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Feedback numbers of Kautz digraphs☆

Jun-Ming Xu*, Ye-Zhou Wu, Jia Huang, Chao Yang

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China

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Abstract

A subset of vertices (resp. arcs) of a graph G is called a feedback vertex (resp. arc) set of G if its removal results in an acyclic subgraph. Let $f(d,n)$ ($f_a(d,n)$) denote the minimum cardinality over all feedback vertex (resp. arc) sets of the Kautz digraph $K(d,n)$. This paper proves that for any integers $d\geqslant 2$ and $n\geqslant 1$

$$f(d,n)=\begin{cases} d & \text{for } n=1,\\ \dfrac{(\varphi\odot\theta)(n)}{n}+\dfrac{(\varphi\odot\theta)(n-1)}{n-1} & \text{for } 2\leqslant n\leqslant 7,\\ \dfrac{d^n}{n}+\dfrac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4}) & \text{for } n\geqslant 8,\end{cases}$$

$$f_a(d,n)=f(d,n+1)\quad \text{for } n\geqslant 1,$$

where $(\varphi\odot\theta)(n)=\sum_{i|n}\varphi(i)\theta(n/i)$, $i|n$ means i divides n, $\theta(i)=d^i+(-1)^i d$, $\varphi(1)=1$ and $\varphi(i)=i\cdot\prod_{j=1}^{r}(1-1/p_j)$ for $i\geqslant 2$, where $p_1,\ldots,p_r$ are the distinct prime factors of i, not equal to 1.

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1. Introduction

The minimum feedback vertex (or arc) set problem is as follows. Given a digraph or an undirected graph $G=(V,E)$, find a smallest subset $F\subset V$ (or $F'\subset E$) whose removal induces an acyclic subgraph.

The problem was originally formulated in the area of combinatorial circuit design [13]. Other applications of the problem are connected with resource allocation mechanisms in operating systems that prevent deadlocks, to the constraint satisfaction problem and Bayesian inference in artificial intelligence, to the study of monopolies in synchronous distributed systems and to converter placement problems in optical networks (see [5,6]).

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* Corresponding author.
E-mail address: xujm@ustc.edu.cn (J.-M. Xu).

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The minimum feedback set problem is known to be NP-hard for general graphs [8] and the best known approximation algorithm is one with an approximation ratio two [1]. The problem has been studied for some graphs, such as hypercubic graphs, meshes, toroids, butterflies, cube-connected cycles, hypercubes and directed split-stars (see [1–3,5–7,10,11,13–15]). In particular, Kralovic and Ruzicka [9] proved that the cardinality of a minimum feedback set of the Kautz undirected graph $\mathrm{UK}(2,n)$ is 2^{n-1}.

In this paper, we consider the Kautz digraph $K(d,n)$ ($d\geqslant 2$, $n\geqslant 1$). The vertex-set of $K(d,n)$ is defined as the set

$$\begin{aligned}V(d,n)=\{x_1x_2\cdots x_n|x_i\in\{1,2,\ldots,d+1\}\text{ for } i=1,2,\ldots,n, \text{ and}\\ x_i\neq x_{i+1}\text{ for } i=1,2,\ldots,n-1\}.\end{aligned}$$

There are d arcs from one vertex $x_1x_2\cdots x_n$ to d other vertices $x_2x_3\cdots x_n\alpha$, where $\alpha\in\{1,2,\ldots,d+1\}\backslash\{x_n\}$. Clearly, $|V(d,n)|=d^n+d^{n-1}$.

The Kautz digraphs have many attractive features superior to the hypercube (see, for example, Section 3.3 in [16]) and, thus, been thought of as a good candidate for the next generation of parallel system architectures, after the hypercube network [4].

Denote the minimum cardinality over all feedback vertex (resp. arc) sets of $K(d,n)$ by $f(d,n)$ (resp. $f_a(d,n)$), and call it the feedback number (resp. edge-feedback number) of $K(d,n)$. In this paper, we prove that for any integers $d\geqslant 2$ and $n\geqslant 1$

$$f(d,n)=\begin{cases} d & \text{for } n=1,\\ \dfrac{(\varphi\odot\theta)(n)}{n}+\dfrac{(\varphi\odot\theta)(n-1)}{n-1} & \text{for } 2\leqslant n\leqslant 7,\\ \dfrac{d^n}{n}+\dfrac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4}) & \text{for } n\geqslant 8,\end{cases}$$
$$f_a(d,n)=f(d,n+1)\quad \text{for } n\geqslant 1,$$

where $(\varphi\odot\theta)(n)=\sum_{i|n}\varphi(i)\theta(n/i)$ is a convolution, $i|n$ means i divides n, $\theta(i)=d^i+(-1)^i d$ and $\varphi(i)$ is the Euler totient function (its definition can be found in any text-book on number theory, for example [12]), that is, $\varphi(1)=1$ and $\varphi(i)=i\cdot\prod_{j=1}^{r}(1-1/p_j)$ for $i\geqslant 2$, where $p_1,\ldots,p_r$ are the distinct prime factors of i, not equal to 1.

