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The Bondage Numbers of Extended de Bruijn and Kautz Digraphs

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Abstract—In this paper, we consider the bondage number $b(G)$ for a digraph G, which is defined as the minimum number of edges whose removal results in a new digraph with larger domination number. This parameter measures to some extent the robustness of an interconnection network with respect to link failures. By constructing a family of minimum dominating sets, we compute the bondage numbers of the extended de Bruijn digraph and the extended Kautz digraph. As special cases, we obtain for the de Bruijn digraph $B(d,n)$ and the Kautz digraph $K(d,n)$ that $b(B(d,n)) = d$ if n is odd and $d \leqslant b(B(d,n)) \leqslant 2d$ if n is even, and $b(K(d,n)) = d+1$. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Domination, de Bruijn digraph, Kautz digraph, Extended de Bruijn digraph, Bondage number, Extended Kautz digraph.

1. INTRODUCTION

It is well known that the topological structure of an interconnection network can be modeled by a connected graph whose vertices represent sites of the network and whose edges represent physical communication links. A minimum dominating set in the graph corresponds to a smallest set of sites selected in the network for some particular uses, such as placing transmitters. Such a set may not work when some communication links happen fault. The fault is possible in real world (hacking, experimental error, terrorism, etc.), so one needs to consider it. What is the minimum number of faulty links which will make all minimum dominating sets of the original network not work any more? Such a minimum number is called the bondage number, which measures the robustness of a network with respect to link failures, wherever a minimum dominating set is required for some application. Motivated by the above relevance of bondage number, one wants to know how to compute it for a network. However, this computation is generally difficult; no

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efficient algorithm has been proposed. Therefore, it is of significance to develop a technique to determine the bondage number for special graphs.

In this paper, we focus on the de Bruijn digraph and the Kautz digraph. These digraphs have many attractive features superior to the hypercube (see, for example, Sections 3.2 and 3.3 in [1]). As a topological architecture of interconnection networks, the de Bruijn digraph and the Kautz digraph were first suggested by Schlumberger [2] in 1974. Some computer systems based on the de Bruijn architecture have been built (see [3]). They have been thought of as good candidates for the next generation of parallel system architectures after the hypercube networks [4]. Therefore, the de Bruijn digraph and the Kautz digraph were widely studied and various generalizations of these digraphs were proposed, including the extended de Bruijn digraph and the extended Kautz digraph, which have more flexible structure than the classical de Bruijn digraph and the Kautz digraph, so that one can choose more suitable networks for prescribed requirements. Merit of these digraphs motivates us to determine their bondage number.

In order to give a precise definition of the bondage number, we need some terminology and notation on graph theory. Let $G = (V, E)$ be a digraph with a vertex-set V and an edge-set E. For a subset $S \subseteq V$, let $E^+(S) = \{(u,v) \in E : u \in S,\ v \notin S\}$, $E^-(S) = \{(u,v) \in E : u \notin S,\ v \in S\}$, and $N^+(S) = \{v \in V : u \in S,\ (u,v) \in E^+(S)\}$, $N^-(S) = \{u \in V : v \in S,\ (u,v) \in E^-(S)\}$. For $v \in V$ and $(u,v), (v,w) \in E$, u and w are called an *in-neighbor* and an *out-neighbor* of v, respectively. The *in-degree* and the *out-degree* of v are the number of its in-neighbors and out-neighbors, denoted by $d^-(v)$ and $d^+(v)$, respectively. The *degree* of v is $d(v) = d^+(v) + d^-(v)$. Denote the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum in-degree (resp., out-degree) of G by $\Delta^-(G)$ and $\delta^-(G)$ (resp., $\Delta^+(G)$ and $\delta^+(G)$).

Given two vertices u and v in G, we say u dominates v if $u = v$ or $(u,v) \in E(G)$. A subset $D \subseteq V(G)$ is called a dominating set if its vertices dominate all vertices of G, i.e., $V(G) = D \cup N^+(D)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets. The bondage number of G, denoted by $b(G)$, is the minimum cardinality over all sets of edges E', such that $\gamma(G - E') > \gamma(G)$. Noting that loops have no effect on the domination number and the bondage number, we need not to consider whether G has loops or not.

It is clear that an undirected graph can be thought a digraph obtained by replacing each undirected edge with a pair of directed edge, one in each direction. The concept of the bondage number was proposed for an undirected graph by Fink *et al.* [5] and for a digraph by Carlson and Develin [6]. There are many research articles on the the bondage number for undirected graphs (see, for example [6–13]). However, to date no research has been done on this concept for digraphs except [6].

Fink *et al.* [5] conjectured that $b(G) \leqslant \Delta(G) + 1$ for an undirected graph G, which was later proved invalid generally. A class of counterexamples is the cartesian product $G_n = K_n \times K_n$, where K_n is the complete undirected graph with n vertices. Hartnell and Rall [8] and Teschner [9] independently proved that $b(G_n) = (3/2)\Delta(G_n)$. Furthermore, Teschner [10] showed that $b(G) \leqslant (3/2)\Delta(G)$ for any undirected graph with $\gamma(G) \leqslant 3$, and proposed the following conjecture.

CONJECTURE 1.1. *$b(G) \leqslant (3/2)\Delta(G)$ for any undirected graph G.*

As far as we know, there is no further results on this conjecture. However, Carlson and Develin [6] showed that Conjecture 1.1 is valid for any digraph. They also obtained $b(G_n) = \Delta(G_n) + 1$ for $G_n = K_n \times K_n$, the cartesian product of complete digraphs, and proposed the following conjecture.

