第36卷第9期 中国科学技术大学学报 Vol. 36, No. 9
2006年9月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Sep. 2006

Article ID: 0253-2778(2006)09-0951-05

On addition and deletion of edges of graphs*

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Abstract: Let $P(t,d)$ (resp. $C(t,d)$) denote the minimum diameter of a graph obtained by adding t extra edges to a path (resp. cycle) of length d. Let $T_P(p,d)$ (resp. $T_C(p,d)$) be the minimum number of edges added to a path (resp. cycle) of length d in order to obtain a graph of diameter not greater than p. Let $f(t,d)$ denote the maximum diameter of a connected graph obtained after deleting t edges from a connected graph of diameter d. Some new lower and upper bounds of these parameters were presented. In particular, it is proved that $T_C(3,d) = d-8$ for $d \geqslant 12$ conjectured by Grigorescu [J. Graph Theory, 2003, 43(2): 299-303], and it is partially proved that $f(t,d) \leqslant (t+1)d-t+1$ conjectured by Schoone *et al* [J. Graph Theory, 1987, 11(3): 409-427].

Key words: diameter; altered graph; edge addition; edge deletion; Schoone *et al*'s conjecture

CLC number: O157.5; TP302.1 **Document code**: A

AMS Subject Classification(2000): 33C80; 03F05; 65Q05.

关于图的边添加和减少

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摘要: 用 $P(t,d)$(或者 $C(t,d)$)表示从长为 d 的路(或者圈)通过添加 t 条边后得到的图的最小直径, $T_P(p,d)$(或者 $T_C(p,d)$)表示为了得到直径最多为 p 的图需要向长为 d 的路(或者圈)中添加的最少边数, $f(t,d)$ 表示从直径为 d 的图中删去 t 条边后得到的连通图的最大直径. 我们给出了这些参数新的上下界. 特别地, 证明了 Grigorescu[J. Graph Theory, 2003, 43(2): 299-303]猜想: $T_C(3,d) = d-8$, 其中 $d \geqslant 12$; 并且部分地解决了 Schoone 等人[J. Graph Theory, 1987, 11(13): 409-427] 的猜想: $f(t,d) \leqslant (t+1)d-t+1$.

关键词: 直径; 变更图; 边添加; 边减少; Schoone 等的猜想

0 Introduction

This paper is the series paper of Refs. [5] and [6]. We also follow Ref. [1] for graph-theoretical terminology and notation not defined here. As Ref. [6], let $G=(V,E)$ be a simple undirected graph with a vertex-set $V=V(G)$ and an edge-set $E=E(G)$.

Let $P(t,d)$ (resp. $C(t,d)$) denote the minimum diameter of a graph obtained by adding t extra edges to a path (resp. cycle) of length d. For some small t's and special d's, the exact values of

* **Received**: 2005-10-11; **Revised**: 2006-06-17

Foundation item: Supported by NNSF of China (10271114).

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$P(t,d)$ and $C(t,d)$ have been determined. For example,

$$P(1,d)=\left\lfloor \frac{d+1}{2} \right\rfloor \text{ for } d \geqslant 2,$$

$$P(2,d)=\left\lceil \frac{d+1}{3} \right\rceil \text{ for } d \geqslant 3,$$

$$P(3,d)=\left\lceil \frac{d+2}{4} \right\rceil \text{ for } d \geqslant 5,$$

$$C(1,d)=\left\lfloor \frac{d}{2} \right\rfloor \text{ for } d \geqslant 2,$$

$$C(2,d)=\left\lceil \frac{d+2}{4} \right\rceil \text{ for } d \geqslant 4,$$

determined by Schoone *et al*[2], and

$$P(t,(2k-1)(t+1)+1)=2k$$

for any positive integer k determined by DENG and XU[3]. Schoone *et al*[2] proved that the problem determining the values of the two parameters for general integers t and d is NP-complete. However, many lower and upper bounds of $P(t,d)$ and $C(t,d)$ have been established by several authors[2~7].