2. Feedback vertex sets

In this section, our main aim is to construct two important sets $\Phi(d,n)$ and $F(d,n)$ in $K(d,n)$, respectively, where the former is a set of some cycles in $K(d,n)$ and the latter is a feedback vertex set of $K(d,n)$ for $n\geqslant 2$, and then to show that the feedback number $f(d,n)$ of $K(d,n)$ satisfies $f(d,n)=|\Phi(d,n)|$ for $2\leqslant n\leqslant 7$ and $|\Phi(d,n)|\leqslant f(d,n)\leqslant|F(d,n)|$ for $n\geqslant 8$.

Definition 2.1. Define a mapping ϕ_n: $V(d,n)\to V(d,n)$ subject to

$$\phi_n(X)=\begin{cases} x_2x_3\ldots x_nx_1, & \text{if } x_1\neq x_n;\\ x_2x_3\cdots x_nx_2, & \text{if } x_1=x_n\end{cases}\quad \text{for } X=x_1x_2x_3\cdots x_n.$$

It is clear that ϕ_n is a bijective mapping. Since $V(d,n)$ is finite, for any $X\in V(d,n)$, there must exist a smallest positive integer t, denoted by $\mathrm{ind}(X)$, such that $\phi_n^t(X)=X$. Moreover, for any integer j, if $\phi_n^j(X)=X$ then $t\mid j$ which means that t divides j. For example for an $X=x_1x_2x_3\cdots x_n\in V(d,n)$, if $x_1\neq x_n$, then $\phi_n^n(X)=X$ and $\mathrm{ind}(X)|n$; if $x_1=x_n$, then $\phi_n^{n-1}(X)=X$ and $\mathrm{ind}(X)\mid(n-1)$. For a given $X\in V(d,n)$, define the sequence

$$[X]_{\phi_n}=(X,\phi_n(X),\ldots,\phi_n^{t-1}(X),X),$$

where $t=\mathrm{ind}(X)$. It is clear that $[X]_{\phi_n}$ is a directed cycle in $K(d,n)$. Since $K(d,n)$ contains no self-loops, $t\geqslant 2$ for any $X\in V(d,n)$. Thus, the sequence $(\phi_n^i(X),\phi_n^{i+1}(X),\ldots,\phi_n^{t-1}(X),X,\ldots,\phi_n^{i-1}(X),\phi_n^i(X))$ is equivalent to $[X]_{\phi_n}$ for any integer i with $1\leqslant i\leqslant t-1$. For short, we will replace $[X]_{\phi_n}$ by $[X]$ in the following discussion. Let

$$\Phi(d,n)=\{\,[X]\mid X\in V(d,n)\,\}. \tag{2.1}$$

Theorem 2.1. *Let F be a feedback set in $K(d,n)$. Then for any vertex $X \in V(d,n)$*

(a) $F \cap [X] \neq \emptyset$ *and* $|F| \geqslant |\Phi(d,n)|$;
(b) $|F| = |\Phi(d,n)|$ *if* $|F \cap [X]| = 1$.

Proof. Since F is a feedback set, $F \cap V(C) \neq \emptyset$ for any directed cycle C of $K(d,n)$, of course, including cycles in $\Phi(d,n)$.

It is clear that either $[X]=[Y]$ or $[X] \cap [Y] = \emptyset$ for any two cycles $[X]$ and $[Y]$ in $\Phi(d,n)$, which means that $\Phi(d,n)$ is a partition of $V(d,n)$. Thus,

$$|F| = \sum_{[X]\in\Phi(d,n)} |F \cap [X]| \geqslant \sum_{[X]\in\Phi(d,n)} 1 = |\Phi(d,n)|.$$

The conclusion (b) follows from (a) immediately. □

For any integers d and n with $d \geqslant 2$ and $n \geqslant 1$, let

$$\Omega_{d,n} = \bigcup_{m\geqslant 1}^{n+d^n+d^{n-1}} V(d,m),$$

where $d^n + d^{n-1} = |V(d,n)|$. For any $X = x_1x_2\cdots x_m \in \Omega_{d,n}$, denote $X(i) = x_1x_2\cdots x_i$, $1 \leqslant i \leqslant m$, where m is called the length of X, denoted by $\ell(X)$.

Theorem 2.2. *For given integers $d \geqslant 2$ and $n \geqslant 1$, let V_d be a subset of $\Omega_{d,n}$ satisfying the conditions*:

(a) *For any* $X \in \Omega_{d,n}$, $V_d \cap [X] \neq \emptyset$;
(b) *For any* $Y \in V_d$ *and for any integer i with* $1 \leqslant i \leqslant m$, $Y(i) \in V_d$, *where* $m = \ell(Y)$.

Then $V_d \cap V(d,n)$ is a feedback vertex set of $K(d,n)$.

