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On Super Edge-Connectivity of Cartesian Product Graphs

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The super edge-connectivity λ' of a connected graph G is the minimum cardinality of an edge-cut F in G such that every component of $G - F$ contains at least two vertices. Let G_i be a connected graph with order n_i, minimum degree δ_i and edge-connectivity λ_i for $i = 1, 2$. This article shows that $\lambda'(G_1 \times G_2) \geq \min\{n_1\lambda_2, n_2\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1+\lambda_2\}$ for $n_1, n_2 \geq 3$ and $\lambda'(K_2 \times G_2) = \min\{n_2, 2\lambda_2\}$, which generalizes the main result of Shieh on the super edge-connectedness of the Cartesian product of two regular graphs with maximum edge-connectivity. In particular, this article determines $\lambda'(G_1 \times G_2) = \min\{n_1\delta_2, n_2\delta_1, \xi(G_1 \times G_2)\}$ if $\lambda'(G_i) = \xi(G_i)$, where $\xi(G)$ is the minimum edge-degree of a graph G. © 2006 Wiley Periodicals, Inc. NETWORKS, Vol. 49(2), 152–157 2007

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1. INTRODUCTION

Throughout this article, a graph $G = (V, E)$ always means a finite undirected graph without self-loops or multiple edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For any edge $uv \in E$, the parameter $\xi_G(uv) = d_G(u) + d_G(v) - 2$ is the *degree of the edge uv* and the parameter $\xi(G) = \min\{\xi_G(uv) \mid uv \in E\}$ is the *minimum edge-degree* of G. The symbols $K_{1,n-1}$ and K_n denote a star graph and a complete graph with n vertices, respectively. For the graph theoretical terminology and notation not defined here, we refer the reader to [13].

It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the edge-connectivity $\lambda(G)$ of G is an important measurement for the fault tolerance of the network. In general, the larger $\lambda(G)$ is, the more reliable the network is. It is well known that $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. A connected graph G is said to be *maximally edge-connected* (in short, *max-λ*) if $\lambda(G) = \delta(G)$. Obviously, the set of edges incident with a vertex of degree $\delta(G)$ is certainly a minimum edge-cut and isolates a vertex when G is max-λ.

A graph G is said to be *super edge-connected* (in short, *super-λ*) if G is max-λ and every minimum edge-cut isolates a vertex of G.

It has been shown that a super-λ network is the most reliable and has the smallest edge failure rate (see, e.g., [17, 18]). Several sufficient conditions for a graph to be max-λ or super-λ have been given in the literature (see, e.g., [6]).

A quite natural problem is that if a connected graph G is super-λ then how many edges have to be removed to disconnect G such that every component of the resulting graph contains no isolated vertices. This problem results in the concept of the super edge-connectivity, introduced first by Fiol et al. in [4].

An edge-cut F is called a *super edge-cut* of G if $G - F$ contains no isolated vertices. In general, super edge-cuts do not always exist. The *super edge-connectivity* $\lambda'(G)$ is the minimum cardinality of a super edge-cut in G if super edge-cuts exist, and, by convention, is $+\infty$ otherwise.

The new parameter λ' in conjunction with λ can provide more accurate measures for the fault tolerance of a large-scale parallel processing system and, thus, has received the attention of many researchers in recent years (see, e.g., [3–9, 11, 14–16]). Esfahanian and Hakimi [3] showed that if G is neither $K_{1,n-1}$ nor K_3, then

$$\lambda(G) \leq \lambda'(G) \leq \xi(G). \tag{1}$$

A connected graph G is called a *λ'-graph* if G is neither $K_{1,n-1}$ nor K_3. It is easy to see that if $\lambda'(G) > \lambda(G)$ then G

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is super-λ. A super-λ graph G is said to be *optimally super edge-connected* (in short, λ'*-optimal*) if $\lambda'(G) = \xi(G)$.

Recently, Chiue and Shieh [1] have given some sufficient conditions for the Cartesian product $G_1 \times G_2$ to be super-λ; Shieh [10] has proved that $G_1 \times G_2$ is super-λ if both G_1 and G_2 are regular and max-λ except for $K_2 \times K_n$, where $n \geq 2$. Ueffing and Volkmann [11] have investigated the λ'-optimality of $G_1 \times G_2$ when both G_1 and G_2 are λ'-optimal. Li and Xu [7] have determined $\lambda'(K_2 \times G) = \min\{n, 2\delta(G), 2\lambda'(G)\}$ for any connected graph G with n vertices.

