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Edge-fault-tolerant hamiltonicity of folded hypercubes*

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Abstract: For any n-dimensional ($n\geqslant 3$) folded hypercube with at most $2n-3$ faulty edges in which each vertex is incident with at least two fault-free edges, it is proved that there exists a fault-free Hamiltonian cycle. The result is optimal.

Key words: hypercube; folded hypercube; fault-tolerance; Hamiltonian cycle

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折叠超立方体网络的边容错哈密顿性

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摘要:证明了在至多具有 $2n-3$ 条故障边的 n 维($n\geqslant 3$)折叠超立方体网络中,如果每个顶点至少与两条非故障边相邻,则存在一个不含故障边的哈密顿圈.这个界是最好的.

关键词:超立方体;折叠超立方体;容错;哈密顿圈

0 Introduction

We follow Bondy and Murty[1] or XU[2] for graph-theoretical terminology and notation which are not defined here. A graph $G=(V,E)$ always means a simple and connected graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G. A cycle is called a Hamiltonian cycle if it contains all vertices of G. A subgraph of G is fault-free if it contains no faulty edges in G.

As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties[2,3]. The n-dimensional hypercube Q_n is a graph with 2^n vertices, each with a distinct binary string $u_n\cdots u_2u_1$ on the set $\{0,1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly a bit. For $n\leqslant i\leqslant 1$, we use u^i to denote the binary string $u_n\cdots \bar{u}_i\cdots u_2u_1$ such that $\bar{u}_i=1-u_i$. Chan and Lee[4] proved the following result.

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Theorem 0.1[4] If Q_n has at most $2n-5$ faulty edges, where each vertex is incident with at least two fault-free edges and $n \geqslant 3$, then there is a fault-free Hamiltonian cycle.

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n proposed first by El-Amawy and Latifi[5], can be obtained from the hypercube Q_n by adding an edge, called a complementary edge, between any pair of complementary vertices $u = u_n \cdots u_2 u_1$ and $\bar{u} = \bar{u}_n \cdots \bar{u}_2 \bar{u}_1$, where $\bar{u}_i = 1 - u_i$ for $i = 1, 2, \cdots, n$.

It has been shown that FQ_n is $(n+1)$-regular $(n+1)$-connected, vertex-transitive and edge-transitive. FQ_n is also superior to Q_n in some properties. For example, it has diameter $\left\lceil \frac{n}{2} \right\rceil$, about half the diameter of Q_n [5]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n. As a result, the study of the folded hypercube has recently attracted much attention of researchers[5~10]. In particular, WANG[10] showed that if FQ_n has at most $n-1$ faulty edges, there exists a fault-free Hamiltonian cycle. In this paper, we generalized this result by proving the following theorem.

1 Theorem

Theorem 1.1 If $FQ_n (n \geqslant 3)$ has at most $2n-3$ faulty edges, where each vertex is incident with at least two fault-free edges, then there exists a fault-free Hamiltonian cycle.

The result is optimal in the following sense. If FQ_n has $2n-2$ faulty edges and every vertex is incident with at least two fault-free edges, then there is no fault-free Hamiltonian cycle. The proof of Theorem 1.1 is in Section 2.

2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we state some terminology and notation that will be used in the proof.

By the definition, for any $i \in \{1, 2, \cdots, n\}$, Q_n can be expressed as $Q_n = L_i \odot R_i$, where L_i and R_i are the two $(n-1)$-subcubes of Q_n induced by the vertices with the i position is 0 and 1, respectively. We call edges between L_i and R_i to be i-dimensional. Obviously, for any edge e of Q_n, there is some $i \in \{1, 2, \cdots, n\}$ such that e is i-dimensional. The following lemma is true clearly.

Lemma 2.1 If Q_3 has exactly one faulty edge, then there exists a fault-free Hamiltonian (u, v)-path for any two adjacent vertices u and v.

The edges added to obtain FQ_n from Q_n are called complementary edges. We denote the set of complementary edges by E_c. To distinguish them from the edges in Q_n, we call edges in Q_n regular edges and denote the set of i-dimensional regular edges by E_i for $i = 1, 2, \cdots, n$. Like Q_n, we can express FQ_n as $L_i \otimes R_i$, where $L_i \cong Q_{n-1}$ and $R_i \cong Q_{n-1}$, the complementary edge $(u, \bar{u})$ is between L_i and R_i for any $u \in V(FQ_n)$. For convenience, we will write L and R instead of L_n and R_n, respectively.

Lemma 2.2 There is an automorphism σ of FQ_n such that $\sigma(E_i) = E_j$ for any $i, j \in \{1, 2, \cdots, n, c\}$.