Proof. Let $F = V_d \cap V(d,n)$ for convenience. Suppose to the contrary that a new graph $K(d,n) - F$ obtained from $K(d,n)$ by removing the vertices in F and the corresponding arcs contains a directed cycle C of length j $(2 \leqslant j \leqslant d^n + d^{n-1})$:

$$C = (x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}, \ldots, x_{n+j-1}x_1\cdots x_{n-1}, x_1x_2\cdots x_n).$$

Then $F \cap C = \emptyset$. Let $X = x_1x_2\cdots x_nx_{n+1}\cdots x_{n+j-1} \in \Omega_{d,n}$, and let $\ell = n + j - 1$. So we can express C as

$$C = (X(n), \phi_\ell(X)(n), \ldots, \phi_\ell^{j-1}(X)(n), X(n)).$$

By the condition (a) there exists an integer k $(0 \leqslant k \leqslant j-1)$ such that $Y = \phi_l^k(X) \in V_d \cap [X]$ and $Y(n) \in C$. By the condition (b) we have $Y(n) \in V_d$, of course, $Y(n) \in V(d,n)$. Then $Y(n) \in F$, that is, $F \cap C \neq \emptyset$, a contradiction. □

Let

$$\Omega'_{d,n} = \bigcup_{m\geqslant 1}^{n+d^n+d^{n-1}} \{x_1x_2\cdots x_m \in \Omega_{d,n} \mid x_1 \geqslant x_i, 1 \leqslant i \leqslant m\}. \tag{2.2}$$

It is not difficult to verify that $\Omega'_{d,n}$ satisfies the two conditions in Theorem 2.2. Then $\Omega'_{d,n} \cap V(d,n)$ is a feedback vertex set of $K(d,n)$ with size $\sum_{i=1}^{d} i^{n-1}$. Moreover, $\Omega'_{d,n} \cap V(d,n)$ is minimum for $n = 2, 3$. For example, $\{323, 313, 321, 312, 212\} = \Omega'_{2,3} \cap V(2,3)$ is a minimum feedback vertex set of $K(2,3)$ shown in Fig. 1 by solid vertices.

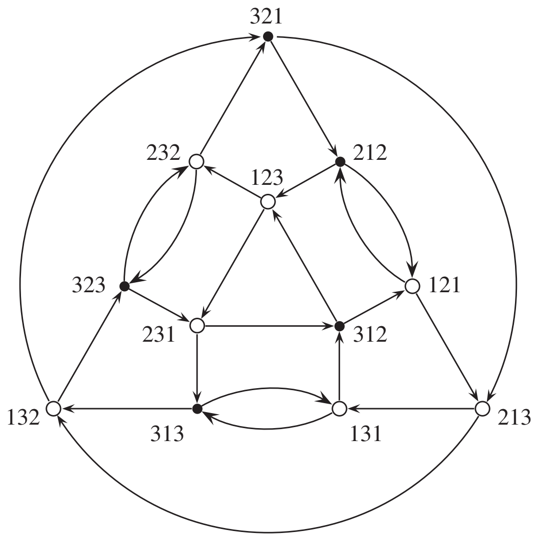

Fig. 1. The Kautz digraph $K(2,3)$.

For an $X = x_1x_2\cdots x_m \in \Omega_{d,n}$, it is convenient to associate X with the integer $\sum_{i=1}^{m} x_i(d+2)^{m-i}$. So, for $Y = y_1y_2\cdots y_{m'}$, $X > Y$ means that

$$\sum_{i=1}^{m} x_i(d+2)^{m-i} > \sum_{i=1}^{m'} y_i(d+2)^{m'-i}.$$

Let $X = x_1x_2\cdots x_m \in \Omega'_{d,n}$ with $x_1 = \max\{x_1, x_2, \ldots, x_m\}$ and let $p = x_1$. Then $2 \leqslant p \leqslant d+1$ and we can write X as

$$X = pX_1pX_2p\cdots pX_r \quad \text{or} \quad X = pX_1pX_2p\cdots pX_rp,$$

where X_i is a non-empty sub-sequence of X between the ith p and the $(i+1)$th p and each digit in X_i is less than p, $1 \leqslant i \leqslant r$.

For example, let $X = 72172172 \in \Omega'_{9,8}$, then $p = 7$ and X can be expressed as $7X_17X_27X_3$, where $X_1 = X_2 = 21$ and $X_3 = 2$.

We are interested in a subset F_d of $\Omega'_{d,n}$. For the sake of our convenience, we give the definition of F_d.

Definition 2.2. Let F_d be a subset of $\Omega'_{d,n}$ such that each $X = x_1x_2\cdots x_m \in F_d$ with $x_1 = p = \max\{x_1, x_2, \ldots, x_m\}$ satisfies one of the following forms:

(1) $X = \underbrace{pX_1pX_1p\cdots pX_1}_{r}$ or $\underbrace{pX_1pX_1p\cdots pX_1}_{r}p$, $r \geqslant 1$;
(2) $X = pX_1pX_2$ or pX_1pX_2p, $X_1 > X_2$;
(3) $X = \underbrace{pX_1pX_1p\cdots pX_1}_{r-1}pX_r$, $r \geqslant 3$ and $X_r = X_1(i)$, where $1 \leqslant i < j$, $i = \ell(X_r)$ and $j = \ell(X_1)$;
(4) $X = pX_1pX_2p\cdots pX_r$ or $pX_1pX_2p\cdots pX_rp$, $X_1 > X_2$, $X_1 \geqslant X_i$, $i = 3, \ldots, r$.