Let G_i be a connected graph of order n_i, minimum degree δ_i and edge-connectivity λ_i for $i = 1, 2$. In this article, we show that $\lambda'(G_1 \times G_2) \geq \min\{n_1\lambda_2, n_2\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$ for $n_1, n_2 \geq 3$ by refining the technique of Chiue and Shieh in [1] and determine that $\lambda'(K_2 \times G_2) = \min\{n_2, 2\lambda_2\}$, which generalizes the result of Shieh [10]. In particular, similar to the proof of Theorem 4.1 in [11], we determine that $\lambda'(G_1 \times G_2) = \min\{n_1\delta_2, n_2\delta_1, \xi(G_1 \times G_2)\}$ if both G_1 and G_2 are λ'-optimal.

The proofs of these results are given in Section 3.

2. PRELIMINARIES

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *union* of two graphs (not necessarily disjoint) G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with the vertex-set $V(G_1 \cup G_2) = V_1 \cup V_2$ and the edge-set $E(G_1 \cup G_2) = E_1 \cup E_2$. The *Cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with the vertex-set $V_1 \times V_2$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2 \in V_1$ with $y_1y_2 \in E_2$ or $y_1 = y_2 \in V_2$ with $x_1x_2 \in E_1$.

By the definition of the Cartesian product $G = G_1 \times G_2$, for any vertex $(x, y) \in V(G)$,

$$d_G(x, y) = d_{G_1}(x) + d_{G_2}(y),$$

and if $x_1x_2 \in E_1$ or $y_1y_2 \in E_2$, then

$$\xi_G((x_1, y_1)(x_2, y_1)) = \xi_{G_1}(x_1x_2) + 2d_{G_2}(y_1),$$
$$\xi_G((x_1, y_1)(x_1, y_2)) = \xi_{G_2}(y_1y_2) + 2d_{G_1}(x_1),$$

respectively, and consequently,

$$\xi(G) = \min\{\xi(G_1) + 2\delta(G_2), \xi(G_2) + 2\delta(G_1)\}.$$

For convenience, we define two kinds of subgraphs G_{1y} and G_{2x} of $G_1 \times G_2$ as follows.

$V(G_{1y}) = \{(x, y) \mid x \in V_1\}$ and

$$E(G_{1y}) = \{(x_1, y)(x_2, y) \mid x_1x_2 \in E_1\} \text{ for any } y \in V_2;$$

$V(G_{2x}) = \{(x, y) \mid y \in V_2\}$ and

$$E(G_{2x}) = \{(x, y_1)(x, y_2) \mid y_1y_2 \in E_2\} \text{ for any } x \in V_1.$$

It is clear that G_{1y} is isomorphic to G_1 for any $y \in V_2$ and G_{2x} is isomorphic to G_2 for any $x \in V_1$. Let

$$V_{1y} = V(G_{1y}),\ E_{1y} = E(G_{1y}),\ V_{2x} = V(G_{2x}),\ E_{2x} = E(G_{2x}).$$

Then

$$E_{1y} \cap E_{1y'} = \emptyset, \quad \text{for any } y, y' \in V_2, y \neq y';$$
$$E_{2x} \cap E_{2x'} = \emptyset, \quad \text{for any } x, x' \in V_1, x \neq x';$$

$$V_{1y} \cap V_{2x} = \{(x, y)\}, \quad E_{1y} \cap E_{2x} = \emptyset \quad \text{for any } x \in V_1, y \in V_2;$$

$$E(G_1 \times G_2) = (\cup_{y \in V_2} E_{1y}) \cup (\cup_{x \in V_1} E_{2x}).$$

To check whether a union graph is connected or not, the following concept and results, due to Chiue and Shieh [1], are useful.