CONJECTURE 1.2. *$b(G) \leqslant \Delta(G) + 1$ for any digraph G.*

In this paper, we mainly consider digraphs. We investigate the domination numbers and the bondage numbers of the extended de Bruijn digraph and the extended Kautz digraph, using a technique of constructing a family of minimum dominating sets. As special cases, we obtain for

the de Bruijn digraph $B(d,n)$ and the Kautz digraph $K(d,n)$ that

$$\begin{aligned} b(B(d,n)) &= d, && \text{if } n \text{ is odd;} \\ d \leqslant b(B(d,n)) &\leqslant 2d, && \text{if } n \text{ is even,} \end{aligned} \quad \text{and } b(K(d,n)) = d+1.$$

That means, the removal of any $d-1$ (resp., d) edges can not enlarge the domination number of $B(d,n)$ (resp., $K(d,n)$). Such a robustness is ensured by the definitional structure of the de Bruijn digraph and the Kautz digraph. Defined in a highly symmetric way, these digraphs possess a large number of minimum dominating sets. As a result, many edges are needed to be removed in order to break down all these minimum dominating sets and enlarge the domination number.

The rest of the paper is organized as follows. In Section 2, we present some bounds of the bondage number in general, and determine it for some simple examples. We give our results for the extended de Bruijn digraphs and the extended Kautz digraphs in Sections 3 and 4, respectively.

2. SOME BOUNDS AND EXAMPLES

The following lemma was established by Carlson and Develin [6].

LEMMA 2.1. *Let G be a loopless digraph and $(u,v) \in E(G)$. Then,*

$$b(G) \leqslant d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|.$$

PROOF. Let $G = (V,E)$ be a loopless digraph and

$$B = E^+(v) \cup E^-(v) \cup E^-(u) - \left\{(w,u) \in E : w \in N^-(u) \cap N^-(v)\right\}$$

and $H = G - B$. Suppose that D is a minimum dominating set of H. Then, D contains v since $d_H(v) = 0$. In order to dominates u, D has to contain u or some $w \in N^-(u) \cap N^-(v)$. However, v dominates only itself in H, and either of u and w dominates v in G. Thus, $D - \{v\}$ is a dominating set of G, which implies $\gamma(H) > \gamma(G)$. It follows that

$$b(G) \leqslant |B| = d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

as required. ∎

We now introduce a parameter $r(G)$ to bound $b(G)$ below. Let e be an edge and D a dominating set in G. We say e *supports* D if $e \in E^+(D)$. Denote by $r(G)$, the minimum number of edges which support all minimum dominating sets in G. One can see that at least $r(G)$ edges must be removed from G in order to invalidate all the minimum dominating sets.

LEMMA 2.2. *For a digraph G, $b(G) \geqslant r(G)$.*

PROOF. Assume $E' \subseteq E(G)$ with $|E'| < r(G)$. Then, E' can not support all minimum dominating sets in G. Let D be a minimum dominating set not supported by E'. We prove by contradiction that D is still a dominating set in $G - E'$.

Suppose to the contrary, that there exists a vertex $v \in V(G) \setminus D$, such that D can not dominate it in $G - E'$. Since D is a dominating set in G, there exists a vertex $u \in D$ which dominates v in G. Hence, $(u,v) \in E(G)$ supports D, which implies that $(u,v) \notin E'$. It follows that $u \in D$

dominates v in $G - E'$, a contradiction. Thus, $\gamma(G - E') = \gamma(G)$ for any set $E' \subseteq E(G)$ with $|E'| < r(G)$, and so $b(G) \geqslant r(G)$. ∎

REMARK. Lemma 2.2 is essential to our computation of $b(G)$. The reason lies in the definition of $b(G)$. Since $b(G)$ is a minimum, every suitable selection of edges will give $b(G)$ an upper bound, whenever the removal of these edges enlarge $\gamma(G)$. However, to bound $b(G)$ well in the opposite direction is often difficult. Thus, we must use Lemma 2.2 to determine the exact value of $b(G)$. That means we have to construct minimum dominating sets of G as many as possible.

Now, we give some examples. A *complete digraph* K_n is a digraph with n vertices $v_1, v_2, \ldots, v_n$ and $n(n-1)$ edges (v_i, v_j), $i \neq j$. Adding a loop at each v_i results in a digraph called the *flowered digraph* and denoted by FK_n.

EXAMPLE 2.3.

$$\gamma(K_n) = \gamma(FK_n) = 1 \qquad \text{and} \qquad b(K_n) = b(FK_n) = n.$$

PROOF. Since loops have no effect on the bondage number, we only consider K_n. It is easy to observe that $\gamma(K_n) = 1$ and $D_i = \{v_i\}$ is a minimum dominating set for $i = 1, 2, \ldots, n$. Since $E^+(D_i) \cap E^+(D_j) = \emptyset$ if $i \neq j$, then every set $E' \subseteq E(G)$ with $|E'| < n$ can not support all $D_1, \ldots, D_n$. By Lemma 2.2, $b(K_n) \geqslant r(K_n) \geqslant n$.

On the other hand, let $E'' = \{(v_n, v_1),\ (v_i, v_{i+1}) \in E(K_n) : i = 1, 2, \ldots, n-1\}$. Then, for each $i = 1, 2, \ldots, n$, the vertex v_i cannot dominate v_{i+1} in $K_n - E''$. As a result, $b(K_n) \leqslant |E''| = n$. ∎

EXAMPLE 2.4. Let $K^-_{1,n}$ be a digraph with vertex set $V = \{x, y_1, \ldots, y_n\}$ and edge set $E = \{(y_i, x) : i = 1, \ldots, n\}$. Then, $\gamma(K^-_{1,n}) = n$ and $b(K^-_{1,n}) = n = \Delta(K^-_{1,n})$.