For example, $\{71217121, 7121765, 71271271, 71271712\} \subset F_6$, in which the vertices satisfy the forms (1)–(4) in Definition 2.2, respectively. By the definition, $71271272 \notin F_6$, since $71271272 = 7X_17X_17X_2$ does not match any form. Let

$$F(d, n) = F_d \cap V(d, n), \tag{2.3}$$

where F_d is defined in Definition 2.2.

Theorem 2.3. *$F(d, n)$ is a feedback vertex set of $K(d, n)$ for $n \geqslant 2$.*

Proof. We only need to prove that F_d defined in Definition 2.2 satisfies the two conditions (a) and (b) in Theorem 2.2.

(a) We need to check that $F_d \cap [Y] \neq \emptyset$ for any $Y \in \Omega_{d,n}$. Let $Y = x_1x_2 \ldots x_m$ be any element in $\Omega_{d,n}$. There exists an integer k such that $x_k = p = \max_{1 \leqslant i \leqslant m}\{x_i\}$. Let $X = x_kx_{k+1} \ldots x_mx_1x_2 \ldots x_{k-1}$, then $[X] = [Y]$. We only need to prove $F_d \cap [X] \neq \emptyset$.

Since $X \in \Omega'_{d,n}$, X can be expressed as either $X = pX_1pX_2p \cdots pX_r$ or $X = pX_1pX_2p \cdots pX_rp$. Without loss of generality, we only consider the former since the latter does not contain form (3) and the proof is similar and simpler.

If $r = 1$ and $X = pX_1$, then $F_d \cap [X] = \{X\}$. If $r = 2$ and $X = pX_1pX_2$, then $F_d \cap [X] = \{X\}$ if $X_1 \geqslant X_2$ and $F_d \cap [X] = \{pX_2pX_1\}$ otherwise, that is, $F_d \cap [X] \neq \emptyset$. Assume $r \geqslant 3$ below.

If $X_1 = X_2 = \cdots = X_r$, then $F_d \cap [X] = \{X\}$. Otherwise there exists an integer j such that $X_j > X_{j+1}$ and $X_j \geqslant X_i$, $1 \leqslant i \neq j + 1 \leqslant r$, then

$$pX_jpX_{j+1}p \cdots pX_rpX_1p \cdots pX_{j-1} = \phi_m^k(X) \in F_d \cap [X],$$

where k is the length of $pX_1pX_2p \cdots pX_{j-1}$, that is, $F_d \cap [X] \neq \emptyset$.

(b) We now check that F_d satisfies the condition (b) in Theorem 2.2. For any $X \in F_d$, either $X = pX_1pX_2p \cdots pX_r$ or $X = pX_1pX_2p \cdots pX_rp$. Then $X(i) = pX_1pX_2p \cdots pX_t$ or $pX_1pX_2p \cdots pX_tp$ or $pX_1pX_2p \cdots pX_{t-1}pX'_t$, where $t \leqslant r$, $X'_t = X_t(j)$ and $j = \ell(X'_t)$. We only need to check the case $X(i) = pX_1pX_2p \cdots pX'_t$ (the other case is similar and simpler).

For $t = 1$ or 2, $X(i)$ satisfies the form either (1) or (2) in Definition 2.2 and the assertion holds obviously. Assume $t \geqslant 3$ below.

If $X_1 = X_2$, X only could be the form either (1) or (3) in Definition 2.2, we have $X_1 = X_2 = \cdots = X_{t-1} = X_t$ and $X'_t = X_t(j) = X_1(j)$. Then $X(i)$ is of the form (3) in Definition 2.2, and so $X(i) \in F_d$.

If $X_1 \neq X_2$, X only could be of the form (4) in Definition 2.2, we have $X_1 > X_2$, $X_1 \geqslant X_j$, $3 \leqslant j \leqslant t - 1$, and $X_1 \geqslant X_t \geqslant X_t(j) = X'_t$. Then $X(i)$ is of the form (4) in Definition 2.2, which also implies $X(i) \in F_d$.

The proof of the theorem is complete. □

Theorem 2.4. *If $2 \leqslant n \leqslant 7$, then $|F(d, n) \cap [X]| = 1$ for any vertex $X \in V(d, n)$.*

Proof. Assume, without loss of generality, $X = pX_1pX_2p \cdots pX_r \in F(d, n)$. (the case $X = pX_1pX_2p \cdots pX_rp$ is similar). We have $r \leqslant 3$ since $n \leqslant 7$. The proof depends that X satisfies which form in Definition 2.2 of F_d.

If X satisfies the form (1) in Definition 2.2, then $\mathrm{ind}(X) = \ell + 1$, where $\ell = \ell(X_1)$. In the directed cycle $[X] = (X, \phi_n(X), \ldots, \phi_n^{\ell-1}(X), X)$, the vertex X is only one whose first digit is p. Thus, $F(d, n) \cap [X] = \{X\}$.

