2006 年 3 月 运筹学学报 第 10 卷 第 1 期
March, 2006 OR Transactions Vol.10 No.1

Distance Domination Numbers of Generalized de Bruijn and Kautz Digraphs*

Tian Fang† Xu Junming†

Abstract

The distance ℓ-domination number $\gamma_\ell(G)$ of a strongly connected digraph G is the minimum number γ for which there is a set $D \subset V(G)$ with cardinality γ such that any vertex $v \notin D$ can be reached within distance ℓ from some vertex in D. In this paper, we establish a lower bound and an upper bound for γ_ℓ of a generalized de Bruijn digraph and a generalized Kautz digraph, and also give a sufficient condition for these digraphs whose γ_2 are equal to the lower bounds. As a consequence, for the de Bruijn digraph $B(d,k)$, we determine that $\gamma_2(B(d,k)) = \left\lceil \frac{d^k}{d^2+d+1} \right\rceil$. At the end of this paper, we conjecture $\gamma_2(K(d,k)) = \left\lceil \frac{d^k+d^{k-1}}{d^2+d+1} \right\rceil$.

Key words: Operation research, distance domination numbers, domination numbers, generalized de Bruijn digraphs, generalized Kautz digraphs

Subject Classification (GB/T13745-92): 110.74

广义 de Bruijn 和 Kautz 有向图的距离控制数

田 方† 徐俊明†

摘 要: 对于任意的正整数 ℓ, 强连通图 G 的顶点子集 D 被称为距离 ℓ- 控制集，是指对于任意顶点 $v \notin D$ ， D 中至少含有一个顶点 u, 使得距离 $d_G(u,v) \leq \ell$. 图 G 距离 ℓ- 控制数 $\gamma_\ell(G)$ 是指 G 中所有距离 ℓ- 控制集的基数的最小者. 本文给出了广义 de Bruijn 和广义 Kautz 有向图的距离 ℓ- 控制数的上界和下界，并且给出当它们的距离 2- 控制数达到下界时的一个充分条件. 从而得到对于 de Bruijn 有向图 $B(d,k)$ 的距离 2- 控制数 $\gamma_2(B(d,k)) = \left\lceil \frac{d^k}{d^2+d+1} \right\rceil$. 在该文结尾，我们猜想 Kautz 有向图 $K(d,k)$ 的距离 2- 控制数 $\gamma_2(K(d,k)) = \left\lceil \frac{d^k+d^{k-1}}{d^2+d+1} \right\rceil$.

关键词: 运筹学，距离控制数，控制数，广义 de Bruijn 有向图，广义 Kautz 有向图

学科分类号(GB/T13745-92): 110.74

1 Introduction

Let $G(V,A)$ be a strongly connected digraph with vertex set $V(G)$ and the arc set $A(G)$. If there is an arc (x,y) from x to y, then the vertex x is called a predecessor of y and y is called

Received: June 9, 2004.

*The work was supported partially by NNSF of China (No.10271114).

†Department of Mathematics, University of Science and Technology of China, Hefei 230026, China; 中国科学技术大学数学系，合肥， 230026.

a successor of x. The distance $d_G(x,y)$ from the vertex x to the vertex y is the minimum length of an (x,y)-directed path in G. The maximum distance between any two vertices in G is called the diameter of G. For a vertex x in G, the k-th out-neighborhood of x, $O_k(x)$, is defined by $\{y \in V(G): \ d_G(x,y)=k\}$, and $O_1(v)$ is usually called the out-neighborhood of x in G. Let S be a proper subgraph of G or a nonempty subset of $V(G)$, we use the symbol $O_\ell(S)$ to denote $\bigcup_{s\in S} O_\ell(s)$. For terminology and notation not given here, the reader is referred to [2].

Let ℓ be a positive integer. A subset $D \subset V(G)$ is called a distance ℓ-dominating set of G if $v \in O_k(D)$ for all $v \notin D$, where $1 \leq k \leq \ell$. The distance ℓ-domination number of G, $\gamma_\ell(G)$, is the minimum cardinality of a distance ℓ-dominating set of G.