PROOF. It is clear that $\gamma(K^-_{1,n}) = n$ and there is a unique minimum dominating set $D = \{y_1, \ldots, y_n\}$. Let $E' \subset E(K^-_{1,n})$ be a proper subset and $H = K^-_{1,n} - E'$. Then, D is still a minimum dominating set of H, since H has at least an edge. Hence, $b(K^-_{1,n}) \geqslant n$. On the other hand, $\gamma(K^-_{1,n} - E(K^-_{1,n})) = n + 1 > \gamma(K^-_{1,n})$. Thus, $b(K^-_{1,n}) \leqslant |E(K^-_{1,n})| = n$. ∎

REMARK. In the undirected case, a tree has bondage number 1 or 2 (see [5]). However, for the directed tree $K^-_{1,n}$, we have $b(K^-_{1,n}) = n$, which means that the bondage number of a directed tree can be arbitrarily large.

EXAMPLE 2.5. Let C_n be a directed cycle of length $n \geqslant 2$. Then,

$$\gamma(C_n) = \lceil n/2 \rceil \qquad \text{and} \qquad b(C_n) = \begin{cases} 3, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

PROOF. Let $C_n = (x_1, \ldots, x_n)$ be a directed cycle. It is clear that every vertex in C_n dominates only one other vertex, which implies that $\gamma(C_n) \geqslant \lceil n/2 \rceil$.

If $n = 2k + 1$ for an integer $k \geqslant 1$, then $D = \{x_1, x_3, \ldots, x_{2k+1}\}$ is dominating set with $|D| = k+1 = \lceil n/2 \rceil$. Thus, $\gamma(C_{2k+1}) = |D| = k+1$. By Lemma 2.1, we have $b(C_{2k+1}) \leqslant 2+1 = 3$. Let $E' = \{(x_i, x_{i+1}), (x_j, x_{j+1})\}$, $i < j$, and $H = C_{2k+1} - E'$. Then,

$$D' = \begin{cases} \{x_{i+1}, x_{i+3}, \ldots, x_j, x_{j+1}, x_{j+3}, \ldots, x_{i+2k}\}, & \text{if } j - i \text{ is odd;} \\ \{x_{i+1}, x_{i+3}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i+2k+1}\}, & \text{if } j - i \text{ is even,} \end{cases}$$

is a minimum dominating set of H with $|D'| = k + 1 = \gamma(C_{2k+1})$, where the subscripts are taken module n. Hence, $b(C_{2k+1}) = 3$.

If $n = 2k$ for some integer $k \geqslant 1$, then $\gamma(C_{2k}) = k$ and $D_1 = \{x_1, x_3, \ldots, x_{2k-1}\}$, $D_2 = \{x_2, x_4, \ldots, x_{2k}\}$ are two disjoint minimum dominating sets with $E^+(D_1) \cap E^+(D_2) = \emptyset$. Then, every single edge can not support $\{E^+(D_1), E^+(D_2)\}$. By Lemma 2.2 we have $b(C_{2k}) \geqslant 2$. On the other hand, any minimum dominating set D of $H = C_{2k} - \{(x_1, x_2), (x_2, x_3)\}$ contains x_2; but x_2 dominates only itself in H. Hence, $|D|-1 \geqslant \lceil (2k-1)/2 \rceil = k$ and so $\gamma(H) = |D| \geqslant k+1 > \gamma(C_{2k})$. Then, $b(C_{2k}) \leqslant 2$. ∎

EXAMPLE 2.6. Let P_n be a directed path with n vertices. Then,

$$\gamma(P_n) = \lceil n/2 \rceil \qquad \text{and} \qquad b(P_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

REMARK. The bondage numbers of directed cycles and paths are the same as those of undirected ones (see [5] or [7]).

3. EXTENDED DE BRUIJN DIGRAPHS

We first recall the definition of the de Bruijn digraph $B(d, n)$ for $d \geqslant 2$ and $n \geqslant 1$. It is a digraph with the vertex-set $V = \{x_1, \ldots, x_n : 0 \leqslant x_i \leqslant d-1\}$; there is a directed edge from x to y if and only if $x = x_1, x_2, \ldots, x_n$ and $y = x_2, \ldots, x_n\alpha$, where $\alpha \in \{0, 1, \ldots, d-1\}$. $B(d, n)$ has d^n vertices, d^{n+1} edges, and is d-regular.

Shibata and Gonda [14] introduced the extended de Bruijn digraph, denoted by $EB(d, n; q_1, \ldots, q_p)$, which was defined as a digraph with the vertex-set as the set of n-dimensional vectors on d elements divided into p blocks of sizes $q_1, \ldots, q_p$, expressed as the following form

$$x = (x_{11}, x_{12}, \ldots, x_{1q_1})(x_{21}, x_{22}, \ldots, x_{2q_2}) \ldots (x_{p1}, x_{p2}, \ldots, x_{pq_p}),$$

where $0 \leqslant x_{ij} \leqslant d-1$, and $q_1 + q_2 + \cdots + q_p = n$. The out-neighbors of x are those vertices having the form

$$(x_{12}, \ldots, x_{1q_1}, \alpha_1)(x_{22}, \ldots, x_{2q_2}, \alpha_2,) \ldots (x_{p2}, \ldots, x_{pq_p}, \alpha_p),$$

where $0 \leqslant \alpha_i \leqslant d-1$, for each $i = 1, 2, \ldots, p$. The extended de Bruijn digraph $EB(d, n; q_1, \ldots, q_p)$ has d^n vertices, d^{n+p} edges and is d^p-regular. From the definition, if $p = 1$, i.e. the vertices are not divided, then $EB(d, n; n) = B(d, n)$, clearly.