Let $T_P(p,d)$ (resp. $T_C(p,d)$) be the minimum number of edges that have been added to a path (resp. cycle) of length d to transform it into a graph of diameter at most p. Schoone *et al*[2] proved that it is NP-complete to determine the $T_P(p,d)$ and $T_C(p,d)$. Alon [8] determined $T_P(2,d)=d-2$ for $d \geqslant 273$, $T_C(2,d)=d-3$, $d-99 \leqslant T_P(3,d)$, $d-100 \leqslant T_C(3,d) \leqslant d-6$ and, in general, $T_P(p,d) \leqslant (d+1)/\lfloor p/2 \rfloor$. Grigorescu[9] proved $d-59 \leqslant T_C(3,d) \leqslant d-8$ and conjectured $T_C(3,d)=d-8$ for $d \geqslant 12$.

Let $f(t,d)$ denote the maximum diameter of a connected graph obtained after deleting t edges from a connected graph of diameter d. Plesnik[10] determined $f(1,d)=2d$. Schoone *et al*[2] proved $f(2,d)=3d-1$, $f(3,d)=4d-2$ for $d>1$, $f(t,2)=t+3$ for $t=1,2,3,4,6$, and $t+2$ otherwise; and conjectured

$$f(t,d) \leqslant (t+1)d-t+1.$$

1 Main results

In this paper we prove $\left\lceil \frac{d-2}{t+1} \right\rceil \leqslant P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1$ for $t \geqslant 4$ and odd $d \geqslant 3$; $P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1$ for $t \geqslant 4$ and $3t+1 \leqslant d \leqslant 3t+3$; $C(t,d) \geqslant \left\lceil \frac{d-1}{t+2} \right\rceil$ for $t \geqslant 3$ and $d \geqslant 2$; $\left\lceil \frac{d}{p} \right\rceil - 1 \leqslant T(p,d) \leqslant \left\lfloor \frac{d-7}{p-3} \right\rfloor - 1$ for $p \geqslant 4$ and $d \geqslant 12$, and $\left\lceil \frac{d-2}{p-1} \right\rceil - 1 \leqslant T(p,d) \leqslant \left\lfloor \frac{d-2}{p-2} \right\rfloor - 1$ for some special integers p and d. In particular, we prove the conjecture of Grigorescu that $T_C(3,d)=d-8$ for $d \geqslant 9$.

For the conjecture of Schoone *et al*, we obtain

$$f(t,d) \leqslant \begin{cases} (t+1)d-t+1 & \text{if } d \geqslant 3 \text{ and is odd or } \\ & P(t,d)=\left\lceil \frac{d-2}{t+1} \right\rceil + 1, \\ (t+1)d-2t & \text{if } P(t,d)=\left\lceil \frac{d-2}{t+1} \right\rceil + 2, \end{cases}$$

which is tight when d is even and

$$P(t,d)=\left\lceil \frac{d-2}{t+1} \right\rceil + 1.$$

2 Several lemmas

Lemma 2.1[5] For any integer $k \geqslant 1$, let $I'(t,k)=\{2k(t+1)+1, 2k(t+1)+2, 2k(t+1)-t+1\} \cup \{2k(t+1)-t+h : h=6,7,\cdots,t\}$. Then

$$P(t,d) \leqslant \begin{cases} \left\lceil \frac{d-2}{t+1} \right\rceil + 2 & \text{if } d \in I'(t,k), \\ \left\lceil \frac{d-2}{t+1} \right\rceil + 1 & \text{otherwise} \end{cases}$$

for any integers $t \geqslant 6$ and $d \geqslant 2$.

Lemma 2.2[6] $P(t,d)=\left\lceil \frac{d-2}{t+1} \right\rceil + 1$, where $t \geqslant 4$, $t+4 \leqslant d \leqslant t+7$, and $t=4$, $d=10k+1$, $k \geqslant 1$. For $t \geqslant 3$, $C(t,d)=3$ where $t+6 \leqslant d \leqslant t+8$.

Lemma 2.3[3] For any positive integers t and $d(\geqslant 2)$, $\left\lceil \frac{d}{t+1} \right\rceil \leqslant P(t,d) \leqslant \left\lfloor \frac{d-2}{t+1} \right\rfloor + 3$. In particular, $P(t,(2k-1)(t+1)+1)=2k$ for any positive integer k, $F(t,d) \leqslant \left\lfloor \frac{d}{t} \right\rfloor + 1$ if d is large enough, and $\left\lceil \frac{d}{t+1} \right\rceil \leqslant P(t,d) \leqslant \left\lceil \frac{d}{t+1} \right\rceil + 1$ for $t=4,5$ and $d \geqslant 4$.