Slater [7] termed a distance ℓ-dominating set as an ℓ-basis and also gave an interpretation for an ℓ-basis in terms of communication networks. Since then many researchers pay much attention to this subject, for example, see [3, 6, 7, 8]. Although digraphs have many applications, there are only a few studies for domination on digraphs [1, 5].

The concept of distance domination in graphs finds applications in many situations and structures which give rise to graphs. A minimum ℓ-dominating set in G may be used to locate a minimum number of facilities (such as utilities, police stations, waste disposal dumps, hospitals, blood banks, transmission towers) such that every intersection is within ℓ city block of a facility.

One is interested in determining the exact value of $\gamma_\ell(G)$ for a given strongly connected digraph G and positive integer ℓ. From definition, $\gamma_\ell(G)=1$ for any positive integer ℓ not less than the diameter of G. Thus let ℓ be any positive integer less than the diameter of G below. Clearly, $\gamma_1(G)$ is the classic domination number $\gamma(G)$ of G. However, Barkauskas and Host [1] showed that the problem of determining $\gamma(G)$ is NP-hard for a general graph G.

In this paper, we consider a generalized de Bruijn digraph $G_B(n,d)$ and a generalized Kautz digraph $G_K(n,d)$ with $d \geq 2$ and $n \geq d$. They have the same vertex set $\{0,\ 1,\ 2, \cdots,\ n-1\ \}$. The edge set of $G_B(n,d)$ is $\{(x,y): y \equiv dx+i \pmod n,\ 0 \leq i < d\}$, while the edge set of $G_K(n,d)$ is $\{(x,y): y \equiv -dx-i \pmod n,\ 0 < i \leq d\}$. In particular, for any positive integer k, $G_B(d^k,d)$ and $G_K(d^k+d^{k-1},d)$ are the well-known de Bruijn digraph $B(d,k)$ and the Kautz digraph $K(d,k)$, respectively.

Kikuchi and Shibata [5] have considered $\gamma(G_B(n,d))$ and $\gamma(G_K(n,d))$. This motivates us to consider $\gamma_\ell(G_B(n,d))$ and $\gamma_\ell(G_K(n,d))$ for $\ell \geq 2$. In this paper, we establish a lower bound and an upper bound for $\gamma_\ell(G_B(n,d))$ and $\gamma_\ell(G_K(n,d))$, and also give a sufficient condition for $\gamma_2(G_B(n,d))$ and $\gamma_2(G_K(n,d))$ attaining the lower bounds. As consequence, we determine that $\gamma_2(B(d,k)) = \left\lceil \frac{d^k}{d^2+d+1} \right\rceil$. At the end of this paper, we conjecture $\gamma_2(K(d,k)) = \left\lceil \frac{d^k+d^{k-1}}{d^2+d+1} \right\rceil$.

2 ℓ-domination numbers of $G_B(n, d)$

Let ℓ be any positive integer less than the diameter of G below.

Lemma 2.1 *For any positive integers n, d and ℓ with $d \geq 2$ and $n \geq d$,*

$$\left\lceil \frac{n}{1+d+\cdots+d^{\ell}} \right\rceil \leq \gamma_{\ell}(G_B(n,d)) \leq \left\lceil \frac{n}{d^{\ell}} \right\rceil .$$

Proof Let D be a minimum ℓ-dominating set of $G_B(n, d)$. From the definition of $G_B(n, d)$, we have $G_B(n, d)$ is d-regular and $|D| + d|D| + \cdots + d^{\ell}|D| \geq n$, from which the lower bound of $\gamma_{\ell}(G_B(n, d))$ can be derived immediately.

To obtain the upper bound of $\gamma_{\ell}(G_B(n, d))$, let $D = \{1, 2, \cdots, \lceil \frac{n}{d^{\ell}} \rceil\}$. Then the elements in

$$O_{\ell}(D) = \{d^{\ell}, d^{\ell}+1, \cdots, d^{\ell} + d^{\ell}\left\lceil \frac{n}{d^{\ell}} \right\rceil - 1\} \pmod n$$

is consecutive and $|O_{\ell}(D)| = \lceil \frac{n}{d^{\ell}} \rceil d^{\ell} \geq n$. Thus D is a ℓ-dominating set of $G_B(n, d)$, and hence, $\gamma_{\ell}(G_B(n, d)) \leq |D| = \lceil \frac{n}{d^{\ell}} \rceil$.

