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The forwarding indices of augmented cubes ☆

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Abstract

For a given connected graph G of order n, a routing R in G is a set of $n(n-1)$ elementary paths specified for every ordered pair of vertices in G. The vertex (resp. edge) forwarding index of G is the maximum number of paths in R passing through any vertex (resp. edge) in G. Choudum and Sunitha [S.A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2002) 71–84] proposed a variant of the hypercube Q_n, called the augmented cube AQ_n and presented a minimal routing algorithm. This paper determines the vertex and the edge forwarding indices of AQ_n as $2^n/9+(-1)^{n+1}/9+n2^n/3-2^n+1$ and 2^{n-1}, respectively, which shows that the above algorithm is optimal in view of maximizing the network capacity.

Keywords: Combinatorial problems; Augmented cubes; Routings; Forwarding index

1. Introduction

A routing R in a connected graph G of order n is a set of $n(n-1)$ elementary paths $R(u,v)$ specified for every (ordered) pair (u,v) of vertices of G. A routing R is said to be minimal if every path $R(u,v)$ in R is a shortest path from u to v in G. To measure the efficiency of a routing deterministically, Chung et al. [3] and Heydemann et al. [13] introduced the concept of the vertex forwarding index and the edge forwarding index of a routing, respectively.

The load $\xi(G,R,x)$ of a vertex x (resp. the load $\pi(G,R,e)$ of an edge e) with respect to R is defined as the number of paths specified by R going through x (resp. e). The parameters

$$\xi(G,R)=\max_{v\in V(G)}\xi(G,R,v)\quad\text{and}$$
$$\pi(G,R)=\max_{e\in E(G)}\pi(G,R,e)$$

are defined as the vertex forwarding index and the edge forwarding index of G with respect to R, respectively; and the parameters

$$\xi(G)=\min_R \xi(G,R)\quad\text{and}\quad \pi(G)=\min_R \pi(G,R)$$

are defined as the vertex forwarding index and the edge forwarding index of G, respectively.

The original study of forwarding indices is motivated by the problem of maximizing network capacity, see [3]. Minimizing the forwarding indices of a routing will result in maximizing the network capacity. Thus, it becomes very significant to determine the vertex and

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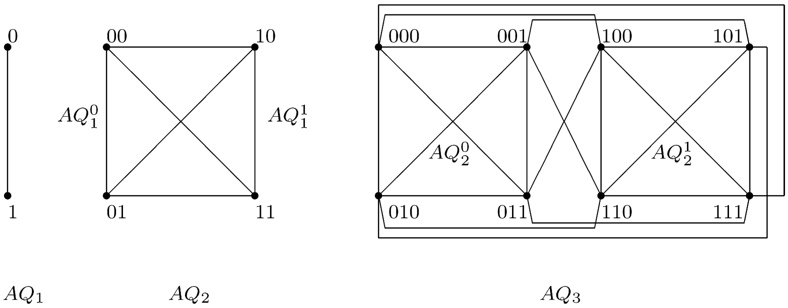

Fig. 1. Augmented cubes AQ_1, AQ_2 and AQ_3.

the edge forwarding indices of a given graph. However, Saad [21] found that for an arbitrary graph determining its vertex-forwarding index is NP-complete even if the diameter of the graph is two. Even so, a number of results have obtained and the forwarding indices of many well-known networks have been determined by several researchers, see, for example, [1,3–23,25–27].

In [2], Choudum and Sunitha proposed a new variant of the hypercube Q_n, called the augmented cube AQ_n, and found some properties not shared by the hypercube. In particular, they presented a minimal routing algorithm, by which they determined the diameter of AQ_n to be $\lceil n/2 \rceil$.

In this paper, we use Choudum and Sunitha's algorithm to determine $\xi(AQ_n) = 2^n/9 + (-1)^{n+1}/9 + n2^n/3 - 2^n + 1$ and $\pi(AQ_n) = 2^{n-1}$, which shows their algorithm is optimal in view of maximizing the network capacity.

The proofs of the results are in Section 4. In Section 2, we recall the definition and some properties of AQ_n. In Section 3, we show a minimal routing of AQ_n.

2. Definition and properties of augmented cubes

We follow the standard terminology of Xu [24]. As with hypercubes, there are many ways to describe the augmented cubes, one of which is follows.