If X satisfies the form (2) in Definition 2.2, then $X = pX_1pX_2$ and $F(d, n) \cap [X] \subseteq \{pX_1pX_2, pX_2pX_1\}$. It is clear that $X_1 \neq X_2$ since X satisfies the form (2). If $X_1 > X_2$, then pX_2pX_1 does not satisfy the form (2); if $X_1 < X_2$, then pX_1pX_2 does not satisfy the form (2). Thus, $|F(d, n) \cap [X]| = 1$.

If X satisfies the form (3) in Definition 2.2, then $X_1 = X_2$ and $X_3 = X_1(i)$, where $i = \ell(X_3) < \ell(X_1)$. Thus, $n \geqslant 3 + 2(i + 1) + i \geqslant 8$, which contradicts our hypothesis $n \leqslant 7$.

If X satisfies the form (4) in Definition 2.2, then $X_1 > X_2$, $X_1 \geqslant X_3$ and $n \geqslant 6$. Thus, both X_2 and X_3 are a single digit. When $n = 6$, X_1 also is a single digit and $X = pX_1pX_2pX_3$. When $n = 7$, X_1 could be either a single digit if $X = pX_1pX_2pX_3p$ or X_1 is a sequence of length two if $X = pX_1pX_2pX_3$. In all the three cases we have $F(d, n) \cap [X] = \{X\}$.

The proof of the theorem is complete. □

Theorem 2.5. *$F(d, n)$ is a minimum feedback vertex set of $K(d, n)$ and $|F(d, n)| = |\Phi(d, n)|$ for $2 \leqslant n \leqslant 7$.*

Proof. The result follows from Theorems 2.2–2.4, immediately. □

3. Feedback numbers

In the preceding section, we construct two important sets $\Phi(d,n)$ and $F(d,n)$ defined in (2.1) and (2.3), respectively. By Theorems 2.1, 2.3 and 2.5, we have that the feedback number $f(d,n)$ of $K(d,n)$ is

$$\begin{aligned} &f(d,n) = |\Phi(d,n)| \quad \text{for } 2\leqslant n\leqslant 7,\\ &|\Phi(d,n)| \leqslant f(d,n) \leqslant |F(d,n)| \quad \text{for } n\geqslant 8. \end{aligned} \tag{3.1}$$

In this section, we determine the value of $|\Phi(d,n)|$ and establish an upper bound of $|F(d,n)|$ for $n\geqslant 8$. In Lemmas 3.1, 3.2 and Theorem 3.1 we assume that the parameter d is fixed since the process of our proofs and calculations will be independent of d.

Lemma 3.1. *Let $W_n = \{X = x_1x_2\cdots x_n \in V(d,n) | x_1 \neq x_n\}$ and $\overline{W}_n = V(d,n)\backslash W_n$.*

(a) $|W_n| = d^n + (-1)^n d$;
(b) $|\Phi(d,n)| = |\{[X]|X \in W_n\}| + |\{[X]|X \in W_{n-1}\}|$.

Proof. We first prove the assertion (a) by induction on $n\geqslant 2$. For $n=2$, then $W_2 = V(d,2)$, $\overline{W}_2=\emptyset$, and so $|W_2| = |V(d,2)| = d^2+d$. Suppose now that $n\geqslant 3$ and the result holds for any integer less than n. By the definition, $|\overline{W}_n| = |W_{n-1}|$ since $|\{x_1x_2\cdots x_{n-1}x_1 \in \overline{W}_n\}| = |\{x_1x_2\cdots x_{n-1} \in W_{n-1}\}|$ for $n\geqslant 3$. Thus, by the induction hypothesis, we have

$$\begin{aligned} |W_n| &= |V(d,n)| - |\overline{W}_n|\\ &= (d^n + d^{n-1}) - |W_{n-1}|\\ &= (d^n + d^{n-1}) - (d^{n-1} + (-1)^{n-1}d)\\ &= d^n + (-1)^n d. \end{aligned}$$

as required.

The assertion (b) follows from $|\{[X]|X \in \overline{W}_n\}| = |\{[X]|X \in W_{n-1}\}|$ immediately. □

Lemma 3.2. *Let $W_1(1)=\emptyset$ and $W_n(i) = \{X \in W_n \,|\, \mathrm{ind}(X) = i\}$ for any $n\geqslant 2$, $1\leqslant i\leqslant n$. Then*

$$|W_n(i)| = \begin{cases} |W_i(i)| & \text{if } i|n,\\ 0 & \text{otherwise}. \end{cases}$$

Proof. If $i \mid n$, for any $X = x_1x_2\cdots x_n \in W_n(i)$, $X = \phi_n^i(X) = x_{i+1}x_{i+2}\cdots x_nx_1\cdots x_i$, where ϕ_n is defined in Definition 2.1. We have $x_j = x_{ki+j}$, $1\leqslant j\leqslant i$, $1\leqslant k\leqslant r = n/i$ and $X = \underbrace{YY\ldots Y}_{r}$, where $Y = x_1x_2\cdots x_i$. It is easy to see $W_i(i) = W_n(i)$ and, hence, $|W_i(i)| = |W_n(i)|$.

