On Super Edge-Connectivity of Cartesian Product Graphs

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The super edge-connectivity $\lambda'$ of a connected graph $G$ is the minimum cardinality of an edge-cut $F$ in $G$ such that every component of $G - F$ contains at least two vertices. Let $G_i$ be a connected graph with order $n_i$, minimum degree $\delta_i$ and edge-connectivity $\lambda_i$ for $i = 1, 2$. This article shows that $\lambda'(G_1 \times G_2) \geq \min\{n_1 \delta_2, n_2 \delta_1\} + 2\lambda_1 + 2\lambda_2$ for $n_1, n_2 \geq 3$ and $\lambda'(K_2 \times G_2) = \min\{n_2, 2\lambda_2\}$, which generalizes the main result of Shieh on the super edge-connectedness of the Cartesian product of two regular graphs with maximum edge-connectivity. In particular, this article determines $\lambda'(G_1 \times G_2) = \min\{n_1 \delta_2, n_2 \delta_1\}$ if $\lambda'(G_1) = \xi(G_1)$, where $\xi(G)$ is the minimum edge-degree of a graph $G$. © 2006 Wiley Periodicals, Inc. NETWORKS, Vol. 49(2), 152–157 2007

Keywords: super edge-connectivity; Cartesian product; super-$\lambda'$-graph; $\lambda'$-optimal graph

1. INTRODUCTION

Throughout this article, a graph $G = (V, E)$ always means a finite undirected graph without self-loops or multiple edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For any edge $uv \in E$, the parameter $\xi(G)(uv) = d_G(u) + d_G(v) - 2$ is the degree of the edge $uv$ and the parameter $\xi(G) = \min\{\xi(G)(uv) | uv \in E\}$ is the minimum edge-degree of $G$. The symbols $K_{1,n-1}$ and $K_n$ denote a star graph and a complete graph with $n$ vertices, respectively. For the graph theoretical terminology and notation not defined here, we refer the reader to [13].

It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where $V$ is the set of processors and $E$ is the set of communication links in the network, the edge-connectivity $\lambda(G)$ of $G$ is an important measurement for the fault tolerance of the network. In general, the larger $\lambda(G)$ is, the more reliable the network is. It is well known that $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$. A connected graph $G$ is said to be maximally edge-connected (in short, max-$\lambda$) if $\lambda(G) = \delta(G)$. Obviously, the set of edges incident with a vertex of degree $\delta(G)$ is certainly a minimum edge-cut and isolates a vertex when $G$ is max-$\lambda$.

A graph $G$ is said to be super edge-connected (in short, super-$\lambda'$) if $G$ is max-$\lambda$ and every minimum edge-cut isolates a vertex of $G$.

It has been shown that a super-$\lambda'$ network is the most reliable and has the smallest edge failure rate (see, e.g., [17, 18]). Several sufficient conditions for a graph to be max-$\lambda'$ or super-$\lambda'$ have been given in the literature (see, e.g., [6]).

A quite natural problem is that if a connected graph $G$ is super-$\lambda'$, then how many edges have to be removed to disconnect $G$ such that every component of the resulting graph contains no isolated vertices. This problem results in the concept of the super edge-connectivity, introduced first by Fiol et al. in [4].

An edge-cut $F$ is called a super edge-cut of $G$ if $G - F$ contains no isolated vertices. In general, super edge-cuts do not always exist. The super edge-connectivity $\lambda'(G)$ is the minimum cardinality of a super edge-cut in $G$ if super edge-cuts exist, and, by convention, is $+\infty$ otherwise.

The new parameter $\lambda'$ in conjunction with $\lambda$ can provide more accurate measures for the fault tolerance of a large-scale parallel processing system and, thus, has received the attention of many researchers in recent years (see, e.g., [3–9, 11, 14–16]). Esfahanian and Hakimi [3] showed that if $G$ is neither $K_{1,n-1}$ nor $K_3$, then

$$\lambda(G) \leq \lambda'(G) \leq \xi(G).$$

A connected graph $G$ is called a $\lambda'$-graph if $G$ is neither $K_{1,n-1}$ nor $K_3$. It is easy to see that if $\lambda'(G) > \lambda(G)$ then $G$
is super-λ. A super-λ graph G is said to be optimally super edge-connected (in short, λ'-optimal) if λ'(G) = \(\xi(G)\).