Definition 1. The n-dimensional augmented cube AQ_n has 2^n vertices, each labeled by an n-bit binary string $a_1a_2\dots a_n$. We define $AQ_1 = K_2$. For $n \geqslant 2$, AQ_n is obtained by taking two copies of the $(n-1)$-dimensional augmented cube AQ_{n-1}, denoted by AQ_{n-1}^0 and AQ_{n-1}^1, and adding $2 \times 2^{n-1}$ edges between the two as follows:

Let $V(AQ_{n-1}^0) = \{0a_2a_3\dots a_n\colon\ a_i = 0 \text{ or } 1\}$ and $V(AQ_{n-1}^1) = \{1b_2b_3\dots b_n\colon\ b_i = 0 \text{ or } 1\}$. A vertex $A = 0a_2a_3\dots a_n$ of AQ_{n-1}^0 is joined to a vertex $B = 1b_2b_3\dots b_n$ of AQ_{n-1}^1 if and only if for each $i = 2, 3, \dots, n$ either

(1) $a_i = b_i$; in this case, AB is called a hypercube edge, or
(2) $a_i = \bar{b}_i$, in this case, AB is called a complement edge.

Fig. 1 shows the augmented cubes AQ_1, AQ_2 and AQ_3.

We write this recursive construction of AQ_n symbolically as $AQ_n = AQ_{n-1}^0 \otimes AQ_{n-1}^1$. The edges between AQ_{n-1}^0 and AQ_{n-1}^1 are called cross edges. Furthermore, we use $AQ_{n-i}^{s_1s_2\dots s_i}$ to denote the subgraph of AQ_n induced by the vertex with prefix $s_1s_2\dots s_i$.

The following two lemmas give the desired property of AQ_n.

Lemma 2. [2] *For $n \geqslant 1$, the augmented cubes AQ_n are Cayley graphs where $AQ_n \cong Cay(Z_2^n, (10\dots000) \cup (01\dots000) \cup \dots \cup (00\dots001) \cup (00\dots011) \cup (00\dots111) \cup \dots \cup (11\dots111))$.*

Lemma 3. [2] *Let $AQ_n = AQ_{n-1}^0 \otimes AQ_{n-1}^1$, $X = x_1x_2\dots x_n$ and $Y = y_1y_2\dots y_n$ be two vertices in AQ_n.*

(1) *If $X, Y \in AQ_{n-1}^0$ (or AQ_{n-1}^1), then there exists a shortest (X, Y)-path in AQ_n with all its vertices in AQ_{n-1}^0 (respectively, AQ_{n-1}^1).*
(2) *Let $X \in AQ_{n-1}^0$ and $Y \in AQ_{n-1}^1$.*
    (a) *There exists a shortest (X, Y)-path T in AQ_n with all its vertices (except X) in AQ_{n-1}^1. Moreover, the second vertex of T (i.e., the neighbor of X in T) is either $1x_2x_3\dots x_n$ or $1\bar{x}_2\bar{x}_3\dots\bar{x}_n$ according to whether $d(x_2x_3\dots x_n, y_2y_3\dots y_n) \leqslant d(\bar{x}_2\bar{x}_3\cdots\bar{x}_n, y_2y_3\dots y_n)$ holds or not.*
    (b) *There exists a shortest (X, Y)-path T in AQ_n with all its vertices (except Y) in AQ_{n-1}^0. Moreover, the penultimate vertex of T (i.e., neighbor of Y in T) is either $0y_2y_3\dots y_n$ or $0\bar{y}_2\bar{y}_3\dots\bar{y}_n$.*
(3) $d(X, Y; AQ_n) \begin{cases} \leqslant d(X, \bar{Y}; AQ_n) & \text{if } x_1 = y_1, \\ \geqslant d(X, \bar{Y}; AQ_n) & \text{if } x_1 \neq y_1. \end{cases}$

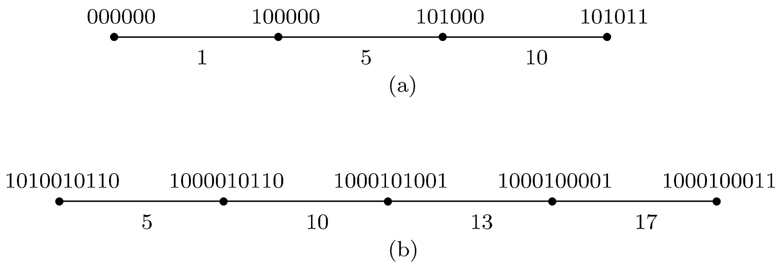

Fig. 2. Routing path (a) from 000000 to 101011 in AQ_6, and (b) from 1010010110 to 1000100011 in AQ_{10}.