Proof If $i, j \in \{1, 2, \cdots, n\}$ and $i < j$, let σ be a mapping from $V(FQ_n)$ to itself defined by $\sigma(x_n \cdots x_j \cdots x_i \cdots x_1) = (x_n \cdots x_i \cdots x_j \cdots x_1)$ for any $x_n \cdots x_2 x_1 \in V(FQ_n)$. Clearly, σ is an automorphism of FQ_n and $\sigma(E_i) = E_j, \sigma(E_j) = E_i$.

If $i = c$, $j = n$, let σ be a mapping from $V(FQ_n)$ to itself defined by

$$\begin{cases} \sigma(0u) = 0u \\ \sigma(1u) = 1\bar{u} \end{cases} \quad \text{for any } u \in V(Q_{n-1}).$$

Clearly, σ is an automorphism of FQ_n, $\sigma(E_c) = E_n$ and $\sigma(E_n) = E_c$.

If $i = c$, $j \neq n$, let σ_n and σ'_n be two automorphisms of FQ_n such that $\sigma_n(E_c) = E_n$ and $\sigma'_n(E_n) = E_j$, respectively, then $\sigma'_n \sigma_n$ is an automorphism of FQ_n and $\sigma'_n \sigma_n(E_c) = E_j$.

Proof of Theorem 1.1 Let F be any subset of $E(FQ_n)$ with $|F| = 2n-3$. Let F_c denote the set of complementary edges in F. For $1 \leqslant i \leqslant n$, let F_i denote the set of i-dimensional edges in F. Thus,

$\sum_{i=1}^{n}|F_i|+|F_c|=|F|$. By Lemma 2.2, without loss of generality, we assume that $|F_c|\geqslant|F_n|\geqslant\cdots\geqslant|F_1|$. We consider three cases depending on $n=3$, $n=4$ and $n\geqslant 5$ as following.

Case 1 $n=3$.

In this case, $\sum_{i=1}^{3}|F_i|+|F_c|=|F|=3$.

Subcase 1.1 $|F_c|\geqslant 2$. Since $FQ_3-E_c\cong Q_3$ has at most one faulty edge. By Theorem 0.1, there is a fault-free Hamiltonian cycle in the faulty FQ_3-E_c.

Subcase 1.2 $|F_c|=|F_3|=|F_2|=1$. Since Q_3 is vertex-transitive and edge-transitive, we assume that $F_3=(000,100)$.

If F_2 is incident with the vertex 000, say $F_2=(000,010)$, then $(000,111)$ is fault-free. Thus $(000,111,101,100,110,010,011,001,000)$ is a fault-free Hamiltonian cycle.

If F_2 is not incident with the vertex 000, without loss of generality, we assume that $F_2=(011,001)$, then $(000,010,011,111,110,100,101,001,000)$ is a fault-free Hamiltonian cycle.

Case 2 $n=4$.

In this case, $\sum_{i=1}^{4}|F_i|+|F_c|=|F|=5$.

Subcase 2.1 $|F_c|\geqslant 3$. FQ_4-E_c has at most 2 faulty edges and every vertex in FQ_4-E_c is incident with at least two fault-free edges. By Theorem 0.1, there exists a fault-free Hamiltonian cycle.

Subcase 2.2 $|F_c|=2$. FQ_4-E_c has at most 3 faulty edges. If every vertex in FQ_4-E_c is incident with at least two fault-free edges, then, by Theorem 0.1, there exists a fault-free Hamiltonian cycle.

Now assume that there is a vertex u in FQ_4-E_c incident with only one fault-free edge and assume that $\{(u,u^i)\mid i=2,3,4\}\subset F$. Since $|F_c|=2$, without loss of generality, assume that $(u^3,\bar{u}^3)$ is fault-free. We can mark the edge (u,u^3) as temporarily fault-free, then $|F_L-(u,u^3)|=1$. By Lemma 2.1, there exists a fault-free Hamiltonian (u,u^3)-path P_L in L. Since $|F_R|=0$, by Lemma 2.1, there exists a fault-free Hamiltonian $(\bar{u},\bar{u}^3)$-path P_R in R. Thus $C=(u,P_L,u^3,\bar{u}^3,P_R,\bar{u},u)$ is a fault-free Hamiltonian cycle.

Subcase 2.3 $|F_c|=|F_4|=|F_3|=|F_2|=|F_1|=1$. We can split the FQ_4 along some dimension i into two Q_3 such that one has two faulty edges and the other has one faulty edge. Without loss of generality, we may assume that $i=4$, $|F_L|=2$ and $|F_R|=1$.