Definition (Separability). *For $G = G_1 \cup G_2 \cup \cdots \cup G_k$, $V(G)$ is called separable if and only if $V(G)$ can be partitioned into two disjoint nonempty sets A and A' such that $A \cup A' = V(G)$ and each $V(G_i)$ is a subset of either A or A' for $i = 1, 2, \ldots, k$.*

Lemma 1. *Suppose $G = \cup_{i=1}^{k} G_i$, where G_i is connected for $i = 1, 2, \ldots, k$. If $V(G)$ is nonseparable, then G is connected.*

Remark 1. Because $V_{1y} \cap V_{2x} = \{(x, y)\}$, $V_{1y} \cup V_{2x}$ is nonseparable for any $x \in V_1$ and $y \in V_2$.

3. MAIN RESULTS

We first introduce some notation used in this section. Let $G = (V, E)$ be a graph. For two disjoint nonempty subsets X and Y of V, denote $(X, Y)_G = \{xy \in E \mid x \in X, y \in Y\}$. If $Y = V \setminus X$, then we write $E_G(X) = (X, Y)_G$ and $d_G(X) = |E_G(X)|$.

A super edge-cut F of G is called a λ'-cut if $|F| = \lambda'(G)$. It is clear that $G - F$ has exactly two components for any λ'-cut F. A nonempty and proper subset X of V is called a λ'-fragment of G if $E_G(X)$ is a λ'-cut of G. The minimum λ'-fragment over all λ'-fragments of G is called a λ'-atom of G.

For $F \subseteq E(G_1 \times G_2)$, let

$$G'_{1y} = G_{1y} - F \text{ for any } y \in V_2,$$
$$G'_{2x} = G_{2x} - F \text{ for any } x \in V_1.$$

Then, it is clear that

$$V(G'_{1y}) = V_{1y},\ V(G'_{2x}) = V_{2x} \text{ for any } x \in V_1 \text{ and } y \in V_2;$$
$$G_1 \times G_2 - F = (\cup_{y \in V_2} G'_{1y}) \cup (\cup_{x \in V_1} G'_{2x}).$$

Let $C = \{x \in V_1 \mid G'_{2x} \text{ is connected}\}$ and $D = \{y \in V_2 \mid G'_{1y} \text{ is connected}\}$.

Throughout this section, we always assume that G_1 and G_2 have m and n vertices, respectively, and $\lambda(G_i) = \lambda_i \geq 1$ for $i = 1, 2$. So $\delta(G_i) \geq 1$ for $i = 1, 2$, which implies $m \geq 2$ and $n \geq 2$.

Lemma 2. *$G = G_1 \times G_2$ is a λ'-graph if $m \geq 2$ and $n \geq 2$.*

Proof. Because $m \geq 2$ and $n \geq 2$, the graph $G = G_1 \times G_2$ has $|V(G_1 \times G_2)| = mn \geq 4$ vertices, and thus G is not K_3. Moreover, $\delta(G) = \delta(G_1) + \delta(G_2) \geq 2$, and thus G is not a star. Therefore, G is a λ'-graph. ∎

Theorem 1. *$\lambda'(G_1 \times G_2) \geq \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$ if $m \geq 3$ and $n \geq 3$.*

Proof. Denote $\mu = \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$. By Lemma 2, $G_1 \times G_2$ is a λ'-graph, so its super edge-cuts always exist. Assume F is a minimum super edge-cut with $|F| < \mu$. We need to show that $G_1 \times G_2 - F$ is connected. Because $|F| < \mu \leq m\lambda_2$, there exists some $x_0 \in V_1$ such that G'_{2x_0} is connected. Because $|F| < \mu \leq n\lambda_1$, there exists some $y_0 \in V_2$ such that G'_{1y_0} is connected. That is to say, $|C| \geq 1$ and $|D| \geq 1$. There are three cases to be considered for us.

CASE 1. $|C| = 1$. This implies that the other $m-1$ subgraphs G'_{2x} are disconnected, where $x \in V_1 \backslash \{x_0\}$. In this case, G'_{1y} is connected for any $y \in V_2$. Otherwise, because $m \geq 3$, we have $|F| \geq (m-1)\lambda_2 + \lambda_1 \geq 2\lambda_2 + \lambda_1 \geq \mu$, a contradiction. Thus, $|D| = n$. By Remark 1, $(\cup_{y \in D} V_{1y}) \cup V_{2x_0} = V_1 \times V_2$ is nonseparable and, thus, $(\cup_{y \in D} G'_{1y}) \cup G'_{2x_0}$ is connected by Lemma 1 and so is $G_1 \times G_2 - F$.