If $i\nmid n$, there must exist integers j and k such that $n = ki + j$ and $1\leqslant j < i$. If there still exists an $X \in W_n(i)$, then $X = \phi_n^i(X) = \phi_n^n(X) = \phi_n^{ki+j}(X) = \phi_n^j(X)$, which contradicts to the definition of $\mathrm{ind}(X)$. Thus, $W_n(i) = \emptyset$. □

Theorem 3.1. *For integer $i\geqslant 1$, let $\theta(i) = d^i + (-1)^i d$ be a function and $\varphi(i)$ the Euler totient function. Then for any $d\geqslant 2$ and $n\geqslant 2$,*

$$|\Phi(d,n)| = \frac{(\varphi\odot\theta)(n)}{n} + \frac{(\varphi\odot\theta)(n-1)}{n-1},$$

where $\odot$ is the convolution, that is, $(\varphi\odot\theta)(n) = \sum_{i|n}\varphi(i)\theta(n/i)$.

Proof. By Lemma 3.1 we only need to prove

$$|\{[X]|X \in W_n\}| = \frac{(\varphi\odot\theta)(n)}{n}.$$

To this purpose, let ω, e, N, μ be arithmetic functions over the set of positive integers defined as

$$\omega(i) = |W_i(i)|, \quad e(i) = 1, \quad N(i) = i,$$

$\mu(i)$ is the Möbius function:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } a^2 | i \text{ for some } a > 1, \\ (-1)^k & \text{if } i = p_1 p_2 \cdots p_k, \text{ distinct prime factors.} \end{cases}$$

It is proved in [12] that for any arithmetic functions f and g,

$$N \odot \mu = \varphi, \quad f = e \odot g \;\Leftrightarrow\; g = \mu \odot f. \tag{3.2}$$

By Lemma 3.2, for any positive integer n we have

$$\begin{aligned} \theta(n) = |W_n| &= \sum_{i=1}^{n} |W_n(i)| = \sum_{i|n} |W_n(i)| \\ &= \sum_{i|n} |W_i(i)| = \sum_{i|n} \omega(i) = (e \odot \omega)(n) \end{aligned}$$

which means $\theta = e \odot \omega$.

For any vertex $X \in W_n(i)$, $\mathrm{ind}(X) = i$ and $[X]$ is a directed cycle with length i and for any vertex $Y \in [X]$, $Y \in W_n(i)$. Thus

$$|\{[X] | X \in W_n(i)\}| = \frac{|W_n(i)|}{i}$$

and we have

$$\begin{aligned} |\{[X] | X \in W_n\}| &= \sum_{i=1}^{n} |\{[X] | X \in W_n(i)\}| \\ &= \sum_{i=1}^{n} \frac{|W_n(i)|}{i} = \sum_{i|n} \frac{|W_n(i)|}{i} = \sum_{i|n} \frac{|W_i(i)|}{i} \\ &= \sum_{i|n} \frac{\omega(i)}{i} = \frac{1}{n} \sum_{i|n} \frac{n}{i}\, \omega(i) = \frac{(N \odot \omega)(n)}{n}. \end{aligned}$$

By (3.2), we have $\theta = e \odot \omega \Leftrightarrow \omega = \mu \odot \theta$, and so

$$N \odot \omega = N \odot (\mu \odot \theta) = (N \odot \mu) \odot \theta = \varphi \odot \theta.$$

Thus,

$$|\{[X] | X \in W_n\}| = \frac{(\varphi \odot \theta)(n)}{n}$$

as required. □

Remark. We have mentioned in Section 2 that $\Omega'_{d,n} \cap V(d,2)$ and $\Omega'_{d,n} \cap V(d,3)$ are minimum feedback vertex sets. This fact can be deduced from Theorem 3.1 immediately as follows:

$$\frac{(\varphi \odot \theta)(n)}{n} + \frac{(\varphi \odot \theta)(n-1)}{n-1} = \sum_{i=1}^{d} i^{n-1} \quad \text{for } n = 2 \text{ or } 3.$$

Let

$$E(d,n) = \{X \in F(d,n) \,|\, |F(d,n) \cap [X]| \geqslant 2\}. \tag{3.3}$$

Then

$$F(d,n)\backslash E(d,n)=\{X\in F(d,n)||F(d,n)\cap[X]|=1\}.$$

For $2\leqslant n\leqslant 7$, it is clear that $E(d,n)=\emptyset$ by Theorem 2.4. For $n\geqslant 8$

$$|F(d,n)|\leqslant|\Phi(d,n)|+|E(d,n)| \tag{3.4}$$

since $|\{X\in F(d,n)||F(d,n)\cap[X]|=1\}|\leqslant|\Phi(d,n)|$.

For example, in $F(2,8)$ we have $[32132132]=[32132321]$, $[31231231]=[31231312]$ and $E(2,8)=\{32132132, 32132321, 31231231, 31231312\}$.