Lemma 2.4[11] Let G be a connected undirected graph, $S \subset E(G)$ and $|S| = t$. If $h = d(G-S)$ is well defined, then $d(G) \geqslant P(t,h)$.

Lemma 2.5[7] Let $t \geqslant 4$. $P(t,d) \leqslant \frac{d-7}{t+1}+3$ for $d \geqslant 2$, and

$$C(t,d) \leqslant \begin{cases} \frac{d-7}{t+2}+3 & \text{if } t \text{ is even,} \\ \left\lceil \frac{d+t-6}{2t+2} \right\rceil + \left\lceil \frac{d+t-1}{2t+2} \right\rceil \leqslant \frac{d-8}{t+1}+3 & \text{if } t \text{ is odd,} \end{cases}$$

and $\left\lceil \frac{d}{4} \right\rceil - 1 \leqslant C(3,d) \leqslant \left\lceil \frac{d}{4} \right\rceil$ for $d \geqslant 5$. Also $f(t,d) \geqslant (t+1)d - 2t + 4$ for any odd $d \geqslant 3$.

3 Proof of main results

3.1 Addition of edges

Theorem 3.1 For $t \geqslant 4$,

$$P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1 = 4,$$

where $3t+1 \leqslant d \leqslant 3t+3$.

Proof Let $P = (x_0, x_1, \cdots, x_d)$ be an (x_0, x_d)-path and G an altered graph with diameter $d(G) = P(t,d)$ obtained from P plus t extra edges, where $d = 3t+1$. From Lemma 2.3, $d(G) \geqslant 3$. So, it is sufficient to prove $d(G) \neq 3$. Assume the contrary $d(G) = 3$. For $0 \leqslant i < d$, let x_i be the smallest numbered vertex that G has no edge (x_i, x_j) with $j > i+1$. Thus, for each $h = 0, 1, \cdots, i-1$, there exists a $j (j \geqslant h+2)$ such that $(x_h, x_j) \in E(G)$ is an extra edge. For each $h = 0, 1, \cdots, i-1$, let A_h be the set of extra edges incident with the vertex x_h and B the set of other extra edges. Then $|A_h| \geqslant 1$ for each $h = 0, 1, \cdots, i-1$, $|\bigcup A_h| \geqslant i$ and $|\bigcup A_h| + |B| = t$.

Suppose that there are three consecutive vertices x_{j-1}, x_j and x_{j+1}, $i+4 \leqslant j \leqslant d-1$ such that none of them is incident with some extra edge. Then x_j needs at least 4 steps to reach x_i, which contradicts the hypothesis of $d(G) = 3$. Thus, at least one of the three consecutive vertices x_{j-1}, x_j and x_{j+1} in $X_2 = \{x_{i+4}, x_{i+5}, \cdots, x_d\}$ is incident with some extra edge. We consider the worst case, that is, exactly one vertex in $\{x_{j-1}, x_j, x_{j+1}\}$ is incident with only one extra edge. Since the distance between x_i and x_j is at least two in G, the only vertex incident with the only extra edge must be x_j. Let the only extra edge be e_j. So, any shortest path P from x_i to x_j with length two must contain either the vertex x_{i+1} or some vertex x_h in $X_1 = \{x_0, x_1, \cdots, x_{i-1}\}$. If the former happens, then $e_j = x_{i+1}x_j$ and $e_j \in B$. If the later happens, then $|A_h| \geqslant 2$ if $h \leqslant i-2$. Thus, the vertex x_i needs at least $\left\lceil \frac{(d+1)-(i+4)}{3} \right\rceil - \delta$ extra edges to reach all vertices in X_2 by at most three steps, where $\delta = 0$ if $i = 0$, and at most one step otherwise. Let E_1 be the set of these edges.

Since every edge in E_1 must reach either the vertex x_{i+1} or some vertex x_h in X_1, then the graph needs at least two new extra edges to become graph of diameter at most three.