CASE 2. $|C| = m$. Because $(\cup_{x \in C} V_{2x}) \cup V_{1y_0} = V_1 \times V_2$ is nonseparable, we have $(\cup_{x \in C} G'_{2x}) \cup G'_{1y_0}$ is connected by Lemma 1, which means $G_1 \times G_2 - F$ is connected.

CASE 3. $2 \leq |C| \leq m-1$. When $|D| = 1$ or $|D| = n$, the connectedness of $G_1 \times G_2 - F$ can be derived in the same way as Case 1 or Case 2. Now assume $2 \leq |D| \leq n-1$. On the other hand, $|D| \geq n-1$ because $|C| \leq m-1$, otherwise $|F| \geq 2\lambda_1 + \lambda_2 \geq \mu$, a contradiction. Thus, $|D| = n-1$. Similarly, $|C| = m-1$. Without loss of generality, assume $G'_{2x'}$ and $G'_{1y'}$ are disconnected for some $x' \in V_1$ and $y' \in V_2$. That is, G'_{2x} is connected for any $x \neq x'$ and G'_{1y} is connected for any $y \neq y'$. Because $(\cup_{y \neq y'} V_{1y}) \cup (\cup_{x \neq x'} V_{2x}) = V_1 \times V_2 \backslash \{(x', y')\}$ is nonseparable, by Lemma 1, $(\cup_{y \neq y'} G'_{1y}) \cup (\cup_{x \neq x'} G'_{2x})$ is connected. Because F is a super edge-cut, the vertex (x', y') is adjacent to $(\cup_{y \neq y'} G'_{1y}) \cup (\cup_{x \neq x'} G'_{2x})$. So $G_1 \times G_2 - F$ is connected and the proof is complete. ∎

Remark 2. The lower bound given above is tight. For example, let $G_1 = K_m$ with the vertex-set $\{x_1, x_2, \ldots, x_m\}$ and let $G_2 = K_{1,n-1}$ with the vertex-set $\{y_1, y_2, \ldots, y_n\}$, where $m \geq 3$ and $n \geq 3$. Then $\lambda_1 = m-1$ and $\lambda_2 = 1$. By Theorem 1, $\lambda'(G_1 \times G_2) \geq \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\} = \min\{m, n(m-1), m+1, 2m-1\} = m$. In addition, if $y_1 y_2 \in E(G_2)$, the edge-set $\{(x_1, y_1)(x_1, y_2), (x_2, y_1)(x_2, y_2), \ldots, (x_m, y_1)(x_m, y_2)\}$ is an edge-cut that isolates no vertex of $K_m \times K_{1,n-1}$. So it is a super edge-cut, which implies $\lambda'(K_m \times K_{1,n-1}) \leq m$. Therefore, $\lambda'(K_m \times K_{1,n-1}) = m$. The lower bound is attained.

Lemma 3 (Hellwig and Volkmann [5]). *If G is a λ'-optimal graph, then $\lambda(G) = \delta(G)$.*

With the proof of Theorem 4.1 in [11], we obtain the super edge-connectivity of the Cartesian product of two λ'-optimal graphs.

Theorem 2. *$\lambda'(G_1 \times G_2) = \min\{m\delta(G_2), n\delta(G_1), \xi(G_1 \times G_2)\}$ if G_1 and G_2 are both λ'-optimal.*

Proof. Denote $\delta(G_i) = \delta_i$, $\lambda(G_i) = \lambda_i$, $\xi(G_i) = \xi_i$, $\lambda'(G_i) = \lambda'_i$ for $i = 1, 2$ and $G = G_1 \times G_2$. Because G_i is λ'-optimal, G_i is a λ'-graph for $i = 1, 2$, which implies $m \geq 4, n \geq 4$.

