Available online at www.sciencedirect.com
ScienceDirect
Information Sciences 177 (2007) 1782–1788
INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL
www.elsevier.com/locate/ins

On reliability of the folded hypercubes ☆

Qiang Zhu [a], Jun-Ming Xu [b,*], Xinmin Hou [b], Min Xu [c]

[a] *Department of Mathematics, XiDian University, Xi'an, Shanxi 710071, China*
[b] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China*
[c] *Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China*

Received 1 November 2005; received in revised form 15 November 2006; accepted 16 November 2006

Abstract

In this paper, we explore the 2-extra connectivity and 2-extra-edge-connectivity of the folded hypercube FQ_n. We show that $\kappa_2(FQ_n) = 3n - 2$ for $n \geqslant 8$; and $\lambda_2(FQ_n) = 3n - 1$ for $n \geqslant 5$. That is, for $n \geqslant 8$ (resp. $n \geqslant 5$), at least $3n - 2$ vertices (resp. $3n - 1$ edges) of FQ_n are removed to get a disconnected graph that contains no isolated vertices (resp. edges). When the folded hypercube is used to model the topological structure of a large-scale parallel processing system, these results can provide more accurate measurements for reliability and fault tolerance of the system.

Keywords: Folded hypercube; Connectivity; Edge-connectivity; Extra connectivity

1. Introduction

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are the two important features determining reliability and fault tolerance of the network [1,2,8,19]. These two parameters, however, have an obvious deficiency, that is, they tacitly assume that either all vertices adjacent to or all edges incident with the same vertex of G can potentially fail at the same time, which happens almost impossible in the practical applications of networks. In other words, in the definitions of κ and λ, absolutely no restrictions are imposed on the components of $G - S$. Consequently, these two measurements are inaccurate for large-scale processing systems in which all processors adjacent to or all links incident with the same processor cannot fail at the same time. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G - S$. Harary [9] first considered this problem by introducing the concept of the conditional connectivity.

☆ The work was supported by NNSF of China (Nos. 10671191, 10626053) and China Postdoctoral Science Foundation.
* Corresponding author.
E-mail address: xujm@ustc.edu.cn (J.-M. Xu).

0020-0255/$ - see front matter
doi:10.1016/j.ins.2006.11.003

Given a graph G and a graph-theoretical property $\mathscr{P}$, he defined the conditional connectivity $\kappa(G;\mathscr{P})$ (resp. edge-connectivity $\lambda(G;\mathscr{P})$) as the minimum cardinality of a set of vertices (resp. edges), if any, whose deletion disconnects G and every remaining component has property $\mathscr{P}$. Clearly, $\kappa(G)$ (resp. $\lambda(G)$) is a special case of $\kappa(G;\mathscr{P})$ (resp. $\lambda(G;\mathscr{P})$) when no condition is restricted to $\mathscr{P}$. The existence and value of $\kappa(G;\mathscr{P})$ ($\lambda(G;\mathscr{P})$) vary depending on the different choice of the property $\mathscr{P}$.

Fàbrega and Fiol [7] considered $\kappa(G;\mathscr{P}_h)$ (resp. $\lambda(G;\mathscr{P}_h)$) for a non-negative integer h and a graph G, where $\mathscr{P}_h$ is the property of having more than h vertices. They called this type of connectivity as the h-extraconnectivity (resp. h-edge-extraconnectivity) of G, denoted by $\kappa_h(G)$ (resp. $\lambda_h(G)$). In other words, $\kappa_h(G)$ (resp. $\lambda_h(G)$) is the minimum cardinality of a set of vertices (resp. edges) of G, if any, whose deletion disconnects G and every remaining component has more then h vertices.

Clearly, $\kappa_0(G) = \kappa(G)$ and $\lambda_0(G) = \lambda(G)$ for any graph G if G is not a complete graph. Thus, the h-extra connectivity is a generalization of the classical connectivity and can provide more accurate measures for the reliability and the tolerance of a large-scale parallel processing system, and so has received much research attention (see, for example, [5,6,10–12,16,21–23,26]) for $h = 1$ in recent years. However, a few results for $h \geqslant 2$ are known in the present literature, for example, [28].

The well-known n-dimensional hypercube is a graph $Q_n = (V,E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is a link between two vertices whenever their binary string representation differ in only one bit position. As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [4], is a graph obtained from the hypercube Q_n by adding 2^{n-1} edges, called complementary edges, each of them is between vertices. $x = (x_1, x_2, \ldots, x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, where $\bar{x}_i = 1 - x_i$.

The graphs shown in Fig. 1 are the folded hypercubes FQ_3 and FQ_4.