In this section, we consider $G = EB(d, n; q_1, \ldots, q_p)$ with $d \geqslant 2$ and $q_1 = \cdots = q_p = q \geqslant 1$. As mentioned in the remark below Lemma 2.2, we must determine $\gamma(G)$ and then construct minimum dominating sets of G, as many as possible in order to determine $b(G)$. The definitional "transitive" structure of $EB(d, n; q_1, \ldots, q_p)$ help us to do so, and the possible value of $\gamma(G)(\geqslant \lceil |V(G)|/(\Delta^+(G)+1) \rceil)$ also gives us a clue to the following construction.

For a given p, a sequence $(i_1, \ldots, i_p)$ on $\{0, 1, \ldots, d-1\}$ and $j \in \{1, 2, \ldots, q\}$, let

$$D^{(j)}_{(i_1, \ldots, i_p,)} = \{(x_{11}, \cdots, x_{1q}) \cdots (x_{p1}, \ldots, x_{pq}) : x_{k1} = \cdots x_{kj} = i_k,\ k = 1, \ldots, p\},$$

and

$$D_{(i_1, \ldots, i_p)} = \begin{cases} D^{(1)}_{(i_1, \ldots, i_p)} - D^{(2)}_{(i_1, \ldots, i_p)} + D^{(3)}_{(i_1, \ldots, i_p)} - \cdots + D^{(q)}_{(i_1, \ldots, i_p)} & \text{if } q \text{ is odd;} \\ D^{(1)}_{(i_1, \ldots, i_p)} - D^{(2)}_{(i_1, \ldots, i_p)} + D^{(3)}_{(i_1, \ldots, i_p)} - \cdots + D^{(q-1)}_{(i_1, \ldots, i_p)} & \text{if } q \text{ is even.} \end{cases} \tag{1}$$

LEMMA 3.1. *Let $G = EB(d, n; q_1, \ldots, q_p)$ with $d \geqslant 2$, $q_1 = \cdots = q_p = q$ and $n = pq$. Then,*

$$\gamma(G) = \begin{cases} \dfrac{(d^n+1)}{(d^p+1)} & \text{if } q \text{ is odd;} \\[2ex] \dfrac{(d^n+d^p)}{(d^p+1)} & \text{if } q \text{ is even,} \end{cases}$$

and $D_{(i_1,\ldots,i_p)}$ defined as (1) is a minimum dominating set for any $0 \leqslant i_1, \ldots, i_p \leqslant d-1$.

PROOF. First we prove that $D_{(i_1,\ldots,i_p)}$ is a dominating set in G.

Let $x = (x_{11}, \ldots, x_{1q}) \ldots (x_{p1}, \ldots, x_{pq})$ be a vertex of G. Then there exists a vertex $y = (i_1, x_{11}, \ldots, x_{1(q-1)}) \ldots (i_p, x_{p1}, \ldots, x_{p(q-1)})$, such that y dominates x. We show that either $x \in D_{(i_1 \ldots i_p)}$ or $y \in D_{(i_1 \ldots i_p)}$.

Since, $D^{(j)}_{(i_1,\ldots,i_p)} \supseteq D^{(j+1)}_{(i_1,\ldots,i_p)}$ for $j = 1, 2, \ldots, q-1$, we assume that $t \in \{1, \ldots, q\}$ is the maximum integer, such that $y \in D^{(t)}_{(i_1,\ldots,i_p)}$. From the construction of $D_{(i_1,\ldots,i_p)}$ in (1), we observe that $y \in D_{(i_1,\ldots,i_p)}$ if $t = 1$. Assume $2 \leqslant t \leqslant q$ below. Then, $i_k = x_{k1} = \cdots = x_{k(t-1)}$ and $x_{kt} \neq i_k$ for $k = 1, 2, \ldots, p$, which implies that $x \in D^{(t-1)}_{(i_1,\ldots,i_p)}$, but $x \notin D^{(t)}_{(i_1,\ldots,i_p)}$. If t is odd, then $y \in D_{(i_1,\ldots,i_p)}$; otherwise $t-1$ is odd and so $x \in D_{(i_1,\ldots,i_p)}$. Thus, either $x \in D_{(i_1,\ldots,i_p)}$ or x is dominated by $y \in D_{(i_1,\ldots,i_p)}$. That means $D_{(i_1,\ldots,i_p)}$ is a dominating set in G.

We show that $D_{(i_1,\ldots,i_p)}$ is minimum. Since G is d^p-regular, every vertex dominates at most d^p other vertices, and so $(d^p+1)\gamma(G) \geqslant d^n$. Since $\gamma(G)$ is an integer, we have

$$\gamma(G) \geqslant \begin{cases} \dfrac{(d^n+1)}{(d^p+1)} = d^{n-p} - d^{n-2p} + d^{n-3p} - \cdots + 1, & \text{if } q \text{ is odd;} \\[2ex] \dfrac{(d^n+d^p)}{(d^p+1)} = d^{n-p} - d^{n-2p} + d^{n-3p} - \cdots + d^p, & \text{if } q \text{ is even.} \end{cases}$$