By Lemma 2, $\lambda'(G)$ is well defined. First, we have $\lambda'(G) \leq \xi(G)$ by (1). Because $m \geq 4$, $E_G(V_1 \times \{y\})$ is a super edge-cut for a vertex $y \in V_2$ with $d_{G_2}(y) = \delta_2$. So $\lambda'(G) \leq m\delta_2$. Analogously, we have $\lambda'(G) \leq n\delta_1$. Thus,

$$\lambda'(G) \leq \min\{m\delta_2, n\delta_1, \xi(G)\}.$$

Assume $\lambda'(G) < \min\{m\delta_2, n\delta_1, \xi(G)\}$. Let F be a λ'-cut. We should show that $G_1 \times G_2 - F$ is connected to deduce a contradiction. Because $|F| < m\delta_2 = m\lambda_2$ by Lemma 3, there exists some $x_0 \in V_1$ such that G'_{2x_0} is connected. Analogously, there exists some $y_0 \in V_2$ such that G'_{1y_0} is connected. That is to say, $|C| \geq 1$ and $|D| \geq 1$. There are three cases to be considered.

CASE 1. $|C| = m$. By Remark 1, $(\cup_{y \in D} V_{1y}) \cup (\cup_{x \in C} V_{2x}) = V_1 \times V_2$ is nonseparable. Then $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ is connected by Lemma 1, and so is $G_1 \times G_2 - F$.

CASE 2. $|C| = m-1$. There is only $x_1 \in V_1$ such that G'_{2x_1} is disconnected. By Remark 1, $(\cup_{y \in D} V_{1y}) \cup (\cup_{x \in C} V_{2x}) = V_1 \times V_2 - \{(x_1, y) | y \in V_2 \backslash D\}$ is nonseparable, and thus by Lemma 1, $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ is connected. To prove that $G_1 \times G_2 - F$ is connected, we only need to show that every vertex in $V' = \{(x_1, y) | y \in V_2 \backslash D\}$ is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$. To this end, let (x_1, y) be any vertex in V_{2x_1} $(y \notin D)$. Note that F is a super edge-cut. If (x_1, y) is an isolated vertex of G'_{2x_1}, then it is adjacent to (x', y) $(x' \neq x_1)$, which is in $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$. In the following, we suppose that (x_1, y) is contained in a component of G'_{2x_1} with at least two vertices. Denote this component by H. We only need to consider the case of $V(H) \subseteq V'$, otherwise (x_1, y) is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$. There are two subcases to be considered.

SUBCASE 2.1. $G_{2x_1} - V(H)$ contains a component with at least two vertices, denoted by H'. Because G_{2x_1} is connected, all the components of $G_{2x_1} - V(H)$ different from H', if any, are connected to H and not connected to H'. So $G_{2x_1} - V(H')$ is connected with $|V(G_{2x_1} - V(H'))| \geq |V(H)| \geq 2$, which implies that $E_{G_{2x_1}}(V(H'))$ is a super edge-cut. Hence, we conclude that $|F \cap E(G_{2x_1})| \geq |E_{G_{2x_1}}(V(H))| \geq |E_{G_{2x_1}}(V(H'))| \geq \lambda'_2$. There is at least one vertex in $V(H)$ with neighbors in $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$. Otherwise, we obtain the following contradiction

$$|F| \geq \lambda'_2 + |V(H)|\delta_1 \geq \lambda'_2 + 2\delta_1 = \xi_2 + 2\delta_1 \geq \xi(G).$$

So the vertex (x_1, y) is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$.

SUBCASE 2.2. $G_{2x_1} - V(H)$ contains only isolated vertices. Then $|F \cap E(G_{2x_1})| \geq |E_{G_{2x_1}}(V(H))| \geq (n - |V(H)|)\delta_2$. Let Δ_2 be the maximum degree of G_2. Obviously, $\Delta_2 \leq n - 1$ and $\xi_2 \leq \Delta_2 + \delta_2 - 2$. We claim that there is at least one vertex in $V(H)$ with neighbors in $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$. Otherwise,

$$
\begin{aligned}
|F| &\geq (n - |V(H)|)\delta_2 + |V(H)|\delta_1 \\
&= \delta_2 + (n - |V(H)| - 1)\delta_2 + (|V(H)| - 2)\delta_1 + 2\delta_1 \\
&\geq \delta_2 + (n - |V(H)| - 1) + |V(H)| - 2 + 2\delta_1 \\
&= \delta_2 + (n - 3) + 2\delta_1 \\
&\geq \delta_2 + \Delta_2 - 2 + 2\delta_1 \\
&\geq \xi_2 + 2\delta_1 \\
&\geq \xi(G),
\end{aligned}
$$

a contradiction. Therefore, the vertex (x_1, y) is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$.