From the above proof, $D = \{1, 2, \cdots, \lceil \frac{n}{d^{\ell}} \rceil\}$ is an ℓ-dominating set, but not minimum. For example, $\{1, 2, 3\}$ is a 2-dominating set of $G_B(12, 2)$. Furthermore, $\{1, 2\}$ is not a 2-dominating set of $G_B(12, 2)$, whereas $\gamma_2(G_B(12, 2)) = 2$ because the vertex set $\{2, 3\}$ is a minimum 2-dominating set of $G_B(12, 2)$. We are interested in the conditions subject to which the lower bound of $\gamma_{\ell}(G_B(n, d))$ given in Lemma 2.1 can be attained. The following theorem gives such a sufficient condition for $\ell = 2$.

Theorem 2.2 *Let n and d be positive integers with $d \geq 2$ and $n \geq d$, and let $m = \lceil \frac{n}{1+d+d^2} \rceil$. If there is a vertex $x \in V(G_B(n, d))$ satisfying*

$$(d-1)x - (m - \ell_1) \equiv 0 \pmod n \tag{2.1}$$

$$(d^2-d)x - (dm - \ell_2) \equiv 0 \pmod n \tag{2.2}$$

for some nonnegative integers $\ell_1 \leq dm$ and $\ell_2 \leq d^2 m$ with $0 \leq \ell_1 + \ell_2 \leq (1+d+d^2)m - n$, then

$$\gamma_2(G_B(n, d)) = m$$

Proof From Lemma 2.1, we only need to construct a 2-dominating set of $G_B(n, d)$ with cardinality m under the given conditions. Let

$$D = \{x, x+1, \cdots, x+m-1\} \pmod n \subseteq V(G_B(n, d)).$$

For short, let $D_1 = O_1(D)$ and $D_2 = O_2(D)$. Then, from the definition of $G_B(n, d)$, we have

$$D_1 = \{dx, dx+1, \cdots, dx+dm-1\} \pmod n,$$
$$D_2 = \{d^2x, d^2x+1, \cdots, d^2x+d^2m-1\} \pmod n.$$

Since $d\,m < d^2\,m < n$, we have

$$|D| = m,\ |D_1| = dm,\ |D_2| = d^2m. \tag{2.3}$$

To show that D is a 2-dominating set of $G_B(n,d)$, it is sufficient to prove that $|D\cup D_1\cup D_2| \geq n$. Let

$$\sigma(D) = |D\cap D_1| + |D\cap D_2| + |D_1\cap D_2| - |D\cap D_1\cap D_2|.$$

If we can prove that

$$\sigma(D) \leq \ell_1 + \ell_2, \tag{2.4}$$

then by the inclusion-exclusion principle, from (3), (4) and the given conditions, we have that

$$\begin{aligned} |D\cup D_1\cup D_2| &= |D| + |D_1| + |D_2| - \sigma(D) \\ &\geq m + dm + d^2m - (\ell_1+\ell_2) \\ &\geq (1+d+d^2)m - [(1+d+d^2)m - n] \\ &= n \end{aligned}$$

We now prove the inequality (4). For this purpose, we rewrite Equations (1) and (2) as follows.

$$\begin{aligned} dx &\equiv x + m - \ell_1 \pmod{n} \\ d^2x &\equiv dx + dm - \ell_2 \pmod{n} \\ &\equiv x + m + dm - (\ell_1+\ell_2) \pmod{n}. \end{aligned}$$

Case 1. $m < \ell_1$.

In this case, we have $dx = x + m - \ell_1 < x$, that is, $dx + \ell_1 - m = x$, and by $1 + (\ell_1 - m) + (m-1) = \ell_1 \leq dm$, which implies that $D \subset D_1$, and so $|D\cap D_1| = |D| = m < \ell_1$.