It has been shown that FQ_n is $(n+1)$-regular $(n+1)$-connected. Moreover, like the Q_n, FQ_n is a Cayley graph and so FQ_n is vertex-transitive. FQ_n is also superior to Q_n in some properties. For example, it has diameter $\lceil \frac{n}{2} \rceil$, about a half of the diameter of Q_n [4]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n. In particular, there are $n+1$ internally disjoint paths of length at most $\lceil \frac{n}{2} \rceil + 1$ between any

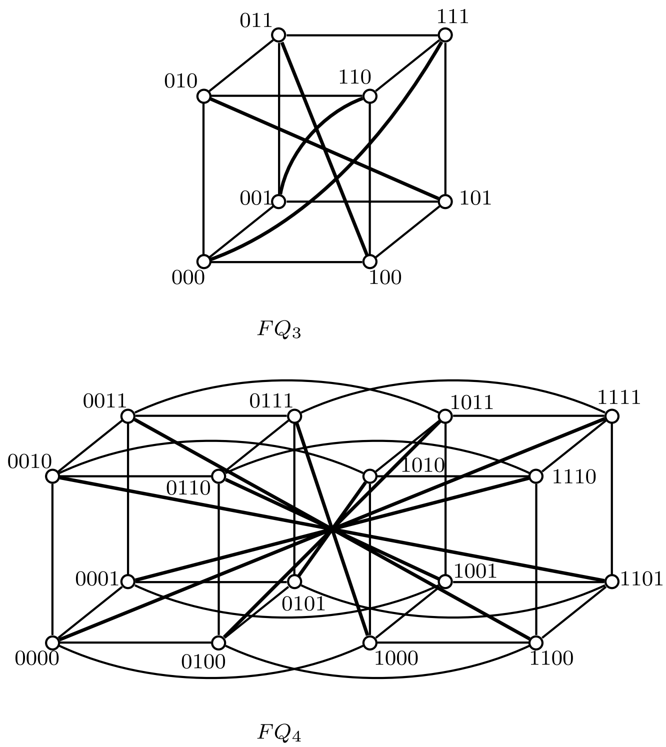

Fig. 1. The folded hypercubes FQ_3 and FQ_4, where the heavy edges are complementary edges.

pair of vertices in FQ_n, the deletion of less than $\lceil \frac{n}{2} \rceil - 2$ vertices or edges does not increase the diameter of FQ_n, and the deletion of up to n vertices or edges increases the diameter by at most one [15,17]. These properties mean that interconnection networks modelled by FQ_n are extremely robust. As a result, the study of the folded hypercube has recently attracted the attention of many researchers [3,13,14,18,24,25].

In [28], we determined $\kappa_1(FQ_n) = 2n$ for $n \geqslant 4$. Since FQ_n is vertex-transitive and contains no triangles, by Theorem 6 in [26] we immediately have $\lambda_1(FQ_n) = 2n$ for $n \geqslant 2$. Also in [28], we determined that the 2-extra-edge-connectivity of hypercubes, twisted cubes, crossed cubes and Möbius cubes are all $3n - 4$ when for $n \geqslant 4$. In this paper, we prove $\kappa_2(FQ_n) = 3n - 2$ for $n \geqslant 8$ and $\lambda_2(FQ_n) = 3n - 1$ for $n \geqslant 5$.

The rest of this paper is organized as follows. In Section 2, we give some notations and lemmas which will be frequently used in the proofs of our main results in Section 3.

2. Preliminaries

For all the terminology and notation not defined here, we follow [20]. For a graph $G = (V, E)$ and $S \subset V(G)$ or $S \subset G$, we use $N_G(S)$ (resp. $E_G(S)$) to denote the set of neighbors (resp. edges) of S in $G - S$. We use $g(G)$ to denote the girth of G, the minimum length of all cycles in G.

By the definition of the folded hypercube, it is easy to see that any $(n+1)$-dimensional folded hypercube FQ_{n+1} can be viewed as $G(Q_n^0, Q_n^1; M_0 + \overline{M})$, where Q_n^0 and Q_n^1 are two n-dimensional hypercubes with the prefix 1 and 0 of each vertex, respectively, and $M_0 = \{0u1u |\ 0u \in V(Q_n^0) \text{ and } 1u \in V(Q_n^1)\}$, $\overline{M} = \{0u1\overline{u} |\ 0u \in V(Q_n^0) \text{ and } 1\overline{u} \in V(Q_n^1)\}$.

For an n-bit binary string u, we use u_i to denote the binary string which differs in the ith bit position with u. Similarly, we use u_{ij} to denote the n-bit binary string which differs in the jth position with u_i. Clearly, $u_{ii} = u$. We use $\overline{u}$ to denote the n-bit binary string which differs with u in every bit position. We use $\overline{e}(u) \in \overline{M}$ to denote the edge in $\overline{M}$ incident with u, and $e_i(u)$ to denote the edge (u, u_i) for $i \in \{1, 2, \ldots, n\}$.