In a word, $G_1 \times G_2 - F$ is connected in this case.

CASE 3. $|C| \leq m - 2$. Similarly, $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ is connected. In this case, to prove that $G_1 \times G_2 - F$ is connected, we only need to show that every vertex of G'_{2x} is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ for $x \notin C$.

Suppose that G'_{2x} contains a component with at least two vertices, denoted by H_x, which has no vertices in $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ for $x \notin C$. If $G_{2x} - V(H_x)$ contains a component with at least two vertices, similar to Subcase 2.1, we have $|F \cap E(G_{2x})| \geq |E_{G_{2x}}(V(H_x))| \geq \lambda'_2$. Hence, $|D| \geq n - 1$, otherwise, noting that G_1 is λ'-optimal, by Lemma 3,

$$|F| \geq |F \cap E(G_{2x})| + 2\lambda_1 \geq \lambda'_2 + 2\delta_1 = \xi_2 + 2\delta_1 \geq \xi(G),$$

a contradiction. The case $|D| \geq n - 1$ can be handled in the same way as Case 1 and Case 2.

If $G_{2x} - V(H_x)$ contains only isolated vertices, then $|F \cap E(G_{2x})| \geq |E_{G_{2x}}(V(H_x))| \geq (n - |V(H_x)|)\delta_2$. Because H_x has no vertices in $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ by our assumption,

$$
\begin{aligned}
|F| &\geq (n - |V(H_x)|)\delta_2 + |V(H_x)|\lambda_1 \\
&= (n - |V(H_x)|)\delta_2 + |V(H_x)|\delta_1 \qquad \text{(by Lemma 3)} \\
&= \delta_2 + (n - |V(H_x)| - 1)\delta_2 + (|V(H_x)| - 2)\delta_1 + 2\delta_1 \\
&\geq \delta_2 + (n - |V(H_x)| - 1) + |V(H_x)| - 2 + 2\delta_1 \\
&= \delta_2 + (n - 3) + 2\delta_1 \\
&\geq \delta_2 + \Delta_2 - 2 + 2\delta_1 \\
&\geq \xi_2 + 2\delta_1 \\
&\geq \xi(G),
\end{aligned}
$$

where Δ_2 is the maximum degree of G_2, a contradiction. So, every vertex of H_x is in or connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$.

Suppose that vertex $(x, y)(y \notin D)$ is isolated in G'_{2x}. Then it is not isolated in G'_{1y}, otherwise, it is isolated in $G_1 \times G_2 - F$, contradicting our hypothesis that F is a super edge-cut. So the vertex (x, y) is contained in a component with at least two vertices of G'_{1y}. We can show that vertex (x, y) is connected to $(\cup_{y \in D} G'_{1y}) \cup (\cup_{x \in C} G'_{2x})$ in the same way as above.

Because all possible cases lead to a contradiction, $\lambda'(G) = \min\{m\delta_2, n\delta_1, \xi(G)\}$ and the proof is complete. ∎

From Theorem 2, we can easily obtain the following corollary.

Corollary 1 (Ueffing and Volkmann [11]). *Let G_1 and G_2 be two disjoint λ'-optimal graphs and let $G = G_1 \times G_2$. Then G is λ'-optimal or the λ'-atoms of G have the form $\{x\} \times V_2$ for a vertex $x \in V_1$ with $d_{G_1}(x) = \delta(G_1)$ or $V_1 \times \{y\}$ for a vertex $y \in V_2$ with $d_{G_2}(y) = \delta(G_2)$.*

The n-dimensional toroidal mesh $C(d_1, d_2, \ldots, d_n)$ ([12]) can be expressed as the Cartesian product $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$, where C_{d_i} is a cycle of length d_i for $i = 1, 2, \ldots, n$.