In fact, only two cycles in $\Phi(2,8)$, each of them intersects with $F(2,8)$ exactly two vertices; the other cycles in $\Phi(2,8)$, each of them intersects with $F(2,8)$ only one vertices. Then from

$$\begin{aligned}|\Phi(2,8)|&=\frac{(\varphi\odot\theta)(8)}{8}+\frac{(\varphi\odot\theta)(7)}{7}\\&=\frac{(2^8+2)+(2^4+2)+2(2^2+2)}{8}+\frac{2^7-2}{7}\\&=54.\end{aligned}$$

we have immediately

$$54\leqslant f(2,8)\leqslant 58.$$

Lemma 3.3. *For any integers $d\geqslant 2$ and $n\geqslant 8$, $|E(d,n)|\leqslant n^2\sum_{i=1}^{d}(i+1)^{n-5}$.*

Proof. Suppose $X=pX_1pX_2p\cdots pX_r$ or $pX_1pX_2p\cdots pX_rp\in E(d,n)$, where $2\leqslant p\leqslant d+1$.

By the definitions of F_d and $E(d,n)$, defined in Definition 2.2 and (3.3), respectively, X only could be of the form either (3) or (4) in Definition 2.2. Thus, $3\leqslant r\leqslant n/2$.

When $r=3$, we have $X=pX_1pX_2pX_1$ or $pX_1pX_1pX_2$ and $F(d,n)\cap[X]=\{pX_1pX_2pX_1, pX_1pX_1pX_2\}$, where $X_2=X_1(i)$ and $i=\ell(X_2)<\ell(X_1)$. Let $E_1(p,n)$ be the set of such X's.

When $r\geqslant 4$, there must exist an integer i, $3\leqslant i\leqslant r-1$ such that $X_i=X_1$. Otherwise, $X_i<X_1$, $3\leqslant i\leqslant r-1$ which leads to $[X]\cap F(d,n)=\{X\}$ and $X\notin E(d,n)$. Then, $\{X,\phi_n^k(X)\}\subset[X]\cap F(d,n)$, where k is the length of $pX_1pX_2p\cdots pX_{i-1}$ and $X=pX_1pX_2p\cdots pX_{i-1}pX_1pX_{i+1}p\cdots pX_r$ or $pX_1pX_2p\cdots pX_{i-1}pX_1pX_{i+1}p\cdots pX_rp$. Let $E_2(p,n)$ be the set of such X's.

Thus,

$$E(d,n)\subseteq\bigcup_{p=2}^{d+1}(E_1(p,n)\cup E_2(p,n)). \tag{3.5}$$

Clearly, $|E_1(p,n)|\leqslant 2p^{(n-4)/2}$. To estimate $|E_2(p,n)|$, let $j=\ell(X_1)$ and $k=\ell(X_2pX_3p\cdots pX_{i-1})$. It is not difficult to get that

$$|E_2(p,n)|\leqslant\sum_{j=1}^{\lfloor(n-6)/2\rfloor}\sum_{k=1}^{n-5-2j}p^jp^kp^{n-4-2j-k}\leqslant\frac{(n-6)(n-7)}{2}p^{n-5}.$$

Thus, by (3.5)

$$\begin{aligned}|E(d,n)|&\leqslant\sum_{p=2}^{d+1}(|E_1(p,n)|+|E_2(p,n)|)\leqslant\sum_{p=2}^{d+1}n^2p^{n-5}\\&=n^2\sum_{i=1}^{d}(i+1)^{n-5}\end{aligned}$$

as required. □

Theorem 3.2. *For any integers $d \geqslant 2$ and $n \geqslant 1$*

$$f(d,n)=\begin{cases} d & \text{for } n=1;\\ \dfrac{(\varphi\odot\theta)(n)}{n}+\dfrac{(\varphi\odot\theta)(n-1)}{n-1} & \text{for } 2\leqslant n\leqslant 7;\\ \dfrac{d^n}{n}+\dfrac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4}) & \text{for } n\geqslant 8,\end{cases}$$

Proof. It is clear that $f(d,1)=d$ since $K(d,1)$ is a complete digraph K_{d+1} and the removal of any $d-1$ vertices from K_{d+1} results in a complete digraph K_2, which is a directed cycle of length two. Assume $n \geqslant 2$ below.

By (3.1) and Theorem 3.1, we only need to prove that for $n \geqslant 8$

$$f(d,n)=\frac{d^n}{n}+\frac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4}).$$

Firstly while $n \geqslant 8$, let k be the biggest nontrivial factor of n, then $k \leqslant n/2$ and

$$(\varphi\odot\theta)(n)=\sum_{i|n}\varphi(i)\theta\left(\frac{n}{i}\right)=\varphi(1)\theta(n)+\mathrm{O}(d^k)=d^n+\mathrm{O}(d^{n/2})$$

and we have

$$|\Phi(d,n)|=\frac{d^n}{n}+\frac{d^{n-1}}{n-1}+\mathrm{O}(d^{n-4}). \tag{3.6}$$

Secondly from Lemma 3.3 and

$$\sum_{i=1}^{d}(i+1)^{n-5}\leqslant\int_0^{d+1}(i+1)^{n-5}=\frac{(d+2)^{n-4}}{n-4},$$

we have $|E(d,n)|=\mathrm{O}(nd^{n-4})$. Then by (3.4), we have

$$|F(d,n)|\leqslant|\Phi(d,n)|+|E(d,n)|=\frac{d^n}{n}+\frac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4}). \tag{3.7}$$