From the definition of $D_{(i_1,\ldots,i_p)}$ in (1), we obtain

$$\left|D_{(i_1,\ldots,i_p)}\right| = \begin{cases} d^{n-p} - d^{n-2p} + d^{n-3p} - \cdots + 1, & \text{if } q \text{ is odd;} \\ d^{n-p} - d^{n-2p} + d^{n-3p} - \cdots + d^p, & \text{if } q \text{ is even.} \end{cases}$$

Thus, $D_{(i_1,\ldots,i_p)}$ is a minimum dominating set in G. ∎

COROLLARY 3.2. *Let $G = EB(d, n; q_1, \ldots, q_p)$ with $d \geqslant 2$, $q_1 = \cdots = q_p = q$, and $n = pq$. Then $b(G) \geqslant d^p$.*

PROOF. By Lemma 3.1, $D_{(i_1,\ldots,i_p)}$ is a minimum dominating set for any $0 \leqslant i_1, \ldots, i_p \leqslant d-1$. The number of such minimum dominating sets is d^p. It is clear that $D_{(i_1,\ldots,i_p)} \cap D_{(j_1,\ldots,j_p)} = \emptyset$ and so $E^+(D_{(i_1,\ldots,i_p)}) \cap E^+(D_{(j_1,\ldots,j_p)}) = \emptyset$, if $(i_1, \ldots, i_p) \neq (j_1, \ldots, j_p)$. Then every subset $E' \subseteq E(G)$ with $|E'| < d^p$ can not support all $D_{(i_1,\ldots,i_p)}$. Thus, $b(G) \geqslant r(G) \geqslant d^p$ by Lemma 2.2. ∎

THEOREM 3.3. *Let $G = EB(d, n; q_1, \ldots, q_p)$ with $d \geqslant 2$, $q_1 = \cdots = q_p = q$ and $n = pq$. Then,*

$$b(G) = d^p, \qquad \textit{if } q \textit{ is odd;}$$

$$d^p \leqslant b(G) \leqslant 2d^p, \qquad \textit{if } q \textit{ is even.}$$

PROOF. We do not need to consider any loop in G. Let $x = (0 \ldots 00) \ldots (0 \ldots 00)$ and $y = (0 \ldots 01) \ldots (0 \ldots 01)$. Then,

$$N^-(x) \cap N^-(y) = \{(i_1 0 \ldots 0) \ldots (i_p 0 \ldots 0) : 0 \leqslant i_1, \ldots, i_p \leqslant d-1\} \setminus \{x\}.$$

By Lemma 2.1 we obtain $b(G) \leqslant 2d^p + (d^p - 1) - (d^p - 1) = 2d^p$. If q is even, then the result follows from Corollary 3.2.

Assume that q is odd below. Let $E' = E^-(x) \cup \{(x,z) \in E(G) : z \in N^+(x)\}$. Then, $|E'| = d^p - 1 + 1 = d^p$. Suppose that D is a minimum dominating set in $H = G - R$. Then D contains x for $N_H^-(\{x\}) = \emptyset$. However, x dominates only $d^p - 1$ vertices in H, which implies that if $|D| = \gamma(G)$, then D dominates at most

$$(|D| - 1)(d^p + 1) + d^p - 1 = d^n + 1 - (d^p + 1) + d^p - 1 < d^n,$$

vertices, a contradiction. Thus, $\gamma(H) = |D| > \gamma(G)$, and so $b(G) \leqslant |R| = d^p$. The result follows from Corollary 3.2. ∎

It has been mentioned that $EB(d,n;n) = B(d,n)$. Thus, we immediately get the results for de Bruijn digraphs if we let $p = 1$ in Theorem 3.3, where the result on domination number has been obtained by Kikuchi and Shibata in [15].

COROLLARY 3.4. *For any $d \geqslant 2$ and $n \geqslant 1$,*

$$\gamma(B(d,n)) = \frac{(d^n+1)}{(d+1)}, \qquad b(B(d,n)) = d, \qquad \text{if } n \text{ is odd;}$$

$$\gamma(B(d,n)) = \frac{(d^n+d)}{(d+1)}, \qquad d \leqslant b(B(d,n)) \leqslant 2d, \qquad \text{if } n \text{ is even.}$$

Shibata and Gonda [14] also considered another extremal case, $p = n$ and $q = 1$. Then $EB(d,n;1,\ldots,1)$ is isomorphic to the flowered complete digraph FK_{d^n}. By Theorem 3.3 we have that $b(FK_{d^n}) = d^n$, which is a special case of Example 2.3.

4. EXTENDED KAUTZ GRAPHS

The Kautz digraph $K(d,n)$ has the vertex-set and the edge-set as follows:

$$V = \{x_1,\ldots,x_n : 0 \leqslant x_i \leqslant d,\ x_i \neq x_{i+1},\ i = 1,\ldots,n-1\}$$

and

$$E = \{(x_1,x_2,\ldots,x_n,\ x_2,\ldots,x_n,\alpha) : 0 \leqslant \alpha \leqslant d,\ \alpha \neq x_n\}.$$

$K(d,n)$ has $d^{n-1}(d+1)$ vertices, $d^n(d+1)$ edges, and is d-regular.