In the following discussion, we use $0u$ to denote a vertex of FQ_{n+1}, which means that $0u \in V(Q_n^0)$. Similarly, we write $1u$, which means that $1u \in V(Q_n^1)$. Moreover, for the sake of convenience, we consider FQ_{n+1} rather than FQ_n.

Lemma 2.1. *Any two vertices in $V(FQ_{n+1})$ exactly have two common neighbors for $n \geqslant 3$ if they have.*

Proof. We prove that any two vertices in $FQ_{n+1} = G(Q_n^0, Q_n^1; M_0 + \overline{M})$ exactly have two common neighbors for $n \geqslant 3$, if they have, according to their location. It is known that any two vertices in Q_n exactly have two common neighbors if they have.

Case 1: Both vertices are located in $V(Q_n^0)$ (or $V(Q_n^1)$), say, without loss of generality, in $V(Q_n^0)$. We suppose that the two vertices are $0u$ and $0v$.

Suppose that $0u$ and $0v$ have two common neighbors in Q_n^0. By the definition of Q_n^0, $0u$ and $0v$ differ in exactly two bit positions. Then $\{1u, 1\overline{u}\} \cap \{1v, 1\overline{v}\} = \phi$ for $n \geqslant 3$. Thus $0u$ and $0v$ have no common neighbors in Q_n^1, and so $0u$ and $0v$ have exactly two common neighbors in FQ_{n+1}.

Suppose that $0u$ and $0v$ have no common neighbors in Q_n^0 below. If $u = \overline{v}$, then $\{1u, 1\overline{u}\} = \{1v, 1\overline{v}\}$. So $0u$ and $0v$ exactly have two common neighbors in FQ_{n+1}. If $u \neq \overline{v}$, then $\{1u, 1\overline{u}\} \cap \{1v, 1\overline{v}\} = \phi$. So $0u$ and $0v$ have no common neighbors in FQ_{n+1}.

Case 2: One of the two vertices is in $V(Q_n^0)$, and the other is in $V(Q_n^1)$. Without loss of generality, we suppose that $0u \in V(Q_n^0)$ and $1v \in V(Q_n^1)$.

If there exists an i such that $v \in \{u_i, \overline{u_i}\}$, then $|N_{FQ_{n+1}}(0u) \cap N_{FQ_{n+1}}(1v)| = |\{0u_1, 0u_2, \ldots, 0u_n, 1u, 1\overline{u}\} \cap \{1v_1, 1v_2, \ldots, 1v_n, 0v, 0\overline{v}\}| = 2$. So $0u$ and $1v$ exactly have two common neighbors in FQ_{n+1}.

If $v \notin \{u_i, \overline{u_i}\}$ for any $i \in \{1, 2, \ldots, n\}$, then $u \notin \{v_i, \overline{v_i} | 1 \leqslant i \leqslant n\}$. Thus $|N_{FQ_{n+1}}(0u) \cap N_{FQ_{n+1}}(1v)| = |\{0u_1, 0u_2, \ldots 0u_n, 1u, 1\overline{u}\} \cap \{1v_1, 1v_2, \ldots 1v_n, 0v, 0\overline{v}\}| = 0$, which implies that $0u$ and $1v$ have no common neighbors in FQ_{n+1}. □

Lemma 2.2. $g(FQ_{n+1}) = 4$ *for* $n \geqslant 2$.

Proof. Since FQ_{n+1} can be viewed as $G(Q_n^1, Q_n^2; M_0 + \overline{M})$ and $g(Q_n) = 4$ for $n \geqslant 2$, we only need to prove that any edge in M_0 or $\overline{M}$ is not contained in a triangle.

Let $e_0 = (0u, 1u)$ be an edge in M_0. Then e_0 is not contained in a triangle since $1u$ is adjacent to neither $1\overline{u}$ for $n \geqslant 2$ nor $0u_i$ for $i \in \{1, 2, \ldots, n\}$. Similarly any edge $\overline{e} = (0u, 1\overline{u})$ is not contained in a triangle. $\square$

Lemma 2.3. *Let* $F \subset V(FQ_{n+1})$, $F_0 = F \cap V(Q_n^0)$ *and* $F_1 = F \cap V(Q_n^1)$. *If* $|F| \leqslant 3n$ *and there are neither isolated vertices nor isolated edges in* $FQ_{n+1} - F$, *then every vertex in* $Q_n^0 - F_0$ *is connected to a vertex in* $Q_n^1 - F_1$.