Recently, Chiue and Shieh [1] have given some sufficient conditions for the Cartesian product \(G_1 \times G_2\) to be super-λ; Shieh [10] has proved that \(G_1 \times G_2\) is super-λ if both \(G_1\) and \(G_2\) are regular and max-λ except for \(K_2 \times K_n\), where \(n \geq 2\). Ueffing and Volkmann [11] have investigated the \(\lambda'\)-optimality of \(G_1 \times G_2\) when both \(G_1\) and \(G_2\) are \(\lambda'\)-optimal. Li and Xu [7] have determined \(\lambda'(K_2 \times G) = \min(n, 2\delta(G), 2\lambda'(G))\) for any connected graph G with \(n\) vertices.

Let \(G_1\) be a connected graph of order \(n_1\), minimum degree \(\delta_1\) and edge-connectivity \(\lambda_1\) for \(i = 1, 2\). In this article, we show that \(\lambda'(G_1 \times G_2) \geq \min[n_1 \lambda_2, n_2 \lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2]\) for \(n_1, n_2 \geq 3\) by refining the technique of Chiue and Shieh in [1] and determine that \(\lambda'(K_2 \times G_2) = \min(n_2, 2\lambda_2)\), which generalizes the result of Shieh [10]. In particular, similar to the proof of Theorem 4.1 in [11], we determine that \(\lambda'(G_1 \times G_2) = \min[n_1 \delta_2, n_2 \delta_1, \xi(G_1 \times G_2)]\) if both \(G_1\) and \(G_2\) are \(\lambda'\)-optimal.

The proofs of these results are given in Section 3.

2. PRELIMINARIES

Let \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\). The union of two graphs (not necessarily disjoint) \(G_1\) and \(G_2\), denoted by \(G_1 \cup G_2\), is the graph with the vertex-set \(V(G_1 \cup G_2) = V_1 \cup V_2\) and the edge-set \(E(G_1 \cup G_2) = E_1 \cup E_2\). The Cartesian product of \(G_1\) and \(G_2\), denoted by \(G_1 \times G_2\), is the graph with the vertex-set \(V_1 \times V_2\) such that two vertices \((x_1, y_1)\) and \((x_2, y_2)\) are adjacent if and only if either \(x_1 = x_2\) or \(y_1 = y_2\) with \(x_1 \neq x_2\) or \(y_1 \neq y_2\) in \(V_1\).

By the definition of the Cartesian product \(G = G_1 \times G_2\), for any vertex \(x, y \in V(G)\),

\[d_G(x, y) = d_{G_1}(x) + d_{G_2}(y),\]

and if \(x_1 \neq x_2\) or \(y_1 \neq y_2\) in \(E_2\), then

\[\xi_G((x_1, y_1)(x_2, y_2)) = \xi_{G_1}(x_1, x_2) + 2d_{G_2}(y_1),\]

\[\xi_G((x_1, y_1)(y_1, x_2)) = \xi_{G_2}(y_1, y_2) + 2d_{G_1}(x_1),\]

respectively, and consequently,

\[\xi(G) = \min[\xi(G_1) + 2\delta(G_2), \xi(G_2) + 2\delta(G_1)].\]

For convenience, we define two kinds of subgraphs \(G_{1y}\) and \(G_{2x}\) of \(G_1 \times G_2\) as follows.

\[V(G_{1y}) = \{(x, y) | x \in V_1\}\] and

\[E(G_{1y}) = \{(x_1, y)(x_2, y) | x_1 \neq x_2 \in V_1\}\]

for any \(y \in V_2\);

\[V(G_{2x}) = \{(x, y) | y \in V_2\}\]

and

\[E(G_{2x}) = \{(x, y)(x, y_2) | y_1 \neq y_2 \in V_2\}\]

for any \(x \in V_1\).