The extended Kautz digraph $EK(d,n;q_1,\ldots,q_p)$ has vertex-set as the set of n-dimensional vectors on d elements divided into p blocks of sizes $q_1,\ldots,q_p$, expressed as the following form,

$$x = (x_{11},x_{12},\ldots,x_{1q_1})(x_{21},x_{22},\ldots,x_{2q_2})\ldots(x_{p1},x_{p2},\ldots,x_{pq_p}),$$

where $0 \leqslant x_{ij} \leqslant d$, $x_{ij} \neq x_{i(j+1)}$, and $q_1 + q_2 + \cdots + q_p = n$. The out-neighbors of x are those vertices having the form

$$(x_{12},\ldots,x_{1q_1},\alpha_1)(x_{22},\ldots,x_{2q_2},\alpha_2)\ldots(x_{p2},\ldots,x_{pq_p},\alpha_p),$$

where $0 \leqslant \alpha_i \leqslant d$ and $\alpha_i \neq x_{iq_i}$ for each $i = 1,2,\ldots,p$. If $p = 1$, i.e. the vertices are not divided, then $EK(d,n;n) = K(d,n)$. It is clear that the extended Kautz graph $G = EK(d,n;q_1,\ldots,q_p)$ has $d^{n-p}(d+1)^p$ vertices, $d^n(d+1)^p$ edges and is d^p-regular.

In this section, we consider $G = EK(d,n;q_1,\ldots,q_p)$ with $q_k \geqslant 2$, $k = 1,2,\ldots,p$. We try to use the same technique to compute $b(G)$ as in Section 3. We first construct a family of dominating sets for G. For given $0 \leqslant i_1,\ldots,i_p \leqslant d$, choose $j_1,\ldots,j_p$, such that $i_k \neq j_k$, for $k = 1,2,\ldots,p$. For given $(i_1,\ldots,i_p)$ and $(j_1,\ldots,j_p)$, let $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}(t)$ be the set of all vertices $(x_{11},\ldots,x_{1q_1})\ldots(x_{p1},\ldots,x_{pq_p})$ with the property that there exists a subset $I \subseteq \{1,2,\ldots,p\}$ with $|I| = t$ ($I = \emptyset$ if $t = 0$), such that $x_{k1} = i_k$ if $k \notin I$, and $x_{k1} = j_k \neq i_k = x_{k_2}$ if $k \in I$. It is

not difficult to observe that

$$\left|D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(t)\right| = \binom{p}{j} d^{n-p-t}, \qquad t = 0, 1, \ldots, p,$$

since there are $p+t$ coordinates fixed in every vertex of $D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(t)$.

THEOREM 4.1. *Let $G = EK(d, n; q_1, \ldots, q_p)$ with $q_k \geqslant 2$ for $k = 1, 2, \ldots, p$. Then,*

$$\left\lceil \frac{d^{n-p}(d+1)^p}{d^p+1} \right\rceil \leqslant \gamma(G) \leqslant d^{n-2p}\left((d+1)^p - 1\right)$$

and for any $i_k, j_k \in \{0, 1, \ldots, d-1\}$, $i_k \neq j_k$, $k = 1, 2, \ldots, p$,

$$D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)} = \bigcup_{t=0}^{p-1} D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(t), \tag{2}$$

is a dominating set in G.

PROOF. Since $G = EK(d, n; q_1, \ldots, q_p)$ is d^p-regular, every vertex dominates d^p other vertices and so $\gamma(G) \geqslant \lceil d^{n-p}(d+1)^p/(d^p+1)\rceil$.

We prove that $D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$ is a dominating set in G. To the end, let

$$v = (x_{11}, \ldots, x_{1q_1}) \ldots (x_{p1}, \ldots, x_{pq_p})$$

be any vertex in G. If $x_{k1} = i_k$ for $k = 1, 2, \ldots, p$, then $v \in D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(0) \subseteq D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$. If $x_{k1} \neq i_k$, for all $k = 1, 2, \ldots, p$, then v is dominated by

$$u = (i_1, x_{11}, \ldots, x_{1(q_1-1)}) \ldots (i_k, x_{p1}, \ldots, x_{p(q_p-1)}) \in D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(0) \subseteq D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}.$$

Now, assume that $x_{k1} = i_k$ if $k \leqslant t$ and $x_{k1} \neq i_k$ if $k \geqslant t+1$, $1 \leqslant t \leqslant p-1$, without loss of generality. Then, v is dominated by

$$u = (j_1, x_{11}, \ldots, x_{1(q_1-1)}) \ldots (j_t, x_{t1}, \ldots, x_{t(q_t-1)}) \\ (i_{t+1}, x_{(t+1)1}, \ldots, x_{(t+1)(q_{t+1}-1)}) \ldots (i_p, x_{p1}, \ldots, x_{p(q_p-1)})$$

and $u \in D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(t) \subseteq D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$. Thus, $D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$ is a dominating set for G. Then, we obtain

$$\gamma(G) \leqslant \left|D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}\right| = \sum_{t=0}^{p-1} \binom{p}{t} d^{n-p-t} = d^{n-2p}\left((d+1)^p - 1\right)$$

from (2). ∎

In [15], Kikuchi and Shibata obtained $\gamma(K(d,n)) = d^{n-1}$, which is a consequence of Theorem 4.1 since $K(d,n) = EK(d,n;n)$.

COROLLARY 4.2. *For any $n \geqslant 1$, $\gamma(K(d,n)) = d^{n-1}$ and*

$$D_i = \{ix_2, \ldots, x_n : x_2 \neq i,\ x_j \neq x_{j+1},\ j = 2, 3, \ldots, n-1\}$$

is a minimum dominating set for $i = 0, 1, \ldots, d$.

PROOF. If $n = 1$, the result follows from Example 2.3.