Proof. Suppose that $0u$ is a vertex in $Q_n^0 - F_0$. If $1u \notin F$ or $1\overline{u} \notin F$, then we are done. So suppose that both $1u$ and $1\overline{u}$ are in F below. For some $i \in \{1, 2, \ldots, n\}$, if $0u_i \notin F$ and $1u_i \notin F$, then we are done. So we suppose for each i, at least one of $0u_i$ and $1u_i$ belongs to F. Let $A = \{0u_i, 1u_i | i \in \{1, 2, \ldots, n\}\} \cap F$. Then $|A| \geqslant n$. Since there are no isolated vertices in $FQ_{n+1} - F$, there exists some j such that $0u_j \notin F$. If $1\overline{u}_j \notin F$, then we are done. So assume $1\overline{u}_j \in F$. For each $i \in \{1, 2, \ldots, n\}$ and $i \neq j$, if $0u_{ji} \notin F$ and $1u_{ji} \notin F$, then we are done. So we suppose that for each j, at least one of $0u_{ji}$ and $1u_{ji}$ belongs to F. Let $B = \{0u_{ji}, 1u_{ji} | i \in \{1, 2, \ldots, n\}$ and $i \neq j\} \cap F$. Then $|B| \geqslant n - 1$. Since there are no isolated edges in $FQ_{n+1} - F$, then $N_{FQ_{n+1}}(0u, 0u_j) - F = N_{Q_n^0}(0u, 0u_j) - F \neq \phi$. Let $0v$ be a vertex in $N_{Q_n^0}(0u, 0u_j) - F$. Then $0v$ is adjacent to $0u$ or $0u_j$. Without loss of generality, we suppose $0v$ is adjacent to $0u$. Assume $0v = 0u_k$ for some k. If $1\overline{u}_k \notin F$, then we are done. So we suppose $1\overline{u}_k \in F$. Let $C = \{0u_{ki}, 1u_{ki} | i \in \{1, 2, \ldots, n\},\ i \neq k, j\}$ and $D = \{1u, 1\overline{u}, 1\overline{u}_j, 1\overline{u}_k\}$. It is clear that any two sets in $\{A, B, C, D\}$ are disjoint to each other. Thus $|C \cap F| \leqslant |F - (A \cup B \cup D)| = |F| - |A| - |B| - |D| \leqslant n - 3$. Since there are $n - 2$ pairs of vertices in C, so there exists an $i \in \{1, 2, \ldots, n\}$ with $i \neq k, j$ such that $0u_{ki} \notin F$ and $1u_{ki} \notin F$. Thus $0u$ can be connected to a vertex in $Q_n^1 - F_1$, we are done. $\square$

Lemma 2.4. *Let* $F \subset E(FQ_{n+1})$, $F_0 = F \cap E(Q_n^0)$, $F_1 = F \cap E(Q_n^1)$, $F_{M_0} = F \cap M_0$ *and* $F_{\overline{M}} = F \cap \overline{M}$. *If* $F \leqslant 3n + 1$ *and there are neither isolated vertices nor isolated edges in* $FQ_{n+1} - F$, *then any vertex in* $Q_n^0 - F_0$ *(resp.* $Q_n^1 - F_1$*) is connected to a vertex in* $Q_n^1 - F_1$ *(resp.* $Q_n^0 - F_0$*).*

Proof. Without loss of generality we only need to prove that any vertex in $Q_n^1 - F_1$ is connected to a vertex in $Q_n^0 - F_0$.

Let $1u$ be any vertex in $Q_k^1 - F_1$. If $e_0(1u)$ or $\overline{e}(1u) \notin F$, then we are done. So assume $e_0(1u) \in F$ and $\overline{e}(1u) \in F$. We define $A = \{e_i(1u), e_0(1u_i) | i \in \{1, 2, \ldots, n\}\} \cap F$. If $|A| < n$, then there exists some i such that $e_i(1u) \notin F$ and $e_0(1u_i) \notin F$, and so we are done. So assume $|A| \geqslant n$. Since there are no isolated vertices in $G - F$, then there exists some $i' \in \{1, 2, \ldots, n\}$ such that $e_{i'}(1u) \notin F$. If $e_0(1u_{i'}) \notin F$ or $\overline{e}(1u_{i'}) \notin F$, then we are done. So assume $e_0(1u_{i'}) \in F$ and $\overline{e}(1u_{i'}) \in F$. Let $B = \{e_j(1u_{i'}), e_0(1u_{i'j}) | j \in \{1, 2, \ldots, n\}, j \neq i'\}$. If $|B| < n - 1$, then there exists some $j \in \{1, 2, \ldots, n\}$, $j \neq i'$ such that $e_j(1u_{i'}) \notin F$ and $e_0(1u_{i'j}) \notin F$, we are done. So assume $|B| \geqslant n - 1$. Since there are no isolated edges in $FQ_{k+1} - F$, there exist some j' such that $e_{j'}(1u_{i'}) \notin F$. If $e_0(1u_{i'j'}) \notin F$ or $\overline{e}(1u_{i'j'}) \notin F$, then we are done. So assume that $e_0(1u_{i'j'}) \in F$ and $\overline{e}(1u_{i'j'}) \in F$. Let $C = \{e_l(1u_{i'j'}), e_0(1u_{i'j'l}) | l \in \{1, 2, \ldots, n\} - \{i', j'\}\}$. Then $|C \cap F| \leqslant |F - (A \cup B \cup \{e_0(1u), \overline{e}(1u), e_0(1u_{i'}), \overline{e}(1u_{i'}), e_0(1u_{i'j'}), \overline{e}(1u_{i'j'})\})| \leqslant n - 4$. Since there exist $n - 2$ pairs of edges in B, so there exists a pair of edges $e_l(1u_{i'j'}), e_0(1u_{i'j'l})$ $(l \in \{1, 2, \ldots, n\} - \{i', j'\})$ which is not in F. Thus $1u$ can be connected to $Q_n^0 - F_0$. $\square$