It is clear that \(G_{1y}\) is isomorphic to \(G_1\) for any \(y \in V_2\) and \(G_{2x}\) is isomorphic to \(G_2\) for any \(x \in V_1\). Let \(V_{1y} = V(G_{1y}), E_{1y} = E(G_{1y}), V_{2x} = V(G_{2x}), E_{2x} = E(G_{2x}).\)

Then

\[E_{1y} \cap E_{1y'} = \emptyset, \quad \text{for any } y, y' \in V_2, y \neq y';\]

\[E_{2x} \cap E_{2x'} = \emptyset, \quad \text{for any } x, x' \in V_1, x \neq x';\]

\[V_{1y} \cap V_{2x} = \{(x, y)\}, \quad E_{1y} \cap E_{2x} = \emptyset\]

for any \(x \in V_1, y \in V_2\);

\[E(G_1 \times G_2) = (\bigcup_{y \in V_2} E_{1y}) \cup (\bigcup_{x \in V_1} E_{2x}).\]

To check whether a union graph is connected or not, the following concept and results, due to Chiue and Shieh [1], are useful.

**Definition (Separability).** For \(G = G_1 \cup G_2 \cup \cdots \cup G_k\), \(V(G)\) is called separable if and only if \(V(G)\) can be partitioned into two disjoint nonempty sets \(A\) and \(A'\) such that \(A \cup A' = V(G)\) and each \(G_i\) is a subset of either \(A\) or \(A'\) for \(i = 1, 2, \ldots, k\).

**Lemma 1.** Suppose \(G = \bigcup_{i=1}^k G_i\), where \(G_i\) is connected for \(i = 1, 2, \ldots, k\). If \(V(G)\) is nonseparable, then \(G\) is connected.

**Remark 1.** Because \(V_{1y} \cap V_{2x} = \{(x, y)\}\), \(V_{1y} \cup V_{2x}\) is nonseparable for any \(x \in V_1\) and \(y \in V_2\).

3. MAIN RESULTS

We first introduce some notation used in this section. Let \(G = (V, E)\) be a graph. For two disjoint nonempty subsets \(X\) and \(Y\) of \(V\), denote \((X, Y)_G = \{xy \in E | x \in X, y \in Y\}\). If \(Y = V \setminus X\), then we write \(E_G(X) = (X, Y)_G\) and \(d_G(X) = |E_G(X)|\).

A super edge-cut \(F\) of \(G\) is called a \(\lambda'\)-cut if \(|F| = \lambda'(G)\). It is clear that \(G - F\) has exactly two components for any \(\lambda'\)-cut \(F\). A nonempty and proper subset \(X\) of \(V\) is called a \(\lambda'\)-fragment of \(G\) if \(E(G)(X)\) is a \(\lambda'\)-cut of \(G\). The minimum \(\lambda'\)-fragment over all \(\lambda'\)-fragments of \(G\) is called a \(\lambda'\)-atom of \(G\).

For \(F \subseteq E(G_1 \times G_2)\), let

\[G_{1y} = G_{1y} - F\]

for any \(y \in V_2\),

\[G_{2x} = G_{2x} - F\]

for any \(x \in V_1\).

Then, it is clear that

\[V(G_{1y}) = V_{1y}, \quad V(G_{2x}) = V_{2x}\]

for any \(x \in V_1\) and \(y \in V_2\);

\[G_1 \times G_2 - F = (\bigcup_{y \in V_2} G_{1y'}) \cup (\bigcup_{x \in V_1} G_{2x'}).\]

Let \(C = \{x \in V_1 | G_{2x} \text{ is connected}\}\) and \(D = \{y \in V_2 | G_{1y} \text{ is connected}\}\).

Throughout this section, we always assume that \(G_1\) and \(G_2\) have \(m\) and \(n\) vertices, respectively, and \(\lambda(G_i) = \lambda_i \geq 1\) for \(i = 1, 2\). So \(\delta(G) \geq 1\) for \(i = 1, 2\), which implies \(m \geq 2\) and \(n \geq 2\).