Now, consider $n \geqslant 2$. Let $p = 1$ in Theorem 4.1. Then $EK(d,n;n) = K(d,n)$ and $d^{n-1} \leqslant \gamma(K(d,n)) \leqslant d^{n-1}$. Thus, $\gamma(K(d,n)) = d^{n-1}$ and

$$D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)} = \bigcup_{t=0}^{p-1} D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}(t) = D_{(i)}^{(j)}(0)$$

$$= \{x_1, x_2, \ldots, x_n \in V(K(d,n)) : x_1 = i\} = D_i.$$

is a minimum dominating set for any $i = 0, 1, \ldots, d$. ∎

THEOREM 4.3. *For any $n \geqslant 1$, $b(K(d,n)) = d+1$.*

PROOF. The results for $n = 1$ follows from Example 2.3. Assume $n \geqslant 2$ below. By Corollary 4.2, $D_0, D_1, \ldots, D_d$ are $d+1$ vertex-disjoint minimum dominating sets in $K(d,n)$, with $E^+(D_i) \cap E^+(D_j) = \emptyset$ if $i \neq j$. Thus, every subset $E' \subseteq E(K(d,n))$ with $|E'| < d+1$ can not support all $D_0, \ldots, D_d$. By Lemma 2.2 we obtain $b(K(d,n)) \geqslant d+1$.

Let $v \in V(K(d,n))$ and u_i, w_i be an in-neighbors and an out-neighbors of v, respectively, for $i = 1, 2, \ldots, d$. Let $E' = \{(u_i, x) \in E(K(d,n)) : i = 1, 2, \ldots, d\} \cup \{(v, w_1)\}$ and $H = K(d,n) - E'$. Then any minimum dominating set D of H contains v, since $d_H^-(v) = 0$. However, v dominates only d vertices in H, which implies that $|D| > d^{n-1}$, for otherwise D dominates at most $(d^{n-1} - 1)(d+1) + d < |V(K(d,n))|$ vertices. Then, $\gamma(H) > d^{n-1} = \gamma(K(d,n))$, and so $b(K(d,n)) \leqslant |E'| = d+1$. The result follows. ∎

Now, we go further with a discussion on $G = EK(d, n; q_1, \ldots, q_p)$ with $p \geqslant 2$. In order to compute $b(G)$ by our technique, we must first show that $D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$ is a minimum dominating set for G. To this aim, we need the hypothesis that $\sum_{t=1}^{p-1} \binom{p}{j} d^{n-p-t} \leqslant d^p$. Provided this, we have

$$\begin{aligned}
\left|D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}\right| (d^p + 1) &= \sum_{t=0}^{p-1} \binom{p}{j} d^{n-p-t} (d^p + 1) \\
&= d^{n-p} (d+1)^p + \sum_{t=1}^{p-1} \binom{p}{t} d^{n-p-t} \\
&< d^{n-p} (d+1)^p + d^p + 1.
\end{aligned}$$

Thus,

$$\left|D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}\right| = \lceil \frac{d^{n-p} (d+1)^p}{(d^p + 1)} \rceil = \gamma(G).$$

However, the hypothesis $\sum_{t=1}^{p-1} \binom{p}{t} d^{n-p-t} \leqslant d^p$ implies that $n - p - 1 < p$, i.e. $n \leqslant 2p$. On the other hand, the hypothesis $q_k \geqslant 2$ for $k = 1, 2, \ldots, p$ yields that $n \geqslant 2p$. Then $n = 2p$ and $G = EK(d, n; 2, \ldots, 2)$. Thus, our technique of computing $b(G)$ may still work only for the case $G = EK(d, 2p; 2, \ldots, 2)$.

COROLLARY 4.4. *Let $G = EK(d, 2p; 2, \ldots, 2)$. Then, for any fixed $p \geqslant 2$, there exists a positive integer A, such that*

$$\gamma(G) = \left\lceil \frac{d^p (d+1)^p}{d^p + 1} \right\rceil = (d+1)^p - 1$$

and $D_{(i_1,\ldots,i_p)}^{(j_1,\ldots,j_p)}$ defined in (2) is a minimum dominating set in G for every integer $d \geqslant A$.

PROOF. The corollary follows from Theorem 4.1 if we can show

$$(d+1)^p - d^p - 1 = \sum_{t=1}^{p-1} \binom{p}{t} d^{n-p-t} \leqslant d^p,$$

i.e. $f(d) = 2d^p - (d+1)^p + 1 \geqslant 0$. Note that $f'(d) = 2pd^{p-1} - p(d+1)^{p-1} > 0$ if $d > (\sqrt[p-1]{2} - 1)^{-1}$, and $f((\sqrt[p-1]{2} - 1)^{-1}) < 0$. Then there exists a positive $A > (\sqrt[p-1]{2} - 1)^{-1}$, such that $f(A) = 0$, and $f(d)$ is monotonically increasing on the interval $((\sqrt[p-1]{2} - 1)^{-1}, \infty)$. Therefore, $f(d) \geqslant 0$ for every $d \geqslant A$. ∎

THEOREM 4.5. *For any fixed $p \geqslant 2$ and $c \in (0,1]$, there exists a positive integer B, such that*

$$d^p + 1 \leqslant b\left(EK\left(d, 2p; 2, \ldots, 2\right)\right) \leqslant (1+c)\, d^p$$

for every integer $d > B$.

PROOF. Let $G = EK(d, 2p; 2, \ldots, 2)$. By Corollary 4.4, $\mathcal{D} = \{D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}\}$ is a family of minimum dominating sets in G, where $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}$ is defined as (2). It is clear that $|\mathcal{D}| = (d+1)^p d^p$. Let

$$x = (x_{11}, x_{12})\,(x_{21}, x_{22}) \ldots, (x_{p1}, x_{p2}),$$

be a vertex and (x, y) be an edge in G. We now estimate the number of sets in $\mathcal{D}$ that is supported by the edge (x, y).