Lemma 2.5 [27]. $\kappa_1(Q_n) = 2n - 2$ *for* $n \geqslant 3$ *and* $\kappa_2(Q_n) = 3n - 5$ *for* $n \geqslant 5$.

3. Main results

Theorem 3.1. $\kappa_2(FQ_{n+1}) = 3n + 1$ *for* $n \geqslant 7$.

Proof. On the one hand, we can choose a cycle C of length four and a path P in C with length two and without complementary edges such that $N_{FQ_{n+1}}(P) = 3n + 1$ since, by Lemmas 2.2 and 2.1, $g(FQ_{n+1}) = 4$ and any two non-adjacent vertices in P have common neighbors exactly two for $n \geqslant 3$. It is easy to check that $FQ_{n+1} - N_{FQ_{n+1}}(P)$ contains neither isolated vertices nor isolated edges for $n \geqslant 6$, which implies that $\kappa_2(FQ_{n+1}) \geqslant 3n + 1$.

On the other hand, let F be a subset of vertices in $FQ_{n+1} = G(Q_n^0, Q_n^1; M_0 + \overline{M})$ with $|F| \leqslant 3n$ and there are no isolated vertices or isolated edges in $FQ_{n+1} - F$. Let $F_0 = F \cap V(Q_n^0)$, $F_1 = F \cap V(Q_n^1)$. Without loss of generality, we may suppose that $|F_0| \geqslant |F_1|$, then $|F_1| \leqslant \frac{3n}{2} < \frac{4n-6}{2} = 2n - 3$ $(n \geqslant 7)$ since $F_0 \cap F_1 = \phi$.

By Lemma 2.3, any vertex in $Q_n^0 - F_0$ can be connected to a vertex in $Q_n^1 - F_1$. So we only need to prove that $Q_n^1 - F_1$ is connected.

If there are no isolated vertices in $Q_n^1 - F_1$ then, since $|F_1| < 2n - 3 < \kappa_1(Q_n^1)$ by Lemma 2.5, so $Q_n^1 - F_1$ is connected.

Suppose below that there exists an isolated vertex $1u$ in $Q_n^1 - F_1$. Since any two vertices in Q_n^1 can share at most two common neighbors, so at least $2n - 2$ vertices are to be removed to get two isolated vertices in Q_n^1. Since $|F_1| < 2n - 3$, so there is just one isolated vertex $1u$ in $Q_n^1 - F_1$. Let $F_1' = F_1 \cup \{1u\}$. Since $|F_1'| < 2n - 3 + 1 < 2n - 2 \leqslant \kappa_1(Q_n^1)$ by Lemma 2.5 and there are no isolated vertices in $Q_n^1 - F_1'$, so $Q_n^1 - F_1'$ is connected.

In the following we will prove that $1u$ is connected to $Q_n^1 - F_1'$ in $FQ_{n+1} - F$. Since there are no isolated vertices in $FQ_{n+1} - F$, at least one of $0u$ and $0\overline{u}$ is not in F. In the following discussion we consider two cases: (1) $0u \notin F$ and $0\overline{u} \notin F$; (2) $0u \in F$, $0\overline{u} \notin F$, or $0u \notin F$, $0\overline{u} \in F$.

Subcase (2.1): $0u \notin F$ and $0\overline{u} \notin F$.