If the two faulty edges are incident with the same vertex in L, then we assume that $F_L=\{(u,v),(u,w)\}$. For at least two fault-free edges incident with u, without loss of generality, we assume that $(u,\bar{u})$ is fault-free. Since

$$|\{(v,\bar{v}),(w,\bar{w})\}\cap F|=1,$$

suppose $(v,\bar{v})\notin F$. We can mark the edge (u,v) as temporarily fault-free. By Lemma 2.1, there are a fault-free Hamiltonian (u,v)-path P_L in L and a fault-free Hamiltonian $(\bar{u},\bar{v})$-path P_R in R. Thus $C=(u,P_L,v,\bar{v},P_R,\bar{v},u)$ is a fault-free Hamiltonian cycle.

Now assume that the two faulty edges are not incident with the same vertex in L. One of the faulty edges is (u,v) such that $(u,\bar{u})$ and $(v,\bar{v})$ are fault-free. We can mark the edge (u,v) as temporarily fault-free. By Lemma 2.1, there are a fault-free Hamiltonian uv-path P_L in L and a fault-free Hamiltonian $(\bar{u},\bar{v})$-path P_R in R. Thus $C=(u,P_L,v,\bar{v},P_R,\bar{u},u)$ is a fault-free Hamiltonian cycle.

Case 3 $n\geqslant 5$.

In this case,

$$\sum_{i=1}^{n}|F_i|+|F_c|=|F|=2n-3\geqslant n+2,$$

we have $|F_c|\geqslant 2$. Moreover, we use F_L and F_R to denote the set $E(L)\cap F$ and $E(R)\cap F$ respectively. Hence, $F=F_L\cup F_n\cup F_c\cup F_R$ and the faulty edges in $FQ_n-E_c\cong Q_n$ is at most $2n-5$.

Subcase 3.1 If every vertex in $FQ_n-E_c\cong Q_n$ is incident with at least two fault-free edges, then, by Theorem 0.1, there exists a fault-free

Hamiltonian cycle in the faulty $FQ_n - E_c$ as well as in $FQ_n - F$.

Subcase 3.2 If there is a vertex u in $FQ_n - E_c \cong Q_n$ incident with only one fault-free edge, then the complementary edge $(u, \bar{u})$ is fault-free. We assume that $u \in L$ and the fault-free edge (u, u^i) is i-dimensional. Since $|F_c| \geqslant 2$ and the number of faulty edges incident with u is $n-1$, there are at most $(2n-3)-(n-1)-2 = n-4$ other faulty regular edges and every vertex of $FQ_n - u$ is incident with more than $n-(n-4)-1 = 3$ fault-free regular edges. So if there is a dimension $j \neq i$ such that $|F_j| \geqslant 2$, then in $FQ_n - E_j$ every vertex incident with more than 2 fault-free edges and the number of the faulty edges in $FQ_n - E_j$ are at most $2n-5$. By Theorem 0.1, there exists a fault-free Hamiltonian cycle in the faulty $FQ_n - E_j$ as well as in $FQ_n - F$.

Next we suppose that $|F_j| \leqslant 1$ for all $j \neq i$. Since there are at least $n-1$ faulty regular edges with distinct dimensions and $|F_c| \geqslant |F_n| \geqslant \cdots \geqslant |F_1|$, we have $i = 1$ or $i = n$.

If $i = 1$, then $|F_j| = 1$, $(j = 2, 3, \cdots, n)$ and $|F_1| \leqslant 1$, hence $|F_c| \leqslant (2n-3)-(n-1) = n-2$, $|F_L| \leqslant n-1$ and $|F_R| \leqslant 1$. We may assume that the faulty edges incident with u is (u, u^j), $(j = 2, 3, \cdots, n)$. Since $F_c \leqslant n-2$, there is a fault-free edge among $(u^j, \bar{u}^j)$, $j \in \{2, 3, \cdots, n\}$. If $j = n$, by Lemma 2.2, there is an automorphism σ of FQ_n such that $\sigma(E_n) = E_k$, $k \notin \{1, n, c\}$.

