2| \geq n \). Let
\[
\sigma(D) = |D \cap D_1| + |D \cap D_2| + |D_1 \cap D_2| - |D \cap D_1 \cap D_2|.
\]

If we can prove that
\[
\sigma(D) \leq \ell_1 + \ell_2, \tag{2.4}
\]
then by the inclusion-exclusion principle, from (3), (4) and the given conditions, we have that
\[
|D \cup D_1 \cup D_2| = |D| + |D_1| + |D_2| - \sigma(D) \\
\geq m + dm + d^2 m - (\ell_1 + \ell_2) \\
\geq (1 + d + d^2)m - [(1 + d + d^2)m - n] \\
= n
\]

We now prove the inequality (4). For this purpose, we rewrite Equations (1) and (2) as follows.
\[
dx \equiv x + m - \ell_1 \pmod{n} \\
d^2x \equiv dx + dm - \ell_2 \pmod{n} \\
\equiv x + m + dm - (\ell_1 + \ell_2) \pmod{n}.
\]

**Case 1.** \( m < \ell_1 \).

In this case, we have \( dx = x + m - \ell_1 < x \), that is, \( dx + \ell_1 - m = x \), and by \( 1 + (\ell_1 - m) + (m - 1) = \ell_1 \leq dm \), which implies that \( D \subseteq D_1 \), and so \( |D \cap D_1| = |D| = m < \ell_1 \).

**Subcase 1.1.** If \( dm < \ell_2 \), then \( d^2x = dx + dm - \ell_2 < dx \), that is, \( d^2x + \ell_2 - dm = dx \), and by \( 1 + (\ell_2 - dm) + (dm - 1) = \ell_2 \leq d^2 m \), which implies that \( D_1 \subset D_2 \), and so \( |D_1 \cap D_2| = |D_1| = dm < \ell_2 \). Since \( D \subset D_1 \subset D_2 \), \( |D \cap D_1 \cap D_2| = |D_1 \cap D_2| = |D| = m \). It follows that \( \sigma(D) < \ell_1 + m + \ell_2 - m = \ell_1 + \ell_2 \).

**Subcase 1.2.** If \( dm \geq \ell_2 \), then \( |D_1 \cap D_2| = \ell_2 \). In fact, if \( \ell_2 \geq 1 \), then \( D_1 \cap D_2 = \{dx + dm - \ell_2, \cdots, dx + dm - 1\} \), and hence, \( |D_1 \cap D_2| = \ell_2 \); if \( \ell_2 = 0 \), then \( D_1 \cap D_2 = \emptyset \) and \( |D_1 \cap D_2| = 0 \).

If \( m + dm < \ell_1 + \ell_2 \), we have \( d^2x = x + m + dm - (\ell_1 + \ell_2) < x \), that is, \( d^2x + (\ell_1 + \ell_2) - (m + dm) = x \), and by \( \ell_1 \leq dm, \ \ell_2 \leq d^2 m \), then \( 1 + (\ell_1 + \ell_2 - m - dm) + (m - 1) = \ell_1 + \ell_2 - dm \leq d^2 m \), which implies \( D \subset D_2 \), and so \( |D \cap D_2| = |D| = m \). Thus by \( D \subset D_1 \), \( |D \cap D_1 \cap D_2| = |D \cap D_2| = |D| = m \). It follows that \( \sigma(D) < \ell_1 + m + \ell_2 - m = \ell_1 + \ell_2 \).

If \( m + dm \geq \ell_1 + \ell_2 \), it also follows that \( \sigma(D) \leq \ell_1 + \ell_2 \). In fact, if \( \ell_1 + \ell_2 \geq dm + 1 \), then \( D \cap D_2 = \{x + m - (\ell_1 + \ell_2 - dm), \cdots, x + m - 1\} \), and so \( |D \cap D_2| = \ell_1 + \ell_2 - dm \). Thus \( |D \cap D_1 \cap D_2| = |D \cap D_2| = \ell_1 + \ell_2 - dm \). It follows that \( \sigma(D) < \ell_1 + (\ell_1 + \ell_2 - dm) + \)}
Corollary For the de Bruijn digraph \( B(d,k) \), \( \gamma_2(B(d,k)) = \left\lceil \frac{d^k}{d^k - 1} \right\rceil \).