**Lemma 2.** \(G = G_1 \times G_2\) is a \(\lambda'\)-graph if \(m \geq 2\) and \(n \geq 2\).
Proof. Because $m \geq 2$ and $n \geq 2$, the graph $G = G_1 \times G_2$ has $|V(G_1 \times G_2)| = mn \geq 4$ vertices, and thus $G$ is not $K_3$-free. Moreover, $\delta(G) = \delta(G_1) + \delta(G_2) \geq 2$, and thus $G$ is not a star. Therefore, $G$ is a $\lambda'$-graph.

**Theorem 1.** $\lambda'(G_1 \times G_2) \geq \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$ if $m \geq 3$ and $n \geq 3$.

Proof. Denote $\mu = \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$. By Lemma 2, $G_1 \times G_2$ is a $\lambda'$-graph, so its super edge-cuts always exist. Assume $F$ is a minimum super edge-cut with $|F| < \mu$. We need to show that $G_1 \times G_2 - F$ is connected. Because $|F| < \mu \leq m\lambda_2$, there exists some $x_0 \in V_1$ such that $G_{x_0}^{2\lambda_2}$ is connected. Because $|F| < \mu < n\lambda_1$, there exists some $y_0 \in V_2$ such that $G_{y_0}^{1\lambda_1}$ is connected. That is to say, $|C| \geq 1$ and $|D| \geq 1$. There are three cases to be considered for us.

**Case 1.** $|C| = 1$. This implies that the other $m-1$ subgraphs $G_{x}^{2\lambda_2}$ are disconnected, where $x \in V_1 \setminus \{x_0\}$. In this case, $G_{y}^{\lambda_1}$ is connected for any $y \in V_2$. Otherwise, because $m \geq 3$, we have $|F| \geq (m-1)\lambda_2 + \lambda_1 \geq 2\lambda_2 + \lambda_1 \geq \mu$, a contradiction. Thus, $|D| = n$. By Remark 1, $(U_{y \in D} G_{y}^{1\lambda_1}) \cup G_{y_0}^{1\lambda_1}$ is connected and, thus, $(U_{y \in D} G_{y}^{1\lambda_1}) \cup G_{x_0}^{2\lambda_2}$ is connected by Lemma 1 and so is $G_1 \times G_2 - F$.

**Case 2.** $|C| = m$. Because $(U_{y \in C} G_{y}^{2\lambda_2}) \cup V_{y_0} = V_1 \times V_2$ is not super edge-cuts, we have $(U_{y \in C} G_{y}^{2\lambda_2}) \cup G_{y_0}^{1\lambda_1}$ is connected by Lemma 1, which means $G_1 \times G_2 - F$ is connected.

**Remark 2.** The lower bound given above is tight. For example, let $G_1 = K_m$ with the vertex-set $\{x_1, x_2, \ldots, x_m\}$ and let $G_2 = K_{1,n-1}$ with the vertex-set $\{y_1, y_2, \ldots, y_{n-1}\}$, where $m \geq 3$ and $n \geq 3$. Then $\lambda_1 = m - 1$ and $\lambda_2 = 1$. By Theorem 1, $\lambda'(G_1 \times G_2) \geq \min\{m\lambda_2, n\lambda_1, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\} = \min\{m, m(n(m-1)), m+1, 2m-1\} = m$. In addition, if $y_1 y_2 \in E(G_2)$, the edge-set $\{(x_1, y), (x_1, y_2), (x_2, y_1), (x_2, y_2), \ldots, (x_m, y_1), (x_m, y_2)\}$ is an edge-cut that isolates no vertex of $K_m \times K_{1,n-1}$. So it is a super edge-cut, which implies $\lambda'(K_m \times K_{1,n-1}) \leq m$. Therefore, $\lambda'(K_m \times K_{1,n-1}) = m$. The lower bound is attained.