Since the distance between $0u$ and $0\overline{u}$ in Q_n^0 is n, so when $n \geqslant 3$ they have no common neighbors in Q_n^0. Thus $|N_{FQ_{n+1}}(0u) \cup N_{FQ_{n+1}}(0\overline{u}) - 1u| = 2n + 1$ since $0u$ and $0\overline{u}$ have exactly two common neighbors. But there are at most $|F| - |N_{Q_n^1}(1u)| \leqslant 2n$ elements of F that may be in these $2n + 1$ vertices. So at least one of them does not belong to F. If $1\overline{u} \notin F$, then we are done. So we suppose that $1\overline{u} \in F$. So at least one of the vertices in $N_{Q_n^0}(0u) \cup N_{Q_n^0}(0\overline{u})$ does not belong to F_0. Without loss of generality, we suppose $0u_i$ is such a vertex.

Since $|F_1| < 2n - 3$ and $1u_j \in F_1 (j = 1, 2, \ldots, n)$, at most $n - 4$ of $1\overline{u}_j$ can be in F_1. For each vertex $1\overline{u}_j \notin F_1$, if one of $0u_j$ or $0\overline{u}_j$ is not in F_0, then $1u$ can be connected to $Q_n^1 - F_1'$, we are done. So we suppose that for any vertex $1\overline{u}_j \notin F_1$ both $0u_j$ and $0\overline{u}_j$ are in F_0. In this case, there are at least $4 * 2 + n - 4 = n + 4$ vertices in F. Let $B = F \cap (N_{Q_n^0}(0u) \cup N_{Q_n^0}(0\overline{u}) \cup N_{Q_n^1}(1\overline{u}))$, then $|B| \geqslant n + 4$.

For each $j \in \{1, 2, \ldots, n\}$ and $j \neq i$, it is clear that both $0u_{ij} \notin B \cup N_{Q_n^1}(1u)$ and $1u_{ij} \notin B \cup N_{Q_n^1}(1u)$. So at most $|F| - |B| - |N_{Q_n^1}(1u)| \leqslant n - 4$ vertices of F may be in these $n - 1$ pairs of vertices, and so there exists an j such that both $0u_{ij} \notin F$ and $1u_{ij} \notin F$. Thus $1u$ can be connected to $Q_n^1 - F'$.

Subcase (2.2): $0u \in F$ and $0\overline{u} \notin F$, or $0u \notin F$ and $0\overline{u} \in F$.

Without loss of generality, we suppose that $0u \in F$ and $0\overline{u} \notin F$.

Since there are no isolated edges in $G - F$, then either $1\overline{u} \notin F$ or $0\overline{u}$ has a neighbor in Q_n^0 which is not in F. If $1\overline{u} \notin F$, then we are done. So we suppose that $1\overline{u} \in F$, thus $0\overline{u}$ has a neighbor in Q_n^0 that is not in F. Suppose that $0\overline{u}_i$ is such a vertex. If $1\overline{u}_i \notin F$, we are done. So we suppose that $1\overline{u}_i \in F$. For any vertex $0v \in N_{Q_n^0}(0\overline{u}, 0\overline{u}_i)$, it is clear that both $0v$ and $1v$ do not belong to $N_{Q_n^1}(1u) \cup \{0u, 1\overline{u}, 1\overline{u}_i\}$. Since there are $2n - 2$ pairs of vertices like $(0v, 1v)$, but at most $|F| - |N_{Q_n^1}(1u) \cup \{0u, 1\overline{u}, 1\overline{u}_i\}| \leqslant 2n - 3$ vertices of F may be in these $2n - 2$ pairs of vertices, so at least one pair of vertices does not belong to F. Thus $1u$ can be connected to $Q_n^1 - F_1'$.

Thus we have proved that all vertices in $Q_n^1 - F_1$ are connected to each other in $G - F$. □

Theorem 3.2. $\lambda_2(FQ_{n+1}) = 3n + 2$ *for* $n \geqslant 4$.

Proof. On the one hand, suppose that P is a path of length 2 in FQ_{n+1}, then, it is clear that $\lambda_2(FQ_n) \leqslant |E_{FQ_n}(P)| = 3n + 2$ for $n \geqslant 2$.

On the other hand, let $F \subset E(FQ_{n+1})$ with $|F| = 3n + 1$ such that there are neither isolated vertices nor isolated edges in $FQ_{n+1} - F$. We want to prove that $FQ_{n+1} - F$ is connected. Let $FQ_{n+1} = G(Q_n^0, Q_n^1; M_0 + \overline{M})$ be a decomposition of FQ_{n+1}.

For each $i = 1, 2, \ldots, n$, let M_i be the set of edges in $E(FQ_{n+1} - \overline{M})$ whose two end-vertices differ in the ith bit position. Then $M_0, M_1, \ldots M_n$ and $\overline{M}$ is a partition of $E(FQ_{n+1})$. Since $2(n + 2) < 3n + 1$ for $n \geqslant 4$, at least one of $|M_0|, |M_1|, \ldots, |M_n|$ and $|\overline{M}|$ is greater than 3. Thus we can relabel the vertices of FQ_{n+1} such that $|F \cap (M_0 \cup \overline{M})| \geqslant 3$.