Subcase 1.1. If $dm < \ell_2$, then $d^2x = dx + dm - \ell_2 < dx$, that is, $d^2x + \ell_2 - dm = dx$, and by $1 + (\ell_2 - dm) + (dm-1) = \ell_2 \leq d^2m$, which implies that $D_1 \subset D_2$, and so $|D_1\cap D_2| = |D_1| = dm < \ell_2$. Since $D \subset D_1 \subset D_2$, $|D\cap D_1\cap D_2| = |D\cap D_2| = |D| = m$. It follows that $\sigma(D) < \ell_1 + m + \ell_2 - m = \ell_1 + \ell_2$.

Subcase 1.2. If $dm \geq \ell_2$, then $|D_1\cap D_2| = \ell_2$. In fact, if $\ell_2 \geq 1$, then $D_1\cap D_2 = \{dx + dm - \ell_2, \cdots, dx + dm - 1\}$, and hence, $|D_1\cap D_2| = \ell_2$; if $\ell_2 = 0$, then $D_1\cap D_2 = \emptyset$ and $|D_1\cap D_2| = 0$.

If $m + dm < \ell_1 + \ell_2$, we have $d^2x = x + m + dm - (\ell_1+\ell_2) < x$, that is, $d^2x + (\ell_1 + \ell_2) - (m+dm) = x$, and by $\ell_1 \leq dm$, $\ell_2 \leq d^2m$, then $1 + (\ell_1 + \ell_2 - m - dm) + (m-1) = \ell_1 + \ell_2 - dm \leq d^2m$, which implies $D \subset D_2$, and so $|D\cap D_2| = |D| = m$. Thus by $D \subset D_1$, $|D\cap D_1\cap D_2| = |D\cap D_2| = |D| = m$. It follows that $\sigma(D) < \ell_1 + m + \ell_2 - m = \ell_1 + \ell_2$.

If $m + dm \geq \ell_1 + \ell_2$, it also follows that $\sigma(D) \leq \ell_1 + \ell_2$. In fact, if $\ell_1 + \ell_2 \geq dm + 1$, then $D\cap D_2 = \{x + m - (\ell_1 + \ell_2 - dm), \cdots, x + m - 1\}$, and so $|D\cap D_2| = \ell_1 + \ell_2 - dm$. Thus $|D\cap D_1\cap D_2| = |D\cap D_2| = \ell_1 + \ell_2 - dm$. It follows that $\sigma(D) < \ell_1 + (\ell_1 + \ell_2 - dm) +$

$\ell_2 - (\ell_1 + \ell_2 - dm) = \ell_1 + \ell_2$. If $\ell_1 + \ell_2 \leq dm$, then $D \cap D_2 = \emptyset$, and so $|D \cap D_2| = 0$. Thus $|D \cap D_1 \cap D_2| = |D \cap D_2| = 0$. It also follows that $\sigma(D) < \ell_1 + 0 + \ell_2 - 0 = \ell_1 + \ell_2$.

Case 2. If $m \geq \ell_1$, then $|D \cap D_1| = \ell_1$. In fact, if $\ell_1 \geq 1$, then $D \cap D_1 = \{x + m - \ell_1, \cdots, x+m-1\}$, and hence, $|D \cap D_1| = \ell_1$. If $\ell_1 = 0$, then $D \cap D_1 = \emptyset$ and $|D \cap D_1| = \ell_1 = 0$.

Subcase 2.1. If $dm < \ell_2$, then $d^2x = dx + dm - \ell_2 < dx$, that is, $d^2x + \ell_2 - dm = dx$, and by $\ell_2 \leq d^2m$, which implies that $D_1 \subset D_2$, and so $|D_1 \cap D_2| = |D_1| = dm$.

If $m + dm < \ell_1 + \ell_2$, we also have $d^2x = x + m + dm - (\ell_1 + \ell_2) < x$, that is, $d^2x + (\ell_1 + \ell_2) - (m + dm) = x$, and by $\ell_1 \leq dm$, $\ell_2 \leq d^2m$, then $\ell_1 + \ell_2 - dm \leq d^2m$, which implies $D \subset D_2$, and so $|D \cap D_2| = |D| = m$. And by $D_1 \subset D_2$, $|D \cap D_1 \cap D_2| = |D \cap D_1| = \ell_1$. It follows that $\sigma(D) = \ell_1 + m + dm - \ell_1 < \ell_1 + \ell_2$.