**Lemma 3** (Hellwig and Volkmann [5]). If $G$ is a $\lambda'$-optimal graph, then $\lambda(G) = \delta(G)$.

With the proof of Theorem 4.1 in [11], we obtain the super edge-connectivity of the Cartesian product of two $\lambda'$-optimal graphs.

**Theorem 2.** $\lambda'(G_1 \times G_2) = \min\{m\delta(G_2), n\delta(G_1), \xi(G_1 \times G_2)\}$ if $G_1$ and $G_2$ are both $\lambda'$-optimal.

Proof. Denote $\delta(G_1) = \delta_1, \lambda(G_1) = \lambda_1, \xi(G_1) = \xi_1, \lambda'(G_1) = \lambda'_1$ for $i = 1, 2$, and $G = G_1 \times G_2$. Because $G_i$ is $\lambda'$-optimal, $G_i$ is a $\lambda'$-graph for $i = 1, 2$, which implies $m \geq 4, n \geq 4$.

By Lemma 2, $\lambda'(G) = \min\{m\delta(G_1), n\delta(G_1), \xi(G_1 \times G_2)\}$ is well defined. First, we have $\lambda'(G) \leq \xi(G)$ by (1). Because $m \geq 4$, $E_G(V_1 \times \{y\})$ is a super edge-cut for a vertex $y \in V_2$ with $d_G(y) = 2\delta$. So $\lambda'(G) \leq m\delta_2$. Analogously, we have $\lambda'(G) \leq n\delta_1$. Thus,

$$\lambda'(G) \leq \min\{m\delta_2, n\delta_1, \xi(G)\}.$$
Suppose that vertex $(x, y)(y \notin D)$ is isolated in $G_{2\xi}$. Then it is not isolated in $G_{1\xi}$, otherwise, it is isolated in $G_1 \times G_2 - F$, contradicting our hypothesis that $F$ is a super edge-cut. So the vertex $(x, y)$ is contained in a component with at least two vertices of $G_{1\xi}$. We can show that vertex $(x, y)$ is connected to $(\cup_{y \in D} G_{1\xi}') \cup (\cup_{x \in E} G_{2\xi}')$ in the same way as above.

Because all possible cases lead to a contradiction, $\lambda'(G) = \min\{m\delta_2, n\delta_1, \xi(G)\}$ and the proof is complete.

From Theorem 2, we can easily obtain the following corollary.

**Corollary 1** (Ueffing and Volkmann [11]). Let $G_1$ and $G_2$ be two disjoint $\lambda'$-optimal graphs and let $G = G_1 \times G_2$. Then $G$ is $\lambda'$-optimal or the $\lambda'$-atoms of $G$ have the form $\{x\} \times V_2$ for a vertex $x \in V_1$ with $d_{G_1}(x) = \delta(G_1)$ or $V_1 \times \{y\}$ for a vertex $y \in V_2$ with $d_{G_2}(y) = \delta(G_2)$.

The $n$-dimensional toroidal mesh $C(d_1, d_2, \ldots, d_n)$ ([12]) can be expressed as the Cartesian product $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$, where $C_{d_i}$ is a cycle of length $d_i$ for $i = 1, 2, \ldots, n$.

**Corollary 2** (Xu and Xu [16]). Let $C(d_1, d_2, \ldots, d_n)$ be the $n$-dimensional toroidal mesh. Then $\lambda'(C(d_1, d_2, \ldots, d_n)) = 4n - 2$ and, thus, $C(d_1, d_2, \ldots, d_n)$ is $\lambda'$-optimal if $d_i \geq 4$ for each $i = 1, 2, \ldots, n$.