Let $F_0 = F \cap E(Q_n^0)$, $F_1 = F \cap E(Q_n^1)$, $F_{M_0} = F \cap M_0$, $F_{\overline{M}} = F \cap \overline{M}$. So $|F_0| + |F_1| \leqslant 3n + 1 - 3 = 3n - 2$.

Since $3n - 2 < 4n - 4$ for $n \geqslant 3$, at least one of $|F_0|$ and $|F_1|$ is less than $2n - 2$. Without loss of generality, we suppose that $|F_0| < 2n - 2$.

Case 1: There are no isolated vertices in $Q_n^0 - F_0$. Then $Q_n^0 - F_0$ is connected since $|F_0| < 2n - 2 = \lambda'(Q_n^0)$. By Lemma 2.4, any vertex in $Q_n^1 - F_1$ is connected to a vertex in $Q_n^0 - F_0$. Thus $FQ_{n+1} - F$ is connected.

Case 2: There is an isolated vertex $0u$ in $Q_n^0 - F_0$. Since $\lambda(Q_n^0 - 0u) \geqslant \kappa(Q_n^0 - 0u) \geqslant n-1$ and $|E(Q_n^0 - 0u) \cap F_0| \leqslant |F_0| - |E_{Q_n^0}(0u)| < n-2$, $Q_n^0 - F_0 - 0u$ is connected. We only need to prove that $0u$ is connected to $Q_n^0 - F_0 - 0u$ in $FQ_{n+1} - F$. Since there are no isolated vertices in $FQ_{n+1} - F$, we have either $e_0(0u) \notin F$ or $\overline{e}(0u) \notin F$.

Without loss of generality we may suppose that $e_0(0u) = (0u, 1u) \notin F$. If $\overline{e}(1u) \notin F$, then we are done. So we suppose that $\overline{e}(1u) \in F$. Let $A = \{e_i(1u), e_0(1u_i) | i \in \{1,2,\ldots,n\}\} \cap F$. If $|A| < n$, then there exists an i such that both $e_i(1u) \notin F$ and $e_0(1u_i) \notin F$, then we are done. So we suppose $|A| \geqslant n$. Since there are no isolated edges in $G - F$, there exist an $i' \in \{1,2,\ldots,n\}$ such that $e_{i'}(1u) \notin F$. If $e_0(1u_{i'}) \notin F$ or $\overline{e}(1u_{i'}) \notin F$, then we are done. So we suppose $e_0(1u_{i'}) \in F$ and $\overline{e}(1u_{i'}) \in F$. Let $B = \{e_j(1u_{i'}), e_0(1u_{i'j}) \mid j \in \{1,2,\ldots,n\}, j \neq i'\}$. Then $|B \cap F| \leqslant |F - (N_{Q_n^0}(0u) \cup A \cup \{\overline{e}(1u), e_0(1u_{i'}), \overline{e}(1u_{i'})\})| \leqslant n-2$. Since there exist $n-1$ pairs of edges in B, so at least one pair of edges does not belong to F, thus $0u$ can be connected to $Q_n^0 - F_0 - 0u$ in $FQ_{n+1} - F$.

Thus the vertices in $Q_n^0 - F_0$ are connected to each other in $FQ_{n+1} - F$. By Lemma 2.4, any vertex in $Q_n^1 - F_1$ is connected to a vertex in $Q_n^0 - F_0$. Thus $FQ_{n+1} - F$ is connected. □

4. Conclusions

In this paper, we consider two new measurement parameters for the reliability and the tolerance of networks, i.e., the 2-extra connectivity $\kappa_2(G)$ and the 2-extra edge-connectivity $\lambda_2(G)$ of a connected graph G, which not only compensate for some shortcomings but also generalize the classical connectivity $\kappa(G)$ and the classical edge-connectivity $\lambda(G)$, and so can provide more accurate measures for the reliability and the tolerance of a large-scale parallel processing system. For the folded hypercube FQ_n, an important variant of the hypercube Q_n, we determine that $\kappa_2(FQ_n) = 3n-2$ for $n \geqslant 8$; and $\lambda_2(FQ_n) = 3n-1$ for $n \geqslant 5$. In other words, for $n \geqslant 8$ (resp. $n \geqslant 5$), at least $3n-2$ vertices (resp. $3n-1$ edges) of FQ_n have to be removed to disconnect FQ_n with each of the remaining components containing no isolated vertices (resp. edges). The two results show that the folded hypercube has a very strong reliability and fault tolerance when it is used to model the topological structure of a large-scale parallel processing system.