By (2), each $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}$ consists of $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}(t)$, $t = 0, 1, \ldots, p-1$. Thus, if (x, y) supports $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}(t)$ for some $t \in \{0, 1, \ldots, p-1\}$, then (x, y) supports $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}$.

Now, suppose that (x, y) supports $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}(t)$, for some $t \in \{0, 1, \ldots, p-1\}$. By the construction of $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}(t)$, there exists a subset $I \subseteq \{1, 2, \ldots, p\}$, $|I| = t$ ($I = \emptyset$ if $t = 0$), such that $x_{k1} = i_k$ if $k \notin I$, and $x_{k1} = j_k \neq i_k = x_{k_2}$ if $k \in I$. That means, every i_k is fixed ($i_k = x_{k1}$ or x_{k2}) by x for $k = 1, \ldots, p$, and only those j_k's with $k \in I$ are also fixed by x. Hence, there are d^{p-t} ways to choose j_k, such that $j_k \neq i_k = x_{k_1}$ for $k \notin I$. Furthermore, there are $\binom{p}{t}$ ways to choose I. Thus, (x, y) supports at most $\binom{p}{t} d^{p-t}$ sets in $\mathcal{D}$, and

$$\sum_{t=0}^{p-1} \binom{p}{t} d^{p-t} = (d+1)^p - 1.$$

Then, (x, y) supports at most $(d+1)^p - 1$ sets in $\mathcal{D}$. Therefore, we need at least

$$\left\lceil \frac{d^p\,(d+1)^p}{((d+1)^p - 1)} \right\rceil$$

edges to supports $\mathcal{D}$. From Lemma 2.2 and

$$\begin{aligned}(d^p + 1)\,((d+1)^p - 1) &= d^p\,(d+1)^p + (d+1)^p - d^p - 1 \\ &\in \left[d^p\,(d+1)^p, d^p\,(d+1)^p + (d+1)^p - 1\right],\end{aligned}$$

we have $b(G) \geqslant \lceil d^p(d+1)^p/((d+1)^p - 1)\rceil = d^p + 1$.

Now, we show that $b(G) \leqslant (1+c)d^p$ for any $c \in (0,1]$. Let $f_c(d) = d^p + cd^p - (d+1)^p$. Then $f_c'(d) = p(1+c)d^{p-1} - p(d+1)^{p-1} > 0$ if $d > (\sqrt[p-1]{1+c} - 1)^{-1}$. Since $f_c(\sqrt[p-1]{1+c} - 1)^{-1}) < 0$, then there exists a positive $B > (\sqrt[p-1]{1+c} - 1)^{-1}$, such that $f_c(B) = 0$, and $f_c(d)$ is monotonically increasing on (B, ∞). Therefore, $f_c(d) > 0$, for every integer $d > B$. From the proof of Corollary 4.4, it is easy to see that $B > A$, since $B > (\sqrt[p-1]{1+c} - 1)^{-1} \geqslant (\sqrt[p-1]{2} - 1)^{-1}$, $f(B) > f_c(B) = 0 = f(A)$ and f is monotonically increasing on the interval $((\sqrt[p-1]{2} - 1)^{-1}, \infty)$.

Suppose $x \in V(G)$ and $y_i \in N^+(x)$ for $i = 1, 2, \ldots, d^p$. Let

$$E' = E^-(x) \cup \{(x, y_i) : i \leqslant \lfloor cd^p \rfloor\} \quad \text{and} \quad H = G - E'.$$

Then, any minimum dominating set D in H contains x for $d_H^-(x) = 0$; but x dominates only $d^p - \lfloor cd^p \rfloor$ other vertices. It follows that $\gamma(H) = |D| > \gamma(G)$, for otherwise Corollary 4.4 yields that D dominates at most

$$\begin{aligned}(|D| - 1)\,(d^p + 1) + d^p - \lfloor cd^p \rfloor + 1 &\leqslant |D|\,(d^p + 1) - cd^p + 1 \\ &= d^p\,(d+1)^p + (d+1)^p - d^p - 1 - cd^p + 1 \\ &= d^p\,(d+1)^p - f_c\,(d) \\ &< d^p\,(d+1),\end{aligned}$$

vertices, a contradiction to $n(G) = d^p(d+1)^p$. Thus,

$$b(G) \leqslant |E'| = d^p + \lfloor cd^p \rfloor \leqslant (1+c)\, d^p,$$

and the upper bound is established.

For general cases, our technique seems not feasible since we can not prove that $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}$ defined in (2) is minimum by showing $|D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}| = \lceil d^{n-p}(d+1)^p/(d^p+1)\rceil$. We suggest the following conjecture.

CONJECTURE 4.6. *The set $D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}$ defined in (2) is a minimum dominating set in $EK(d,n;q_1,\ldots,q_p)$ with $q_k \geqslant 2$ for $k = 1,2,\ldots,p$.*

If this conjecture is valid, then

$$\gamma(G) = \left|D^{(j_1,\ldots,j_p)}_{(i_1,\ldots,i_p)}\right| = d^{n-2p}\left((d+1)^p - 1\right).$$

In addition, we can obtain bounds for $b(G)$ by Lemmas 2.1 and 2.2.

CONJECTURE 4.7. *Let $G = EK(d,n;q_1,\ldots,q_p)$ with $p \geqslant 2$ and $q_k \geqslant 2$ for $k = 1, 2, \ldots, p$. Then $\gamma(G) = d^{n-2p}((d+1)^p - 1)$ and $d^p + 1 \leqslant b(G) \leqslant 3d^p$.*

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