Thus we have

$$t \geqslant i + \left\lceil \frac{(d+1)-(i+4)}{3} \right\rceil + 2 - \delta \geqslant t + \left\lceil \frac{2i+4-3\delta}{3} \right\rceil \geqslant t+1.$$

Thus $P(t, 3t+1) \geqslant 4$. Since $P(t,d) \leqslant P(t,d')$ if $d \leqslant d'$, $P(t,d) \geqslant 4$ for $d \geqslant 3t+1$. On the other hand, since $3t+1, 3t+2, 3t+3 \in I'(t,k)$ for any k, from Lemma 2.1, $P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1 = 4$ for $t \geqslant 6$. For $t = 4, 5$, from Lemma 2.3,

$$P(t,d) \leqslant \left\lceil \frac{d}{t+1} \right\rceil + 1 = 4 \text{ for } d \geqslant 4.$$

Thus, $P(t,d) = 4$ for $3t+1 \leqslant d \leqslant 3t+3$ and $t \geqslant 4$.

Remark From Theorem 3.1 and Lemma 2.3, $3 \leqslant P(t,d) \leqslant 4$ for $t+8 \leqslant d \leqslant 3t$ and $t \geqslant 4$.

Theorem 3.2 For $t \geqslant 3$ and $d \geqslant 2$, $C(t,d) \geqslant \left\lceil \frac{d-1}{t+2} \right\rceil$, which is tight when t is even, $k(t+2)+2 \leqslant d \leqslant k(t+2)+6$ and $k \geqslant 0$. Furthermore, $C(t,d) = 4$, where $(t,d) \in \{(3,13), (3,14), (3,15), (4,16), (4,17), (4,18), (5,19), (6,22)\}$.

Proof It is easy to verify that

$$C(t, d+1) \geqslant P(t+1, d), \tag{1}$$

since one way of adding $t+1$ edges to a path P_{d+1} is to first add one edge joining two end vertices of

P_{d+1} and then to add t edges in an optimal way to the resulting cycle C_{d+1}. Then from Lemma 2.3 we have

$$C(t,d) \geqslant \left\lceil \frac{d-1}{t+2} \right\rceil.$$

Clearly, from Lemma 2.4, the boundary above is tight for any even $t \geqslant 3$,

$$k(t+2)+2 \leqslant d \leqslant k(t+2)+6$$

and $k \geqslant 0$.

From Theorem 3.1 and Inequality (1) we have

$$C(t,d+1) \geqslant 4 \text{ for } 3t+4 \leqslant d \leqslant 3t+6, t \geqslant 3.$$

Also from Lemma 2.4 we have that for $t \geqslant 3$,

$$C(t,d+1) \leqslant 4,$$

where $(t,d) \in \{(3,13),(3,14),(3,15),(4,16),(4,17),(4,18),(5,19),(6,22)\}$.

So the theorem follows. □

Theorem 3.3 For $p \geqslant 4$ and $d \geqslant 12$,

$$\left\lceil \frac{d}{p} \right\rceil - 1 \leqslant T_P(p,d) \leqslant \left\lfloor \frac{d-7}{p-3} \right\rfloor - 1.$$

In particular,

$$\left\lceil \frac{d-2}{p-1} \right\rceil - 1 \leqslant T_P(p,d) \leqslant \left\lfloor \frac{d-2}{p-2} \right\rfloor - 1$$

for $(p=3, d \geqslant 7)$, $(p=4, d \geqslant 12)$, $(p=2k, d=10k-8)$, and $(p=2k+1, d=10k-3)$, $k \geqslant 1$.