Corollary 2 (Xu and Xu [16]). *Let $C(d_1, d_2, \ldots, d_n)$ be the n-dimensional toroidal mesh. Then $\lambda'(C(d_1, d_2, \ldots, d_n)) = 4n - 2$ and, thus, $C(d_1, d_2, \ldots, d_n)$ is λ'-optimal if $d_i \geq 4$ for each $i = 1, 2, \ldots, n$.*

Proof. We prove the corollary by induction on n. It is easy to see that a cycle C_{d_i} is λ'-optimal for $d_i \geq 4$. Now we assume $n \geq 2$. Suppose $G_{n-1} = C(d_1, d_2, \ldots, d_{n-1})$ is λ'-optimal, which implies $\lambda'(G_{n-1}) = 4(n-1) - 2 = 4n - 6 = \xi(G_{n-1})$. $|V(G_{n-1})| = d_1 + d_2 + \cdots + d_{n-1}$ and $\lambda(G_{n-1}) = 2(n-1)$. Denote $G = C(d_1, d_2, \ldots, d_n) = G_{n-1} \times C_{d_n}$. Noting $d_i \geq 4$ for each $i = 1, 2, \ldots, n$, by Theorem 2, we have

$$
\begin{aligned}
\lambda'(G) &= \min\{|V(G_{n-1})|\delta(C_{d_n}),\ |V(C_{d_n})|\delta(G_{n-1}),\ \xi(G)\} \\
&= \min\{2(d_1 + d_2 + \cdots + d_{n-1}), 2d_n(n-1), 4n - 2\} \\
&= 4n - 2 = \xi(G).
\end{aligned}
$$

Thus, G is λ'-optimal. ∎

Theorem 3. *Let G_0 be a connected graph with n vertices and $\lambda(G_0) = \lambda$. Then $\lambda'(K_2 \times G_0) = \min\{n, 2\lambda\}$.*

Proof. Let $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ and $G = K_2 \times G_0$. By the definition of the Cartesian product, $K_2 \times G_0$ is obtained from two copies of G_0 by connecting (via a new edge) vertex v_i in one copy to the vertex v_i in the other copy of G_0, $1 \leq i \leq n$. These new edges are called cross edges. Denote the two copies by G_1 and G_2, respectively, and let $V_1 = V(G_1), V_2 = V(G_2)$.

If $|V(G_0)| \geq 2$, then $E_G(V_1)$ is a super edge-cut of G, and hence, $\lambda'(G) \leq |E_G(V_1)| = |V_1| = n$. Suppose $X_1 \subseteq V_1$ with $d_{G_1}(X_1) = \lambda$ and let $X_2 \subseteq V_2$ be the set of those vertices adjacent to X_1. It is easy to see $E_G(X_1 \cup X_2)$ is a super edge-cut of G, and hence, $\lambda'(G) \le |E_G(X_1 \cup X_2)| = 2|E_{G_1}(X_1)| = 2\lambda$. It follows that

$$\lambda'(G) \le \min\{n, 2\lambda\}. \qquad (2)$$

We will show that the equality in (2) holds. Suppose to the contrary that $\lambda'(G) < \min\{n, 2\lambda\}$. We will show $G - F$ is connected for any λ'-cut F of G.

Let F be a λ'-cut of G. Denote $G_1' = G_1 - F$ and $G_2' = G_2 - F$. Since $|F| < 2\lambda$, at least one of G_1' and G_2' is connected. Without loss of generality, assume G_2' is connected. If G_1' is also connected, then G_1' is connected to G_2' by one cross edge because $|F| < n$. Hence, $G - F$ is connected.

In the following we suppose that G_1' is not connected. For any vertex v_i of G_1', denote the crossedge incident with it by e. If $e \notin F$, then v_i is connected to G_2'. If $e \in F$, then v_i is adjacent to other vertices of G_1' as F is a super edge-cut. Denote the component of G_1' containing v_i by G_{11} and $X = V(G_{11})$. Let $t = |X|$. Clearly, $t \ge 2$. If some vertex of G_{11} is connected to G_2' in $G - F$, then the vertex v_i is also connected to G_2'. Otherwise, $|F| \ge d_{G_1}(X) + |X| \ge \lambda + t$. Because $|F| < 2\lambda$, we have $t < 2\lambda - \lambda = \lambda \le \delta$. That is to say, $t \le \delta - 1$ and $\delta \ge 3$. Every vertex in X has at least $\delta - (t-1)$ neighbors in $V_1 \setminus X$, so