If $m + dm \geq \ell_1 + \ell_2$, it also follows that $\sigma(D) \leq \ell_1 + \ell_2$. In fact, since $\ell_2 > dm$, then $\ell_1 + \ell_2 \geq dm + 1$, thus $D \cap D_2 = \{x + m - (\ell_1 + \ell_2 - dm), \cdots, x + m - 1\}$, and so $|D \cap D_2| = \ell_1 + \ell_2 - dm > \ell_1$. Thus $D \cap D_1 \cap D_2 = D \cap D_1 = \{x + m - \ell_1, \cdots, x + m - 1\}$, and $|D \cap D_1 \cap D_2| = |D \cap D_1| = \ell_1$. It follows that $\sigma(D) \leq \ell_1 + (\ell_1 + \ell_2 - dm) + dm - \ell_1 = \ell_1 + \ell_2$.

Subcase 2.2. If $dm \geq \ell_2$, then $|D_1 \cap D_2| = \ell_2$. In fact, if $\ell_2 \geq 1$, then $D_1 \cap D_2 = \{dx + dm - \ell_2, \cdots, dx + dm - 1\}$, and so $|D_1 \cap D_2| = \ell_2$. If $\ell_2 = 0$, then $D_1 \cap D_2 = \emptyset$ and $|D_1 \cap D_2| = 0$.

Since $m + dm \geq \ell_1 + \ell_2$, it also follows that $\sigma(D) \leq \ell_1 + \ell_2$. In fact, if $\ell_1 + \ell_2 \geq dm + 1$, then $D \cap D_2 = \{x + m - (\ell_1 + \ell_2 - dm), \cdots, x + m - 1\}$, and so $|D \cap D_2| = \ell_1 + \ell_2 - dm \leq \ell_1$. Thus $|D \cap D_1 \cap D_2| \leq |D \cap D_2| = \ell_1 + \ell_2 - dm \leq \ell_1$. It follows that $\sigma(D) \leq \ell_1 + (\ell_1 + \ell_2 - dm) + \ell_2 - (\ell_1 + \ell_2 - dm) = \ell_1 + \ell_2$. If $\ell_1 + \ell_2 \leq dm$, then $D \cap D_2 = \emptyset$, and so $|D \cap D_2| = 0$. Thus $|D \cap D_1 \cap D_2| = 0$. It also follows that $\sigma(D) = \ell_1 + 0 + \ell_2 - 0 = \ell_1 + \ell_2$.

The proof of the theorem is completed.

Corollary *For the de Bruijn digraph $B(d,k)$, $\gamma_2(B(d,k)) = \left\lceil \frac{d^k}{d^2+d+1} \right\rceil$.*

Proof For $k = 1, 2$, $B(d, 1) = K_d^+$ and $B(d, 2) = L(K_d^+)$, that is, the line digraph of $B(d, 1)$. We could easily verify that $\gamma_2(B(d, 1)) = 1$ since K_d^+ is obtained from a complete digraph K_d by appending a loop at each vertex. For $k = 2$, let a be any vertex in $V(B(d, 2))$, then we will prove that for any other vertex b in $V(B(d, 2))$, b can be reached from a within distance 2. Since $B(d, 2)$ is the line digraph of K_d^+, then there are $u, v, z, w \in V(K_d^+)$ such that $a = (u, v)$, $b = (z, w)$. If $v = z$, then b is adjacent from a in $B(d, 2)$. If $v \neq z$, then there must exist an edge (v, z) in K_d^+. Let $P = \{a\} \cup (v, z) \cup \{b\}$, then the line digraph $L(P)$ is a path from a to b with distance 2.

For $k \geq 3$, we only need to find a common solution to satisfy the equations (1) and (2) in Theorem 2.2 under the given conditions. We first assume that $k = 3h$, where h is a positive integer. Then the value of $\ell_1 + \ell_2$ in Theorem 2.2 is an integer between 0 and $d + d^2$. From the definition of $B(d,k)$, we have $d \geq 2$. The equations

$$
\begin{aligned}
x &\equiv d^{3h-3} + d^{3h-6} + \cdots + d^3 + 1 \pmod{d^k} \\
\ell_1 &= 1 \\
\ell_2 &= d
\end{aligned}
$$

leads x to satisfy Equations (1) and (2) in Theorem 2.2.

