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Fault diameter of product graphs ☆

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Abstract

The $(k-1)$-fault diameter $D_k(G)$ of a k-connected graph G is the maximum diameter of an induced subgraph by deleting at most $k-1$ vertices from G. This paper considers the fault diameter of the product graph $G_1 * G_2$ of two graphs G_1 and G_2 and proves that $D_{k_1+k_2}(G_1 * G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$ if G_1 is k_1-connected and G_2 is k_2-connected. This generalizes some known results such as Banič and Žerovnik [I. Banič, J. Žerovnik, Fault-diameter of Cartesian graph bundles, Inform. Process. Lett. 100 (2) (2006) 47–51].

Keywords: Combinatorics; Diameter; Fault diameter; Product graphs; Cartesian graphs; Cartesian graph bundles; Permutation graphs

1. Introduction

For graph-theoretical terminology and notation not defined here, we follow [8]. Let $d(G)$ denote the diameter of a graph G, and let $d(G) = \infty$ if G is not connected. The $(k-1)$-fault diameter of a graph G is defined as

$$D_k(G) = \max\big\{d(G-F)\colon\ F \subseteq V(G), |F| < k\big\}.$$

Note that $D_k(G) < \infty$ if and only if G is k-connected. Since nodes of a network do not always work, if some nodes are fault, the information cannot be transmitted by these nodes and the efficiency of network must be affected. The fault diameter is an important measurement for reliability and efficiency of an interconnection network. For some well-known graphs, the fault diameters have been determined, some of which can be found in Section 4.2 in [7].

The concept of fault diameter is first introduced by Krishnamoorthy and Krishnamurthy [5], who gave an upper bound of the fault diameter of the Cartesian product graph $G_1 \times G_2$, that is, $D_{k_1+k_2}(G_1 \times G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2)$. However, Xu et al. [9] pointed out that this bound is not correct by considering $C_4 \times C_4$, where C_4 is a cycle of length four, and established a sharp upper bound, that is, $D_{k_1+k_2}(G_1 \times G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$.

Very recently, Banič and Žerovnik in [2] have considered the fault diameter of the Cartesian graph bundle $G_1 \cdot G_2$, which contains Cartesian product graphs as its special case, and generalized Xu et al.'s result to $D_{k_1+k_2}(G_1 \cdot G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$.

In this paper, we consider the product graphs, a more general class of graphs that contains Cartesian graph bundles as its special case.

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Definition 1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. For each edge $xy = yx \in E_1$, assign two permutations φ_{xy} and φ_{yx} of V_2 such that $\varphi_{xy} = \varphi_{yx}^{-1}$. The product graph $G_1 * G_2$ has $V_1 \times V_2$ as the vertex set, two vertices (x, x') and (y, y') being adjacent if and only if either

$$x = y \quad \text{and} \quad x'y' \in E_2 \quad \text{or}$$
$$xy \in E_1 \quad \text{and} \quad y' = \varphi_{xy}(x').$$

By the definition, the product graph $G_1 * G_2$ can be viewed as formed by $|V_1|$ disjoint copies G_2^x ($x \in V_1$ here) of G_2 plus a perfect matching between copies G_2^x and G_2^y determined by φ_{xy} for each edge $xy \in E_1$.

The product graph is a good method in constructing large graphs with given degree and diameter there and first proposed by Bermond et al. in [3], in which the connectivity and diameter of $G_1 * G_2$ is discussed. In the paper by Balbuena et al. [1], the connectivity of $G_1 * G_2$ is discussed deeper.

The product graphs certainly contain a lot of graphs as its special cases.

For example, it is clear that the Cartesian product graphs are a subclass of product graphs by taking the identity mapping as the permutation φ_{xy} for every edge $xy \in E_1$. But unlikely the Cartesian product, the product graphs do not satisfy commutative law generally, namely $G_1 * G_2$ may be not isomorphic to $G_2 * G_1$.

The so-called Cartesian graph bundles, proposed by Pisanski et al. [6], are a larger subclass (compared to Cartesian product) of product graphs, since the permutation φ_{xy} in Cartesian graph bundles must be chosen to be an automorphism of G_2.

In addition, another family of graphs which often appears in literature, the permutation graphs introduced by Chartrand and Harary [4], can also be referred to as a special case of product graphs where $G_1 = K_2$.

In this paper, we show the following result, which contains two above-mentioned results, clearly.

Theorem 1. *Let G_i be a k_i-connected graph and $k_i \geqslant 1$ for $i = 1, 2$. Then*

$$D_{k_1+k_2}(G_1 * G_2) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1.$$

2. Proof of Theorem 1

Lemma 1. *Let $P_n = (x_0, x_1, \ldots, x_n)$ be a path of length n and G_2 a k_2-connected graph. Let $u = x_0y_1$ and $v = x_ny_2$ be two vertices of $P_n * G_2$, and $X \subseteq V(P_n * G_2) \setminus \{u, v\}$ with $|X| \leqslant k_2$. Then*

$$d_{(P_n*G_2)\setminus X}(u, v) \leqslant D_{k_2}(G_2) + n + 1,$$

where $(P_n * G_2) \setminus X$ denotes a graph obtained from $P_n * G_2$ by removing the vertices in X and the incident edges.

Proof. If $|X \cap V(G_2^{x_n})| = k_2$, then $X \cap (V(P_n * G_2) \setminus V(G_2^{x_n})) = \emptyset$. Let $Q = (x_ns_n, x_{n-1}s_{n-1}, \ldots, x_0s_0)$ be a path in $(P_n * G_2) \setminus X$ with $s_n = y_2$ (namely $x_ns_n = v$) and $s_{i-1} = \varphi_{x_ix_{i-1}}(s_i)$ for $1 \leqslant i \leqslant n$. Then Q is a path in $P_n * G_2$ avoiding X from v to some vertex $v_0 = x_0s_0 \in V(G_2^{x_0})$ of length n. Moreover, there is a path from v_0 to u in $G_2^{x_0}$ avoiding X with length at most $d(G_2)$. Therefore, $d_{(P_n*G_2)\setminus X}(u, v) \leqslant n + d(G_2)$.