Proof From Lemma 2.1 and Lemma 2.5 we have $\left\lceil \frac{d}{t+1} \right\rceil \leqslant P(t,d) \leqslant \frac{d-7}{t+1}+3$ for $t \geqslant 4$ and $d \geqslant 2$. Put $p = P(t,d)$. Let $p \geqslant 4$ then we get $d \geqslant t+8$, which means that $d \geqslant 12$. Since

$$p \leqslant \frac{d-7}{t+1}+3,$$

we have

$$t \leqslant \left\lfloor \frac{d-7}{p-3} \right\rfloor - 1.$$

Since $p \geqslant \left\lceil \frac{d}{t+1} \right\rceil \geqslant \frac{d}{t+1}$, we have

$$t \geqslant \left\lceil \frac{d}{p} \right\rceil - 1.$$

So, for $p \geqslant 4$ and $d \geqslant 12$,

$$\left\lceil \frac{d}{p} \right\rceil - 1 \leqslant T_P(p,d) \leqslant \left\lfloor \frac{d-7}{p-3} \right\rfloor - 1.$$

From Theorem 3.1, and Lemmas 2.2 and 2.3 we have $P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1$ for $t \geqslant 4$, $3t+1 \leqslant d \leqslant 3t+4$, $t+4 \leqslant d \leqslant t+7$, and $d = (2k-1)(t+1)+1$, and for $t=4$, $d=10k+1$, $k \geqslant 1$. Let $p=3$ then $\frac{d-2}{5}+2 \geqslant 3$, this means $d \geqslant 7$. Also in same way we have $d \geqslant 12$, $10k-8$, $10k-3$ when $p=4$, $2k, 2k+1$, respectively. Since $p \leqslant \frac{d-2}{t+1}+2$, we have

$$t \leqslant \left\lfloor \frac{d-2}{p-2} \right\rfloor - 1.$$

Also since $p \geqslant \frac{d-2}{t+1}+1$, we have

$$\left\lceil \frac{d-2}{p-1} \right\rceil - 1 \leqslant t.$$

So the theorem follows. □

Theorem 3.4 For $d \geqslant 11$, $T_C(3,d) = d-8$ and $d-7 \leqslant T_P(3,d) \leqslant d-3$.

Proof From Lemma 2.2,

$$C(t,d) = 3 \text{ for } t+6 \leqslant d \leqslant t+8 \text{ and } t \geqslant 3.$$

Then we have $C(t,d)=3$ for $d-8 \leqslant t \leqslant d-6$ and $d \geqslant 11$, which means

$$T_C(3,d) \geqslant d-8.$$

Since Grigorescu[9] proved $T_C(3,d) \leqslant d-8$, we have

$$T_C(3,d) = d-8. \tag{2}$$

It is easy to verify that

$$T_C(p,d+1) \leqslant T_P(p,d).$$

Then from Equality (2) and Theorem 3.3 we have

$$d-7 \leqslant T_P(3,d) \leqslant d-3.$$

The theorem follows. □

3.2 Deletion of edges

Theorem 3.5 For any integers $t \geqslant 2$ and $d \geqslant 1$,

$$f(t,d) \leqslant \begin{cases} (t+1)d-t+1 & \text{if } d \geqslant 3 \text{ and is odd or } P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1, \\ (t+1)d-2t & \text{if } P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 2. \end{cases}$$

This bound is tight when d is even and $P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + 1$.

Proof It is clear that for any $t \geqslant 4$,

$$\left\lceil \frac{d-2}{t+1} \right\rceil \leqslant \left\lceil \frac{d}{t+1} \right\rceil \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1.$$

From Lemma 2.1 and Lemma 2.3 we have for $t \geqslant 4$ and $d \geqslant 2$,

$$P(t,d) = \left\lceil \frac{d-2}{t+1} \right\rceil + i, \quad \text{for some } i = 0,1,2. \tag{3}$$

Let G be an undirected graph with diameter d, $S \subset E(G)$ and $|S| = t$ such that

$$d(G-S) = h = f(t,d).$$

Then from Lemma 2.4 and Equality (3) there exists some i such that

$$\left\lceil \frac{h-2}{t+1} \right\rceil + i = P(t,h+1) \leqslant d.$$

Then

$$\frac{h-2+it+i}{t+1} \leqslant d.$$

Thus

$$f(t,d) = h \leqslant (t+1)d - it - i + 2$$

So, from Lemma 2.5, the theorem follows. □

From Lemma 2.5 and Theorem 3.5, we immediately have

Corollary 3.6 If $t \geqslant 4$, and $d \geqslant 3$ and is odd, then $\left\lceil \frac{d-2}{t+1} \right\rceil \leqslant P(t,d) \leqslant \left\lceil \frac{d-2}{t+1} \right\rceil + 1$.

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