**Proof.** We prove the corollary by induction on $n$. It is easy to see that a cycle $C_{d_i}$ is $\lambda'$-optimal for $d_i \geq 4$. Now we assume $n \geq 2$. Suppose $G_{n-1} = C(d_1, d_2, \ldots, d_{n-1})$ is $\lambda'$-optimal, which implies $\lambda'(G_{n-1}) = 4(n-1) - 2 = 4n - 6 = \xi(G_{n-1})$. $|V(G_{n-1})| = d_1 + d_2 + \cdots + d_{n-1}$ and $\lambda(G_{n-1}) = 2(n-1)$. Denote $G = C(d_1, d_2, \ldots, d_n) = G_{n-1} \times C_{d_n}$. Noting $d_i \geq 4$ for each $i = 1, 2, \ldots, n$, by Theorem 2, we have

$$\lambda'(G) = \min\{|V(G_{n-1})|\delta(C_{d_n}), |V(C_{d_n})|\delta(G_{n-1}), \xi(G)\} = \min(2d_1 + d_2 + \cdots + d_{n-1}, 2d_n(n-1), 4n - 2) = 4n - 2 = \xi(G).$$

Thus, $G$ is $\lambda'$-optimal.

**Theorem 3.** Let $G_0$ be a connected graph with $n$ vertices and $\lambda(G_0) = \lambda$. Then $\lambda'(K_2 \times G_0) = \min\{n, 2\lambda\}$.

**Proof.** Let $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ and $G = K_2 \times G_0$. By the definition of the Cartesian product, $K_2 \times G_0$ is obtained from two copies of $G_0$ by connecting (via a new edge) vertex $v_i$ in one copy to the vertex $v_i$ in the other copy of $G_0$. $1 \leq i \leq n$. These new edges are called cross edges. Denote the two copies by $G_1$ and $G_2$, respectively, and let $V_1 = V(G_1)$, $V_2 = V(G_2)$.

If $|V(G_0)| \geq 2$, then $E_G(V_1)$ is a super edge-cut of $G$, and hence, $\lambda'(G) \leq |E_G(V_1)| = |V_1| = n$. Suppose $X_1 \subseteq V_1$ with $d_{G_1}(x_1) = \lambda$ and let $X_2 \subseteq V_2$ be the set of those vertices adjacent to $X_1$. It is easy to see $E_G(X_1 \cup X_2)$ is a super edge-cut.
of $G$, and hence, $\lambda'(G) \leq |E_G(X_1 \cup X_2)| = 2|E_G(X_1)| = 2\lambda$.

It follows that

$$\lambda'(G) \leq \min(n, 2\lambda). \quad (2)$$

We will show that the equality in (2) holds. Suppose to the contrary that $\lambda'(G) < \min(n, 2\lambda)$. We will show $G - F$ is connected for any $\lambda'$-cut $F$ of $G$.

Let $F$ be a $\lambda'$-cut of $G$. Denote $G'_1 = G - F$ and $G'_2 = G - F$. Since $|F| < 2\lambda$, at least one of $G'_1$ and $G'_2$ is connected. Without loss of generality, assume $G'_1$ is connected. If $G'_1$ is also connected, then $G'_1$ is connected to $G'_1$ by one cross edge because $|F| < n$. Hence, $G - F$ is connected.

In the following we suppose that $G'_1$ is not connected. For any vertex $v_i$ of $G'_1$, denote the cross edge incident with it by $e$. If $e \notin F$, then $v_i$ is connected to $G'_2$. If $e \in F$, then $v_i$ is adjacent to other vertices of $G'_1$ as $F$ is a super edge-cut. Denote the component of $G'_1$ containing $v_i$ by $G_{11}$ and $X = V(G_{11})$. Let $t = |X|$. Clearly, $t \geq 2$. If some vertex of $G_{11}$ is connected to $G'_2$ in $G - F$, then the vertex $v_i$ is also connected to $G'_2$. Otherwise, $|F| \geq d_{G_1}(X) + |X| \geq \lambda + t$. Because $|F| < 2\lambda$, we have $t < 2\lambda - \lambda = \lambda \leq \delta$. That is to say, $t \leq \delta - 1$ and $\delta \geq 3$. Every vertex in $X$ has at least $\delta - (t - 1)$ neighbors in $V_1 \setminus X$, so

$$|F| \geq d_{G_1}(X) + t \geq (\delta - (t - 1))t + t = -t^2 + (\delta + 2)t.$$ 

Define a function $f(t) = -t^2 + (\delta + 2)t$. It is easy to see the function $f(t)$ reaches the minimum value at an end-point of the interval $[2, \delta - 1]$. Because $f(2) = 2\delta$ and $f(\delta - 1) = 3\delta - 3 = 2\delta + (\delta - 3) \geq 2\delta$, we obtain a contradiction that $|F| \geq 2\delta \geq \lambda$. Therefore, the equality in (2) follows.