3. Routing of AQ_n

In this section, we show a minimal routing which proposed by Choudum and Sunitha in [2].

We recall the logical OR operation $\oplus_2$ on $\{0,1\}$ which means $0 \oplus_2 0 = 0$, $0 \oplus_2 1 = 1 \oplus_2 0 = 1$ and $1 \oplus_2 1 = 1$.

A message from a vertex S (source) to another vertex D (destination) along this shortest (S, D)-path, any "current" vertex B performs three tasks:

(1) Compute its $tag(B \oplus_2 D) = (b_1 \oplus_2 d_1, b_2 \oplus_2 d_2, \cdots, b_n \oplus_2 d_n)$.
(2) Scans $tag(B \oplus_2 D)$ for the least i such that $b_i \oplus_2 d_i = 1$.
(3) (a) If $b_{i+1} \oplus_2 d_{i+1} = 0$, it changes the ith entry of B to d_i and routes the message to the next current vertex $B' = (d_1 d_2 \dots d_i b_{i+1} b_{i+2} \dots b_n)$ along the hypercube edge of weight $2i - 1$.
    (b) If $c_{i+1} \oplus_2 d_{i+1} = 1$, it changes the ith entry of B to d_i and routes the message to the next current vertex $B' = (d_1 d_2 \dots d_i \bar{b}_{i+1} \bar{b}_{i+2} \dots \bar{b}_n)$ along the complement edge of weight $2i$.

They also give two illustrations as shown in Fig. 2.

We use R_n to denote the routing of AQ_n defined above. By Lemma 3, we can verify that R_n is a minimum routing in AQ_n.

4. Main results

In this section, we will give the vertex and the edge forwarding indices of the augmented cube of AQ_n. The proofs of our results depend on the following lemma strongly, which is due to Heydemann et al. [13].

Lemma 4.

(1) *If $G = (V, E)$ is a Cayley graph of order n, then for any u we have,*

$$\xi(G) = \sum_{v \in V} d(u, v) - (n - 1).$$

(2) *Let $G = (V, E)$ be a simple connected graph of order n. Then*

$$\frac{1}{|E(G)|} \sum_{(u,v) \in V \times V} d(u, v) \leqslant \pi(G) \leqslant \pi_m(G).$$

The equalities hold if and only if there exists a minimal routing in G for which the load of all edges is the same.

Theorem 5. *The vertex forwarding index of AQ_n is*

$$\xi(AQ_n) = \frac{2^n}{9} + \frac{(-1)^{n+1}}{9} + \frac{n2^n}{3} - 2^n + 1.$$

Proof. By Lemma 4, in order to prove the theorem, we only need to compute the sum D_n of all distances from the fixed vertex $u = (00\dots00)$ to any other vertex v since AQ_n is a Cayley graph.

The distances between $u = (\overbrace{00\dots00}^{n})$ and the vertex $0v_2v_3\dots v_n$ in AQ^0_{n-1} is

$$d(u, v) = d(\overbrace{0\dots0}^{n-1}, v_2v_3\dots v_n).$$

Then the sum of all distances from vertex $u = (\overbrace{00\dots00}^{n})$ to the vertices in AQ^0_{n-1} is D_{n-1}. The vertex set of AQ^1_{n-1} can be partitioned into $V(AQ^{10}_{n-2})$ and $V(AQ^{11}_{n-2})$. The distance between $u = (\overbrace{00\dots00}^{n})$ and the vertex $10v_3\dots v_n$ in AQ^{10}_{n-2} is

$$\begin{aligned} d(u, v) &= d(u, 1\overbrace{0\dots0}^{n-1}) + d(\overbrace{0\dots0}^{n-2}, v_3\dots v_n) \\ &= 1 + d(\overbrace{0\dots0}^{n-2}, v_3\dots v_n). \end{aligned}$$