Next, we assume that $|X \cap V(G_2^{x_n})| < k_2$. We find $k_2 + 1$ paths from u to some vertices in $G_2^{x_n}$ as follows. Let $Q_0 = (x_0w_0, x_1w_1, \ldots, x_nw_n)$ be a path in $P_n * G_2$ with $x_0w_0 = x_0y_1 = u$ and $w_i = \varphi_{x_{i-1}x_i}(w_{i-1})$, for $1 \leqslant i \leqslant n$. Since G_2 is k_2-connected, u has at least k_2 neighbors $u_1, u_2, \ldots, u_{k_2}$ in $G_2^{x_0}$. For each u_j ($1 \leqslant j \leqslant k_2$), we find a path $Q_j = (u, x_0t_0, x_1t_1, \ldots, x_nt_n)$ with $x_0t_0 = u_j$ and $t_i = \varphi_{x_{i-1}x_i}(t_{i-1})$, for $1 \leqslant i \leqslant n$. It is easy to check that these $k_2 + 1$ paths are disjoint except the vertex u. Furthermore, the length of Q_0 is n and the lengths of other k_2 paths are $n + 1$. Since $|X| \leqslant k_2$, there is at least one path avoiding X, denoted by Q', and the last vertex of Q' by z. Because $|X \cap V(G_2^{x_n})| < k_2$, there is a path from z to v avoiding X with length at most $D_{k_2}(G_2)$. Thus, we have $d_{(P_n*G_2)\setminus X}(u, v) \leqslant (n + 1) + D_{k_2}(G_2)$. □

Proof of Theorem 1. Let $G = G_1 * G_2$. Let $X \subseteq V(G)$ with $|X| < k_1 + k_2$ and u, v be two vertices in $V(G) \setminus X$. It is sufficient to show that $d_{G\setminus X}(u, v) \leqslant D_{k_1}(G_1) + D_{k_2}(G_2) + 1$.

If there is some $x \in V(G_1)$ such that both $u \in G_2^x$ and $v \in G_2^x$, we consider two subcases. If $|X \cap V(G_2^x)| < k_2$, then there is a path from u to x within $G_2^x \setminus X$ with length at most $D_{k_2}(G_2)$. If $|X \cap V(G_2^x)| \geqslant k_2$, then $|X \cap (V(G) \setminus V(G_2^x))| \leqslant k_1 - 1$. As x has at least k_1 neighbors in G_1, there is a neighbor x' of x such that $G_2^{x'}$ avoids X. Hence we can find a path from u to v through $G_2^{x'}$ of length at most $1 + d(G_2) + 1$.

So we may assume that u and v lie in different copies of G_2, say $u \in V(G_2^{x_1})$ and $v \in V(G_2^{x_2})$. Let $K \subseteq V(G_1) \setminus \{x_1, x_2\}$ be a set of $k_1 - 1$ vertices with $\sum_{x\in K} |X \cap V(G_2^x)|$ as large as possible. Obviously, there is a path Q from x_1 to x_2 in $G_1 \setminus K$ with length at most $D_{k_1}(G_1)$. Let

$$a = \sum_{x\in K} \left|X \cap V(G_2^x)\right|.$$

Case 1: $a \geqslant k_1 - 1$, then $\sum_{x\in V(G_1)\setminus K} |X \cap V(G_2^x)| \leqslant k_2$. By Lemma 1, we can find a path from u to v in

$Q * G_2$ avoiding X with length at most $D_{k_2}(G_2) + D_{k_1}(G_1) + 1$.

Case 2: $a < k_1 - 1$, by our choice of K, we have $X \cap V(G_2^x) = \emptyset$ for each $x \in V(G_1) \setminus (K \cup \{x_1, x_2\})$. Furthermore, there exists some $x^* \in K$ that $X \cap V(G_2^{x^*}) = \emptyset$. Let $K_1 = K \cup \{x_1\} \setminus \{x^*\}$ and $K_2 = K \cup \{x_2\} \setminus \{x^*\}$, then $|K_1| = |K_2| = |K| = k_1 - 1$. Let x_0 be a neighbor of x_1 in G_1 outside K_2, and x_0 exists because G_1 is k_1-connected. Then, there is a path R from x_0 to x_2 in $G_1 \setminus K_1$, of length at most $D_{k_1}(G_1)$. As before, we can find a path along $R * G_2$ of length at most $D_{k_1}(G_1)$ from v to some vertex v' in $G_2^{x_0}$, and let u' be the neighbor of u in $G_2^{x_0}$. Since $X \cap V(G_2^{x_0}) = \emptyset$, the distance between u' and v' are at most $d(G_2)$ in $G_2^{x_0} \setminus X$. Thus, we have found a path from u to v in $G \setminus X$ with length at most $D_{k_1}(G_1) + d(G_2) + 1$.

The proof of the theorem is complete. □

In the proof of Theorem 1, we find a path of length at most $D_{k_1}(G_1) + D_{k_2}(G_2) + 1$ between any two vertices u and v in $G - X$, which implies that $G - X$ is still connected, where X is any subset of vertices with $|X| \leqslant k_1 + k_2 - 1$. Thus, we obtain the following corollary.

Corollary 1. *If G_i is k_i-connected for $i = 1, 2$, then $G_1 * G_2$ is $(k_1 + k_2)$-connected.*

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