It follows from (3.1), (3.6) and (3.7) that

$$f(d,n)=\frac{d^n}{n}+\frac{d^{n-1}}{n-1}+\mathrm{O}(nd^{n-4})\quad\text{for } n\geqslant 8$$

as required. The theorem follows. □

4. Arc-Feedback numbers

To discuss the edge-feedback number of the Kautz digraph $K(d,n)$, we need another equivalent definition of $K(d,n)$ by the line digraph.

Let $G=(V,E)$ be a digraph with $E(G)\neq\emptyset$. The line graph of G, denoted by $L(G)$, is a directed graph, in which $V(L(G))=E(G)$, and there is an arc (a,b) if and only if there are vertices $x, y, z \in V(G)$ with $a=(x,y)$ and $b=(y,z)$. For a given integer $n\geqslant 1$, the nth iterated line graph of G, denoted by $L^n(G)$, is recursively defined as $L(L^{n-1}(G))$ if $E(L^{n-1}(G))\neq\emptyset$, where $L^0(G)$ and $L^1(G)$ denote G and $L(G)$, respectively. By the line digraph, the Kautz digraph $K(d,n)$ can be recursively defined as follow (see Section 3.3 in [16]).

$$K(d,1)=K_{d+1};\qquad K(d,n)=L^{n-1}(K_{d+1}),\quad n\geqslant 2.$$

Let $f_a(d,n)$ denote the minimum cardinality over all feedback arc sets of the Kautz digraph $K(d,n)$, called the arc-feedback number of $K(d,n)$.

Theorem 4.1. *For any integers $d \geqslant 1$ and $n \geqslant 1$, $f_a(d,n)=f(d,n+1)$.*

Proof. Let F be a minimum feedback vertex set of $K(d, n+1)$. We need to prove that there exist a minimum feedback arc set F_a of $K(d, n)$ such that $|F_a| = f(d, n+1)$.

For any vertex $X = x_1x_2\cdots x_{n+1} \in F$, $(x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1})$ is an arc of $K(d, n)$. Let

$$F_a = \{(x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}) | x_1x_2\cdots x_{n+1} \in F\}.$$

Clearly,

$$|F_a| = |F| = f(d, n+1).$$

We first prove F_a is a feedback arc set of $K(d, n)$. Suppose to the contrary that $K(d, n) - F_a$ obtained from $K(d, n)$ by removing the arcs in F_a contains a directed cycle C of length j:

$$C = (x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}, \ldots, x_{n+j-1}x_1\cdots x_{n-1}, x_1x_2\cdots x_n).$$

Then $F_a \cap C = \emptyset$ and we get a directed cycle C' of $K(d, n+1)$:

$$C' = (x_1x_2\cdots x_nx_{n+1}, x_2x_3\cdots x_{n+1}x_{n+2}, \ldots, x_{n+j-1}x_1\cdots x_{n-1}x_n, x_1x_2\cdots x_nx_{n+1}).$$

Since, F is a feedback vertex set of $K(d, n+1)$, we have $F \cap C' \neq \emptyset$.

Assume, without loss of generality, $x_1x_2\cdots x_nx_{n+1} \in F \cap C'$. Then by the definition of F_a, $e = (x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}) \in F_a$. Since e is an arc in C, $F_a \cap C \neq \emptyset$, a contradiction. The contradiction means that F_a is a feedback arc set of $K(d, n)$.

We now prove F_a is minimum. Suppose to the contrary that there exists a feedback arc set F'_a of $K(d, n)$ such that $|F'_a| < |F_a|$. Let

$$F' = \{x_1x_2\cdots x_{n+1} | (x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}) \in F'_a\}.$$

Then $|F'| = |F'_a| < |F_a|$. Let

$$D = (x_1x_2\cdots x_nx_{n+1}, x_2x_3\cdots x_{n+1}x_{n+2}, \ldots, x_{n+j-1}x_1\cdots x_{n-1}x_n, x_1x_2\cdots x_nx_{n+1})$$

be any directed cycle of $K(d, n+1)$. Then

$$D' = (x_1x_2\cdots x_n, x_2x_3\cdots x_{n+1}, \ldots, x_{n+j-1}x_1\cdots x_{n-1}, x_1x_2\cdots x_n)$$

is a directed cycle of $K(d, n)$. Since F'_a is a feedback arc set of $K(d, n)$ we have $F'_a \cap D' = \emptyset$. Then, $F' \cap D = \emptyset$ and F' is a feedback vertex set of $K(d, n+1)$. Since F is also a minimum feedback vertex of $K(d, n+1)$, we have

$$|F_a| = |F| = |F'| < |F_a|,$$

a contradiction.

The proof of the theorem is complete. □

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