Then the sum of all distances from vertex $u = (\overbrace{00\dots00}^{n})$ to the vertices in AQ^{10}_{n-2} is $2^{n-2} + D_{n-2}$. The distance between $u = (\overbrace{00\cdots00}^{n})$ and the vertex $11v_3\dots v_n$ in AQ^{11}_{n-2} is

$$d(u,v) = d(u, 11\overbrace{1\ldots1}^{n-2}) + d(\overbrace{1\ldots1}^{n-2}, v_3\ldots v_n)$$
$$= 1 + d(\overbrace{1\ldots1}^{n-2}, v_3\ldots v_n).$$

Then the sum of all distances from vertex $u = (\overbrace{00\ldots00}^{n})$ to the vertices in AQ^{11}_{n-2} is $2^{n-2} + D_{n-2}$. So we have $D_n = D_{n-1} + 2 \times (2^{n-2} + D_{n-2})$. Since $D_1 = 1$, $D_2 = 3$, we have

$$D_n = \frac{2^n}{9} + \frac{(-1)^{n+1}}{9} + \frac{n2^n}{3}.$$

By Lemma 4, we have $\xi(AQ_n) = 2^n/9 + (-1)^{n+1}/9 + n2^n/3 - 2^n + 1$. The theorem follows. □

Theorem 6. *The edge forwarding index of AQ_n is $\pi(AQ_n) = 2^{n-1}$.*

Proof. We prove the theorem by induction. Since $\pi(AQ_2) = 2 = 2^{2-1}$, the theorem is true when $n = 2$. Assume that the theorem is true for every k with $2 \leqslant k < n$.

Let $AQ_n = AQ^0_{n-1} \otimes AQ^1_{n-1}$ and CR_n denote the paths between the $V(AQ^0_{n-1})$ and $V(AQ^1_{n-1})$ in R_n. Then $|CR_n| = 2 \times 2^{n-1} \times 2^{n-1}$. Since there are 2^n cross edges between AQ^0_{n-1} and AQ^1_{n-1} and every path in CR_n uses at least one cross edge, we have $\pi(AQ_n) \geqslant |CR_n|/2^n = 2^{n-1}$.

On the other hand, since $\pi(AQ_n) \leqslant \pi(AQ_n, R_n)$ clearly, we only need to show that $\pi(AQ_n, R_n) \leqslant 2^{n-1}$.

Let $e = (u_1u_2\ldots u_n, v_1v_2\ldots v_n)$ be any edge in AQ_n. If e is a cross edge then, by the definition of R_n, the path $R_n(x, y)$ passes through the edge e if and only if $x_i = u_i$ for $1 \leqslant i \leqslant n$, $y_j = v_j$ for $1 \leqslant j \leqslant 2$ or $x_i = v_i$ for $1 \leqslant i \leqslant n$, $y_j = u_j$ for $1 \leqslant j \leqslant 2$. Because each path passes through the edge e only once, we have $\pi(AQ_n, R_n, e) = 2^{n-1}$.

We now assume that the edge e is in AQ^0_{n-1} or AQ^1_{n-1}. If the path $R_{n-1}(x, y)$ passes through the edge $(u_2\ldots u_n, v_2\ldots v_n)$, then the path $R_n(u_1x, u_1y)$ must pass through the edge e. When $u_2 = v_2$, if the path $R_{n-2}(x, y)$ passes through the edge $(u_3\ldots u_n, v_3\ldots v_n)$, then the path $R_n(\bar{u}_1u_2x, u_1u_2y)$ and the path $R_n(\bar{u}_1\bar{u}_2\bar{x}, u_1u_2y)$ passes through the edge e. And there are all the path which pass through the edge e. Then by induction hypothesis, we have

$$\pi(AQ_n, R_n, e) \leqslant \pi(AQ_{n-1}, R_{n-1}) + 2\pi(AQ_{n-2}, R_{n-2})$$
$$= 2^{n-2} + 2 \times 2^{n-3} = 2^{n-1}.$$

So, we have $\pi(AQ_n, R) \leqslant 2^{n-1}$. Based on the above discussion, we get the result. □

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