$$|F| \ge d_{G_1}(X) + t \ge (\delta - (t-1))t + t = -t^2 + (\delta + 2)t.$$

Define a function $f(t) = -t^2 + (\delta + 2)t$. It is easy to see the function $f(t)$ reaches the minimum value at an end-point of the interval $[2, \delta - 1]$. Because $f(2) = 2\delta$ and $f(\delta - 1) = 3\delta - 3 = 2\delta + (\delta - 3) \ge 2\delta$, we obtain a contradiction that $|F| \ge 2\delta \ge 2\lambda$. Therefore, the equality in (2) follows. ∎

Corollary 3 (Li and Xu [7]). *Let G_0 be a connected graph of order n (≥ 2). Then*

$$\lambda'(K_2 \times G_0) = \min\{n, 2\delta(G_0), 2\lambda'(G_0)\}.$$

Proof. If G_0 is super edge-connected, $\lambda(G_0) = \delta(G_0) < \lambda'(G_0)$. If G_0 is not super edge-connected, then $\lambda(G_0) = \lambda'(G_0)$. Noting that $\lambda(G_0) \le \delta(G_0)$, we have $\min\{\delta(G_0), \lambda'(G_0)\} = \lambda(G_0)$. Thus, $\min\{\delta(G_0), \lambda'(G_0)\} = \lambda(G_0)$ for any connected graph G_0, and thus the corollary holds by Theorem 3. ∎

Corollary 4 (Esfahanian [2]). *Let Q_n be an n-dimensional cube. Then $\lambda'(Q_n) = 2n - 2$ and, thus, Q_n is λ'-optimal for $n \ge 2$, and is super-λ for $n \ge 3$.*

Proof. Because $Q_n = K_2 \times Q_{n-1}$ (see Section 3.1 in [12]), by Theorem 3, $\lambda'(Q_n) = \min\{|V(Q_{n-1})|, 2\lambda(Q_{n-1})\} = \min\{2^{n-1}, 2n-2\} = 2n - 2 = \xi(Q_n)$ and thus Q_n is λ'-optimal for $n \ge 2$. In addition, $\lambda'(Q_n) = 2n - 2 > \lambda(Q_n) = n$ for $n \ge 3$, so Q_n is super-λ. ∎

Combining Theorem 1 with Theorem 3, we obtain the main result in [1, 10].

Corollary 5. *Assume $G_1 \times G_2 \ncong K_2 \times K_n$ for $n \ge 2$. If G_i are regular and max-λ for $i = 1, 2$, then $G_1 \times G_2$ is super-λ.*

Proof. Note that $\lambda(G_1 \times G_2) \le \delta(G_1 \times G_2) = d_1 + d_2$, where d_i is the regular degree of G_i for $i = 1, 2$. Let $m = |V(G_1)|, n = |V(G_2)|$ and $\lambda_i = \lambda(G_i)$ for $i = 1, 2$.

When $m \ge 3$ and $n \ge 3$, we have $d_i > 1$ for $i = 1, 2$ and

$$m\lambda_2 \ge (d_1 + 1)\lambda_2 = (d_1 + 1)d_2 = d_1d_2 + d_2 > d_1 + d_2$$
$$n\lambda_1 \ge (d_2 + 1)\lambda_1 = (d_2 + 1)d_1 = d_1d_2 + d_1 > d_1 + d_2$$
$$\lambda_1 + 2\lambda_2 = d_1 + 2d_2 > d_1 + d_2$$
$$2\lambda_1 + \lambda_2 = 2d_1 + d_2 > d_1 + d_2$$

When $m = 2$, we have $d_1 = 1$ and $1 < d_2 < n - 1$ (because $G_1 \times G_2 \ncong K_2 \times K_n$), and

$$n > 1 + d_2 = d_1 + d_2$$
$$2\lambda_2 = 2d_2 > d_1 + d_2.$$

By Theorem 1 and Theorem 3, the corollary holds. ∎

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