Acknowledgements

The authors express their gratitude to the anonymous referees for their constructive suggestions.

References

[1] Z. Chen, K.S. Fu, On the connectivity of clusters, Inf. Sci. 8 (4) (1975) 283–299.
[2] S.R. Das, C.L. Sheng, Strong connectivity in symmetric graphs and generation of maximal minimally strongly connected subgraphs, Inf. Sci. 14 (3) (1978) 181–187.
[3] D.R. Duh, G.H. Chen, J.F. Fang, Algorithms and properties of a new two-level network with folded hypercubes as basic modules, IEEE Trans. Parallel Distrib. Syst. 6 (7) (1995) 714–723.
[4] A. El-Amawy, S. Latifi, Properties and performance of folded hypercubes, IEEE Trans. Parallel Distrib. Syst. 2 (1991) 31–42.
[5] A.H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Trans. Comput. 38 (11) (1989) 1586–1591.
[6] A.H. Esfahanian, S.L. Hakimi, On computing a conditional edge-connectivity of a graph, Inf. Process. Lett. 27 (1988) 195–199.
[7] J. Fàbrega, M.A. Fiol, Extraconnectivity of graphs with large girth, Discrete Math. 127 (1994) 163–170.
[8] J.-S. Fu, Longest fault-free paths in hypercubes with vertex faults, Inf. Sci. 176 (7) (2006) 759–771.
[9] F. Harary, Conditional connectivity, Networks 13 (1983) 346–357.
[10] A. Hellwig, D. Rautenbach, L. Volkmann, Note on the connectivity of line graphs, Inf. Process. Lett. 91 (1) (2004) 7–10.
[11] A. Hellwig, L. Volkmann, Sufficient conditions for λ'-optimality in graphs of diameter 2, Discrete Math. 283 (2004) 113–120.
[12] A. Hellwig, L. Volkmann, Sufficient conditions for graphs to be λ'-optimal, super-edge-connected and maximally edge-connected, J. Graph Theory 48 (2005) 228–246.
[13] X.-M. Hou, M. Xu, J.-M. Xu, Forwarding indices of folded n-cubes, Discrete Appl. Math. 145 (3) (2005) 490–492.
[14] C.N. Lai, G.H. Chen, D.R. Duh, Constructing one-to-many disjoint paths in folded hypercubes, IEEE Trans. Comput. 51 (1) (2002) 33–45.
[15] S.C. Liaw, G.J. Chang, Generalized diameters and Rabin numbers of networks, J. Combin. Optim. 2 (1998) 371–384.
[16] J.-X. Meng, Y.-H. Ji, On a kind of restricted connectivity of graphs, Discrete Appl. Math. 117 (2002) 183–193.

[17] E. Simó, J.L.A. Yebra, The vulnerability of the diameter of folded *n*-cubes, Discrete Math. 174 (1997) 317–322.
[18] D. Wang, Embedding Hamiltonian cycles into folded hypercubes with faulty links, J. Parallel Distrib. Comput. 61 (2001) 545–564.
[19] J.-M. Xu, Toplogical Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers., Dordrecht/Boston/London, 2001.
[20] J.-M. Xu, Theory and Application of Graphs, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[21] J.-M. Xu, M. Lü, On restricted arc-connectivity of regular digraphs, Taiwan J. Math. 9 (4) (2005) 661–670.
[22] J.-M. Xu, M. Lü, Super connectivity of line graphs and digraphs, Acta Math. Appli. Sinica 22 (1) (2006) 43–48.
[23] J.-M. Xu, M. Lü, Min, M.-J. Ma, A. Hellwig, Super connectivity of line graphs, Inf. Process. Lett. 94 (4) (2005) 191–195.
[24] J.-M. Xu, M.-J. Ma, Cycles in folded hypercubes, Appl. Math. Lett. 19 (2) (2006) 140–145.
[25] J.-M. Xu, M.-J. Ma, Z.-Z. Du, Edge-fault-tolerant properties of hypercubes and folded hypercubes, Australasian J. Combinatorics 35 (1) (2006) 7–16.
[26] J.-M. Xu, K.-L. Xu, On restricted edge-connectivity of graphs, Discrete Math. 243 (1–3) (2002) 291–298.
[27] J.-M. Xu, Q. Zhu, X.-M. Hou, T. Zhou, On restricted connectivity and extra connectivity of hypercubes and folded hypercubes, J. Shanghai Jiaotong Univ. (Sci.) E-10 (2) (2005) 208–212.
[28] Q. Zhu, J.-M. Xu, M. Lü, Edge fault tolerance analysis of a class of networks, Appl. Math. Comput. 172 (1) (2006) 111–121.