Judging from the above discussion, there is a fault-free edge $(u^k, \bar{u}^k)$ among $(u^j, \bar{u}^j)$, $(j = 2, 3, \cdots, n-1)$. We can mark the edge (u, u^k) as temporarily fault-free, then $|F_L| \leqslant n-2 \leqslant 2n-7$, for $n \geqslant 5$ and every vertex is incident with at least two fault-free edges in L. By Theorem 0.1, there is a fault-free Hamiltonian cycle H_L in L. Let P_L be the path obtained by deleting the edge (u, u^k) from H_L. We can mark the $(n-3)$ edges $(\bar{u}, \bar{u}^j)$, $(j \in \{2, 3, \cdots, n-1\} - \{k\})$ as temporarily faulty, then $|F_R| \leqslant n-2 \leqslant 2n-7$ and every vertex is incident with at least two fault-free edges in R. By Theorem 0.1, there is a fault-free Hamiltonian cycle H_R in R. Let P_R be the path obtained by deleting the edge $(\bar{u}, \bar{u}^k)$ from H_R. Thus $H = (u, P_L, u^k, \bar{u}^k, P_R, \bar{u}, u)$ is a fault-free Hamiltonian cycle.

If $i = n$, then $|F_j| = 1$, $(j = 1, 2, \cdots, n-1)$. In order not to fall into the case $i = 1$, we assume that $|F_n| \geqslant 2$, hence

$$|F_c| \leqslant (2n-3)-(n-1)-2 = n-4.$$

Since

$$|F_c| + |F_n| = (2n-3)-(n-1) = n-2$$

and

$$|F_c| \geqslant |F_n|, |F_c| \geqslant \left\lceil \frac{n-2}{2} \right\rceil$$

and

$$|F_n| \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor.$$

We have

$$\left\lfloor \frac{n-2}{2} \right\rfloor \geqslant 2,$$

thus $n \geqslant 6$. We can express $FQ_n = L_1 \otimes R_1$. Without loss of generality, we assume that u is in L_1. Then

$$|F_L| \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor + (n-2) \leqslant 2n-6$$

and

$$|F_R| \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor.$$

There must exist a fault-free edge among $(u^i, \bar{u}^i)$, $(i = 2, 3, \cdots, n-1)$ since $|F_c| \leqslant n-4$. Assume such an edge is $(u^k, \bar{u}^k)$. We can mark the edge (u, u^k) as temporarily fault-free, then

$$|F_L| \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor + (n-3) \leqslant 2n-7$$

and each vertex is incident with at least two fault-free edges in L_1, by Theorem 0.1, there is a fault-free Hamiltonian cycle H_L in L_1. Let P_L be the path obtained by deleting the edge (u, u^k) from H_L. We can mark the $(n-3)$ edges $(\bar{u}, \bar{u}^j)$, $(j \in \{2, 3, \cdots, n-1\} - \{k\})$ as temporarily faulty, then

$$|F_R| \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor + (n-3) \leqslant 2n-7$$

and each vertex is incident with at least two fault-free edges in R_1. By Theorem 0.1, there is a fault-free Hamiltonian cycle H_R in R_1. Let P_R be the path obtained by deleting the edge $(\bar{u}, \bar{u}^k)$ from H_R. Thus $C = (u, P_L, u^k, \bar{u}^k, P_R, \bar{u}, u)$ is a fault-free

Hamiltonian cycle.

The proof of Theorem 1.1 is complete. □

The result is optimal in the following sense. Let (a,b,c,d) be a cycle in FQ_n and F a set of edges in FQ_n incident with a or c different from edges (a,b), (a,d), (b,c) and (c,d). Clearly, $|F| = 2n-2$ and every vertex is incident with at least two fault-free edges. However, $FQ_n - F$ contains no Hamiltonian cycles, as any Hamiltonian cycle must contain the four edges (a,b), (b,c), (c,d) and (d,a).

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If there is a surviving pair of vertices in $S_6 \cup S_7 \cup S_8$, without loss of generality, assume that such a pair of vertices is $\{x_{1s}x_{2t}, x_{1s}y_{2t}\}$. By Lemma 2.2,

$$\kappa(G_1 \boxtimes G_2;\ x_{1s}x_{2t}, x_{1s}y_{2t}) \geqslant \min\{\kappa_1(\delta_2+1), \kappa_2(\delta_1+1)\},$$

so the vertices $x_{1s}x_{2t}$ and $x_{1s}y_{2t}$ are connected in $G-S$. Similarly, the vertices $y_{1s}y_{2t}$ and $x_{1s}y_{2t}$ are connected in $G-S$ too. Because the vertex $x_{1s}x_{2t}$ is vertex x or adjacent to the vertex x and the vertex $y_{1s}y_{2t}$ is vertex y or adjacent to the vertex y, the vertices x and y are connected in $G-S$.

So the vertices x and y are connected in $G-S$ and the connectivity of G is at least $\min\{\kappa_1(\delta_2+1), \kappa_2(\delta_1+1)\}$. □

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