**Corollary 3** (Li and Xu [7]). Let $G_0$ be a connected graph of order $n \geq 2$. Then

$$\lambda'(K_2 \times G_0) = \min(n, 2\delta(G_0), 2\lambda'(G_0)).$$

**Proof.** If $G_0$ is super edge-connected, $\lambda(G_0) = \delta(G_0) < \lambda'(G_0)$. If $G_0$ is not super edge-connected, then $\lambda(G_0) = \lambda'(G_0)$. Noting that $\lambda(G_0) \leq \delta(G_0)$, we have $\min(\delta(G_0), \lambda'(G_0)) = \lambda'(G_0)$ for any connected graph $G_0$, and thus the corollary holds by Theorem 3.

**Corollary 4** (Esfahanian [2]). Let $Q_n$ be an $n$-dimensional cube. Then $\lambda'(Q_n) = 2n - 2$ and, thus, $Q_n$ is $\lambda'$-optimal for $n \geq 2$, and is super-$\lambda$ for $n \geq 3$.

**Proof.** Because $Q_n = K_2 \times Q_{n-1}$ (see Section 3.1 in [12]), by Theorem 3, $\lambda'(Q_n) = \min(|V(Q_{n-1})|, 2\lambda'(Q_{n-1})) = \min(2^{n-1}, 2n - 2) = 2n - 2 = \xi(Q_n)$ and thus $Q_n$ is $\lambda'$-optimal for $n \geq 2$. In addition, $\lambda'(Q_n) = 2n - 2 > \lambda'(Q_n) = n$ for $n \geq 3$, so $Q_n$ is super-$\lambda$.

Combining Theorem 1 with Theorem 3, we obtain the main result in [1, 10].

**Corollary 5.** Assume $G_1 \times G_2 \not\cong K_2 \times K_n$ for $n \geq 2$. If $G_i$ are regular and max-$\lambda$ for $i = 1, 2$, then $G_1 \times G_2$ is super-$\lambda$.

**Proof.** Note that $\lambda(G_1 \times G_2) \leq \delta(G_1 \times G_2) = d_1 + d_2$, where $d_i$ is the regular degree of $G_i$ for $i = 1, 2$. Let $m = |V(G_1)|, n = |V(G_2)|$ and $\lambda_i = \lambda(G_i)$ for $i = 1, 2$.

When $m \geq 3$ and $n \geq 3$, we have $d_i > 1$ for $i = 1, 2$ and

$$m\lambda_2 \geq (d_1 + 1)\lambda_2 = (d_1 + 1)d_2 = d_1d_2 + d_2 > d_1 + d_2, \quad n\lambda_1 \geq (d_2 + 1)\lambda_1 = (d_2 + 1)d_1 = d_1d_2 + d_1 > d_1 + d_2.$$ 

$$\lambda_1 + 2\lambda_2 = d_1 + 2d_2 > d_1 + d_2, \quad 2\lambda_1 + \lambda_2 = 2d_1 + d_2 > d_1 + d_2.$$

When $m = 2$, we have $d_1 = 1$ and $1 < d_2 < n - 1$ (because $G_1 \times G_2 \not\cong K_2 \times K_n$), and

$$n > 1 + d_2 = d_1 + d_2, \quad 2\lambda_2 = 2d_2 > d_1 + d_2.$$ 

By Theorem 1 and Theorem 3, the corollary holds.

**Acknowledgment**

The authors thank the anonymous referees for their helpful comments and suggestions.

**REFERENCES**


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