Next let us assume that $k = 3h + 1$, then the value of $\ell_1 + \ell_2$ in Theorem 2.2 is an integer between 0 and $1 + d^2$. We put

$$\begin{aligned} x &\equiv d^{3h-2} + d^{3h-5} + \cdots + d^4 + d \pmod{d^k} \\ \ell_1 &= 1 \\ \ell_2 &= d \end{aligned}$$

then x satisfies Equations (1) and (2) in Theorem 2.2.

At last we assume $k = 3h + 2$, then the value of $\ell_1 + \ell_2$ in Theorem 2.2 is an integer between 0 and $1 + d$. Let

$$\begin{aligned} x &\equiv d^{3h-1} + d^{3h-4} + \cdots + d^5 + d^2 \pmod{d^k} \\ \ell_1 &= 1 \\ \ell_2 &= d \end{aligned}$$

then x satisfies Equations (1) and (2) in Theorem 2.2.

3 ℓ-domination numbers of $G_K(n, d)$

Consider a generalized Kautz digraph $G_K(n, d)$, we can prove analogies of a generalized de Bruijn digraph $G_B(n, d)$ by a similar argument. We state them as follows, but the proofs are omitted here.

Lemma 3.1 *For any positive integers n, d and ℓ with $d \geq 2$, we have that*

$$\left\lceil \frac{n}{1 + d + \cdots + d^\ell} \right\rceil \leq \gamma_\ell(G_K(n, d)) \leq \left\lceil \frac{n}{d^\ell} \right\rceil.$$

Theorem 3.2 *Let n and d be positive integers with $d \geq 2$ and $n \geq d$, and let $m = \lceil \frac{n}{1+d+d^2} \rceil$. If there is a vertex $x \in V(G_K(n, d))$ satisfying*

$$\begin{aligned} (d+1)x + (d+1)m - \ell_1 &\equiv 0 \pmod{n} \\ d(d+1)x + \ell_2 &\equiv 0 \pmod{n} \end{aligned} \tag{3.5}$$

for some nonnegative integers $\ell_1 \leq dm$ and $\ell_2 \leq d^2 m$ with $0 \leq \ell_1 + \ell_2 \leq (1 + d + d^2)m - n$, then $\gamma_2(G_K(n, d)) = m$.

For the Kautz digraph $K(d, k)$, it seems to have an analogy of the de Bruijn digraph $B(d, k)$ in Corollary of Theorem 2.2. However, unfortunately, we find that Equations (5) have no solution for $K(d, k)$ in general. We propose the following conjecture.

Conjecture *For the Kautz digraph $K(d, k)$, $\gamma_2(K(d, k)) = \left\lceil \frac{d^k + d^{k-1}}{d^2+d+1} \right\rceil$.*

References

[1] A. E. Barkuskas, L.H. Host, Finding efficient dominating sets in oriented graphs, *Congr,Numer.*, **98** (1993), 27-32.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications. New York: North Holland, 1976.

[3] M. Fischermann and L. Volkmann, Graphs having distance-n domination number half their orer. *Discrete Applied Math.*, **120** (2002), 97-107.

[4] J. Ghoshal, R. Laskar, D. Pillone, Topics on domination in directed graphs, in *domination in Graphs* (eds. T.W. Haynes, S.T. Hedetniemi, P.J. Slater), Marcel Dekker, New York, (1998), pp. 401-437.

[5] Y. Kikuchi and Y. Shibata, On the domination numbers of generalized de Bruijn digraphs and generalized Kautz digraphs. *Inform. process. Lett.*, **86** (2003) 79-85.

[6] A. Meir and and J. W. Moon, Relation between packing and covering number of a tree. *Pacific Journal of Mathematics*, **61**(1) (1975), 225-233.

[7] P. J. Slater, R-dominations in graphs. *J. Assoc. Comput. Macg.*, **23** (1976), 446-460.

[8] N. Sridharan, V. S. A. Subramanian and M. D. Elias, Bounds on the Distance Two-Domination Number of a Graph. *Graphs and Combinatorics.*, **18** (2002), 667-675.