Proof For \( k = 1, 2 \), \( B(d,1) = K_d^+ \) and \( B(d,2) = L(K_d^+) \), that is, the line digraph of \( B(d,1) \). We could easily verify that \( \gamma_2(B(d,1)) = 1 \) since \( K_d^+ \) is obtained from a complete digraph \( K_d \) by appending a loop at each vertex. For \( k = 2 \), let \( ab \) be any vertex in \( V(B(d,2)) \), then we will prove that for any other vertex \( b \) in \( V(B(d,2)) \), \( b \) can be reached from \( ab \) within distance 2. Since \( B(d,2) \) is the line digraph of \( K_d^+ \), then there are \( u, v, z, w \in V(K_d^+) \) such that \( a = (u,v) \), \( b = (z,w) \). If \( v = z \), then \( b \) is adjacent from \( ab \) in \( B(d,2) \). If \( v \neq z \), then there must exist an edge \((v,z)\) in \( K_d^+ \). Let \( P = \{a\} \cup (v,z) \cup \{b\} \), then the line digraph \( L(P) \) is a path from \( a \) to \( b \) with distance 2.

For \( k \geq 3 \), we only need to find a common solution to satisfy the equations (1) and (2) in Theorem 2.2 under the given conditions. We first assume that \( k = 3h \), where \( h \) is a positive integer. Then the value of \( \ell_1 + \ell_2 \) in Theorem 2.2 is an integer between 0 and \( d + d^2 \). From the definition of \( B(d,k) \), we have \( d \geq 2 \). The equations
\[
x = d^{3h-3} + d^{3h-6} + \cdots + d^3 + 1 \pmod{d^k} \\
\ell_1 = 1 \\
\ell_2 = d
\]
leads \( x \) to satisfy Equations (1) and (2) in Theorem 2.2.

Next let us assume that \( k = 3h + 1 \), then the value of \( \ell_1 + \ell_2 \) in Theorem 2.2 is an integer between 0 and \( 1 + d^2 \). We put

\[
\begin{align*}
  x & \equiv d^{3h-2} + d^{3h-5} + \cdots + d^1 + d \pmod{d^k} \\
  \ell_1 & = 1 \\
  \ell_2 & = d
\end{align*}
\]

then \( x \) satisfies Equations (1) and (2) in Theorem 2.2.

At last we assume \( k = 3h + 2 \), then the value of \( \ell_1 + \ell_2 \) in Theorem 2.2 is an integer between 0 and \( 1 + d \). Let

\[
\begin{align*}
  x & \equiv d^{3h-1} + d^{3h-4} + \cdots + d^5 + d^2 \pmod{d^k} \\
  \ell_1 & = 1 \\
  \ell_2 & = d
\end{align*}
\]

then \( x \) satisfies Equations (1) and (2) in Theorem 2.2.

## 3 \( \ell \)-domination numbers of \( G_K(n,d) \)

Consider a generalized Kautz digraph \( G_K(n,d) \), we can prove analogies of a generalized de Bruijn digraph \( G_B(n,d) \) by a similar argument. We state them as follows, but the proofs are omitted here.

**Lemma 3.1** For any positive integers \( n, d \) and \( \ell \) with \( d \geq 2 \), we have that

\[
\left\lceil \frac{n}{1 + d + \cdots + d^\ell} \right\rceil \leq \gamma_\ell(G_K(n,d)) \leq \left\lceil \frac{n}{d^\ell} \right\rceil.
\]

**Theorem 3.2** Let \( n \) and \( d \) be positive integers with \( d \geq 2 \) and \( n \geq d \), and let \( m = \left\lceil \frac{n}{1 + d + \cdots + d^\ell} \right\rceil \).

If there is a vertex \( x \in V(G_K(n,d)) \) satisfying

\[
\begin{align*}
  (d + 1)x + (d + 1)m - \ell_1 & \equiv 0 \pmod{n} \\
  d(d + 1)x + \ell_2 & \equiv 0 \pmod{n}
\end{align*}
\]

for some nonnegative integers \( \ell_1 \leq dm \) and \( \ell_2 \leq d^2m \) with \( 0 \leq \ell_1 + \ell_2 \leq (1 + d + d^2)m - n \), then \( \gamma_2(G_K(n,d)) = m \).

For the Kautz digraph \( K(d,k) \), it seems to have an analogy of the de Bruijn digraph \( B(d,k) \) in Corollary of Theorem 2.2. However, unfortunately, we find that Equations (5) have no solution for \( K(d,k) \) in general. We propose the following conjecture.

**Conjecture** For the Kautz digraph \( K(d,k) \), \( \gamma_2(K(d,k)) = \left\lceil \frac{d^{k+1}-1}{d^k+d+1} \right\rceil \).
References


